Uncertainty under hyperbolic discounting: the cost of untying your hands

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Abstract

Purpose – The relevance of present consumption bias on personal finance has been confirmed in several studies and has important theoretical and practical implications. It has important, measurable implications when analyzing commitment or self-control, adherence to healthy habits (e.g. exercising or dieting), procrastination tendencies or savings. The purpose of this paper is to contribute to our understanding of these issues by postulating a model of income uncertainty within a hyperbolic discounting framework that measures the cost of financial intertemporal inconsistencies related to this bias. The emphasis is on the analysis of this cost. We also propose experimental designs and consistent estimation methods, as well as agent-based modelling extensions.

Design/methodology/approach – The authors develop a finite-horizon model with hyperbolic preferences. Individuals have a present bias distinct from their discount rate so their choices face intertemporal inconsistencies. The authors further extend the analysis with uncertainty about future incomes. Specifically, individuals live for three periods, and the authors find the optimal consumption levels in the perfect-information benchmark by backward induction. They then proceed to add biases and uncertainty to characterize their implications and measure the costs of the intertemporal inconsistencies they cause.

Findings – The authors measure how an agent's utility is greater when they "tie their hands" than when they are free to re-evaluate and change their consumption schedule. This "cost of being vulnerable to falling into temptation" only depends (increasingly) on the measure of the present bias and (decreasingly) on the discount factor. They analyze the varying effects on utility and consumption of changes in impatience and optimism. They conclude by discussing theoretical and practical implications; they also propose agent-based simulations, as well as empirical and experimental designs, to further test the relevance and applications of the results.

Practical implications – This model has important, measurable implications when analyzing commitment or self-control, adherence to healthy habits (e.g. exercising or dieting), procrastination tendencies or savings.

Social implications – The results enhance the estimation of the costs of present biases such that employers can better identify the incentives required to acquire and retain human capital. The authors provide evidence that workers are vulnerable to contract renegotiations and about the need for a regulator that restores ex-ante efficiency. Similarly, in the private sector, firms could recognize the postulated consumer profiles and focus their resources on anxious, too-optimistic or potentially addictive consumers; this, again, provides some justification about the need for a regulator.

Originality/value – In traditional exponential discounting, the marginal rate of substitution of consumption between two points depends only on their distance; thus, it allows none of the intertemporal

JEL classification – C73, D14, D81, G02

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inconsistencies we often observe in real life. Therefore, hyperbolic discounting better fits the data. The authors model choice under uncertainty and focus on the costs caused when present biases (ex-post) push behaviour away from ex-ante optimality. They conclude by proposing experimental designs to further enhance the estimation and implications of these costs. The postulated refinements have the potential to improve previous analyses on commitment devices and commitment-related regulation.

Keywords Uncertainty, Behavioural economics, Bounded rationality, Experimental economics, Hyperbolic discounting, Personal finances, Present bias, Renegotiation

Paper type Research paper

"O Lord, . . .
Give me chastity
and continency!
But not yet!"
Saint Augustine, *Confessions*,
Book VIII, Chapter VII

1. Introduction

Recent literature provides evidence about how the *hyperbolic discounting model* (Laibson, 1997) is a good predictor of human and animal choices (Dasgupta and Maskin, 2005; Redden, 2007; Thaler and Benartzi, 2004)[1]. It is a generalization of the exponential discounting model that assumes agents have a preference for present consumption (a *present bias*) and, therefore, when lacking self-control, will face intertemporal inconsistencies in their choices inasmuch as these choices are evaluated differently through different periods. Within the traditional intertemporal models of exponential discounting, the marginal rate of substitution between any two points in time depends only on their distance so intertemporal inconsistencies are not allowed. On the other hand, the hyperbolic discounting model indeed includes them–two examples are heterogeneous susceptibility to addictive behaviours and to procrastination–so it is important to have access to hyperbolic models on income uncertainty where choices do not only depend on expectations about payments.

In this article, we create a model that includes uncertainty on future incomes within the framework of this behavioural bias. Our main results deal with the cost agents suffer after changing the present expenditure that was maximized during previous periods. Besides present bias, these changes do not depend on bounded rationality, imperfect information or exogenous shocks as we are simply measuring the effects of lack of self-control. To this effect, we compare *optimal* behaviour (*tying up your hands* for future consumption) with *predicted behaviour* after generalizing our model by relaxing the assumptions that prevent intertemporal inconsistencies.

We find that individuals pay a cost when they are victims of present bias temptation and we analyze the decision paths agents follow when their behavioural characteristics vary: impatience, optimism, wealth and other initial conditions. We focus on how consumption paths change when parametric behavioural characteristics vary; this allows us to describe the real-world implications of our model. Finally, we propose agent-based model (ABM) simulations as well as experimental designs to test our hypotheses and permit the estimation of the utility costs we analyze in order to refine the policies we suggest.

2. Literature review

Redden (2007) finds that the hyperbolic discounting model is a good fit for agent behaviour as it reflects the tendency to choose immediate rewards because of impatience. He shows this model can be applied to self-control and addiction issues and, in general, contributes to the understanding of the difference between active vs. passive interest rates. In their seminal text on behavioural economics, Mullainathan and Thaler (2000) analyze individuals in everyday contexts, basing their analysis on Simon (1955), who postulates rationality might be limited and, thus, individuals should not be generally expected to follow the rational behaviour of simple theoretical models. Similarly, Tversky and Kahneman (1985) recognize the existence of various determinants of agents' choices such as oversimplification, heuristics, and overconfidence.

Likewise, Yaari (1965) and Thaler and Benartzi (2004) suggest that agents with hyperbolic preference procrastinate because they wrongly assume that future actions cannot be as important as present behaviour. Harris and Laibson (2001) assert that economists wrongfully assume rationality when studying individual choice. In contrast, hyperbolic preference models untangle inefficiencies by extending the concepts of rationality. Mazur and Coe (1987) study indifference between varying time intervals and rewards for animals. They find that hyperbolic discounting yields the best fit for the empirical data.

Experimental literature comparing exponential with hyperbolic discounting provides strong evidence in favour of the latter within realistic scenarios. Madden *et al.* (1999) and Tang *et al.* (2016) corroborate its better fit, suggesting that future theories should incorporate this extension to previous discounting models and focus their analysis on the influence of risk and uncertainty on individual conduct. Romano (2015) also highlights how in real-life settings, hyperbolic utility functions imply dynamic inconsistencies so time can change relative preferences. He describes how if agents are aware of these inconsistencies, they can use two kinds of strategies to deal with them: agents can use commitment devices (such as compulsory retirement savings) or restrict themselves to actions not affected by time inconsistencies. These strategies are at the core of our investigation.

Sozou (1998) and Azfar (1999) show that if agents are uncertain about their discount rates, it is rational to discount hyperbolically. They prove that uncertainty on discount rates can sometimes imply time inconsistencies like deviating from optimally determined consumption paths by overconsuming during the initial periods. Takahashi *et al.* (2007) find that the subjective probability of obtaining a reward is discounted hyperbolically. Thus, the valuation of rewards is related to delay time but not with discounts on subjective probability. They conclude that the difference between these two is a measure of present biases.

Rachlin *et al.* (1991) run an experiment that involves varying payoffs through time to analyze individual behaviour under uncertainty on future income. Participants face trade-offs between reduced risk and payoff delay. Their results provide evidence in favour of hyperbolic discounting: risk investment decisions and consumption delaying in these experimental conditions imply preferences for present consumption that go beyond the results of exponential discounting.

Finally, Jon Elster (2002) states that human rationality is imperfect, and this is illustrated in what today is known as the *Ulysses Pact*. In Homer's *Odyssey*, the main character, aware of his weaknesses, requests his men to tie him to his ship's mast before they approach the Sirens. Elster (1979) states that phenomena like altruism and indiscipline are related to alternative forms of rationality which commonly direct individual behaviour. It should be noted that the literature does not generally consider hyperbolic preferences *irrational* but an extension of rationality modelling.

2.1 Consumption and savings through time

Individual choices on consumption and savings are linked to how utility relates to time. Under Samuelson's (1937) framework, discounting is stationary, which implies no time inconsistencies and a constant discount rate. Therefore, the probability distribution of consumption outcomes at any point in time is constant. However, experimental, field and theoretical research have shown that time preferences are not stationary, generating a new subfield of academic research in charge of extending previous discount models to better reflect individual preferences.

2.1.1 Exponential discounting. We proceed by describing classical discounting under Samuelson's (1937) framework. The utility is discounted in a simple and discrete fashion so if consumption levels are constant, the geometric distance between the instantaneous utility of any two consecutive periods (and, thus, between any two-time intervals of the same distance) is constant too. The exponential discount parameter $\delta \in [0,1]$ therefore defines the present value of consumption as $\tilde{U}\left(\{c_t\}_{t=1}^{\infty}\right) := \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$, where c_t is consumption at time t and $u()$ is the instantaneous utility function. However, as Samuelson states, this can be falsified by empirical data, if intertemporal inconsistencies are observed. His research promoted further studies looking for counterexamples and extensions to exponential discounting; subsequent economics, and psychology studies on human and animal behaviour have found that the relative value of rewards is negatively related to its time, but not in a constant way.

2.1.2 Hyperbolic discounting. Aware of the caveats of exponential discounting models, social scientists provided extensions using hyperbolic functions. These make exponential discounting a particular, with stronger assumptions, case of hyperbolic discounting, which is less restrictive. Simple, discrete modelling allows the introduction of intertemporal inconsistencies generated by present bias, enabling forecasts of procrastination and impatience. We now have better basic explanations of why, for instance, gyms are overcrowded in January or why Odysseus begged to be tied (and then untied) to the mast.

Hyperbolic discounting introduces the parameter $\beta \in [0,1]$ as an additional discount factor reflecting present-day preferences. The discounted utility is now $U\left(\{c_t\}_{t=1}^{\infty}\right) := u(c_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(c_t)$. Thus, when $\beta = 1$, we are facing exponential discounting. If $\beta = 0$ the effect is the same as if $\delta = 1$; agents do not care about future consumption and $\forall t > 1$: $\partial U / \partial c_t = 0$.

As Green and Myerson (2004, 2010) emphasize, contemporary behavioural analysis benefits from including risk measures within hyperbolic modelling. Adding uncertainty to the hyperbolic framework better reflects the conditions behind agents' decisions because the choice between an immediate and a future reward typically requires modelling of risk aversion. Therefore, following Greene's influence, we postulate that modelling future income uncertainty constitutes a tool to enhance our understanding of preferences and human behaviour, highlighting its vulnerabilities.

3. Model: hyperbolic discounting under uncertainty

Our contribution is related to the effects of future expectations on income and the costs generated by the deviation from optimal future decisions. Our findings enhance consumption modelling under different expectation profiles and contribute to the consolidation of models with time inconsistencies which, according to empirical and experimental data, most closely resembles the behaviour of humans and animals. The premise behind present bias is that *people prefer immediate* versus *future rewards, independently of the discount factor* δ.

Therefore, utility maximization problems solved at the start of a payoff sequence will generally not hold through time so agents will tend to deviate from initial consumption paths, even without external shocks or changes in preferences. Our first enhancement to the model above is the inclusion of future income uncertainty, reflecting real-life scenarios. This allows us to measure how variations on individual emotional profiles and on estimations of future outcomes impact present consumption and consumption schedules.

3.1 Timing

There are three periods: infancy ($t = 1$), adulthood ($t = 2$) and old age ($t = 3$). When he is born, the agent receives an initial endowment $w > 0$ and generates an additional income $r > 0$ when he is an adult. However, with a probability $p \in [0,1]$, an external shock destroys r (either he gets ill or he becomes a behavioural economist). The agent consumes c_t during each period and dies at the end of his old age.

Figure 1 illustrates all possible consumption paths where our starting point is Harris and Laibson (2001). The first decision node (c_1) represents consumption during $t = 1$. Afterward, the agent observes nature's move: at the start of $t = 2$, the outcome of the random variable r is revealed, and the agent knows if he received this extra reward. The next decision nodes are the consumption values during $t = 2$ where c_2^1 represents consumption without an additional income (with probability p) and c_2^2 is the second-period consumption after receiving the bonus. Lastly, the consumer enjoys either c_3^1 or c_3^2 during his old age.

The temptation is at the core of our model, so we proceed by backward induction since it is essential that we allow the agent to constantly re-evaluate his behaviour. We can use this technique since this is a finite-horizon problem where all scenarios fall within a well-defined decision tree (Gibbons, 1997). First, we identify optimal consumptions during the last period, and we continue until reaching the first. Note our modelling includes both of the riskless (p = 0, 1) as well as the simple exponential discounting ($\beta = 1$) cases. This allows us to analyze how different probability scenarios relate to different behavioural profiles (positive and optimistic, neutral, or negative and pessimistic). The consumer solves the following maximization problems during his lifetime:

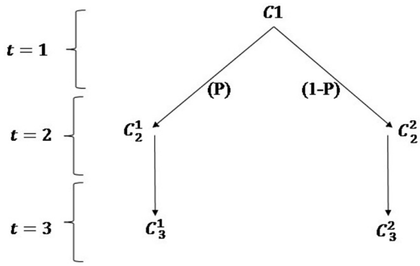


Figure 1.
Game tree

Source: The authors

$$\mathbf{t=3}: c_3^{1*} = w - c_1^* + c_2^{1*},\ c_3^{2*} = w + r - c_1^* + c_2^{2*}$$

$$\mathbf{t=2}: \max_{c_2^i}\left\{u\left(c_2^i\right) + \beta\delta u\left(c_3^{i*}\right)\right\}, i = 1,2, \qquad s.t.: w = c_1^* + c_2^1 + c_3^{1*}, w + r = c_1^* + c_2^2 + c_3^{2*}$$

$$\mathbf{t=1}: \max_{c_1}\left\{u(c_1) + \beta\delta\,\mathbb{E}\left[u\left(c_2^{i*}\right) + \delta u\left(c_3^{i*}\right)\right]\right\}\ s.t.: w = c_1 + c_2^{1*} + c_3^{1*}, w + r = c_1 + c_2^{2*} + c_3^{2*}$$

Note the interest rate is zero. A strictly positive interest complicates the analysis without fundamentally changing our main results. To solve the model, we assume utility is logarithmic, $u(c) = \ln c$ such that $u' > 0$ and $u'' < 0$. Since negative consumption values are not defined in our utility and given that $\lim_{c\downarrow 0} u(c) = -\infty$, then $\forall t$: $c_t \in (0, w)$ (borrowing is allowed but it is never optimal), for any initial values. Thus, consumption at $t = 1$ cannot exceed the initial endowment because otherwise, with a p probability, the consumer would be left without resources for future periods.

3.2 Untied hands model: Backward induction predictions

We start by modelling the behaviour of an individual affected by a present bias. At the beginning of each period, consumption schedules can be re-evaluated. As long as β is strictly less than one, actual behaviour is different than planned behaviour. *We decide to start exercising next week but the morning we are supposed to go to the gym, we postpone our plans.* This inconsistency is costly and, to calculate it, we proceed by backward induction.

P1. In the unique equilibrium, the marginal utility of the first period's consumption is equal to the expected value of the marginal discounted utility of savings a_1^*:

$$u'\left(c_1^*\right) = -\beta\delta(1+\delta)\mathbb{E}\left[u'\left(a_1^*\right)\right]$$

Proof. In the final period ($t = 3$) the individual consumes all that is left. Therefore, in the second period ($t = 2$) the optimization problem is given by the following restrictions, solvable by substitution:

$$\begin{aligned}
c_2^{1*} &= \underset{c_2^1}{\operatorname{argmax}}\{\ln c_2^1 + \beta\delta\ln c_3^{1*}\}, \quad s.t.: c_3^{1*} = w - c_1^* - c_2^1 \\
&= \underset{c_2^1}{\operatorname{argmax}}\{\ln c_2^1 + \beta\delta\ln\left(w - c_1^* - c_2^1\right)\}; \\
c_2^{2*} &= \underset{c_2^2}{\operatorname{argmax}}\{\ln c_2^2 + \beta\delta\ln c_3^{2*}\}, \quad s.t.: c_3^{2*} = w + r - c_1^* - c_2^2 \\
&= \underset{c_2^2}{\operatorname{argmax}}\{\ln c_2^2 + \beta\delta\ln\left(w + r - c_1^* - c_2^2\right)\}.
\end{aligned}$$

The first order conditions (FOC) are:

$$\frac{1}{c_2^{1*}} = \frac{\beta\delta}{w - c_1^* - c_2^{1*}}, \frac{1}{c_2^{2*}} = \frac{\beta\delta}{w + r - c_1^* - c_2^{2*}}.$$

Marginal utility at $t=2$ is in both cases equal to the discounted marginal utility of $t=3$ consumption. Relative consumption c_2^{1*}/c_2^{2*} after observing the realization of r does *not* depend on β, δ nor p. The difference $c_2^{1*} - c_2^{2*}$ depends on $\beta\,\delta$ and this is the source of our agent's intertemporal inconsistencies since, as we discuss below, by maximizing consumption at $t=1$, the consumer *wrongfully behaves as if* relative consumption only depends on δ. This overestimation of self-control only disappears when $\beta = 1$.

Throughout our analysis, given the concavity of u, we assume that the FOCs are sufficient and necessary to reach a local and global maximum. After solving c_2^{i*}, the consumption paths during the first period are described by the two new restrictions added from the maximization process of the second period:

$$c_1^* = \text{argmax}_{c_1}\left\{\ln c_1 + p\left(\beta\delta\ln c_2^{1*} + \beta\,\delta^2\ln c_3^{1*}\right) + (1-p)\left(\beta\delta\ln c_2^{2*} + \beta\,\delta^2\ln c_3^{2*}\right)\right\}$$

$$s.t.: c_3^{1*} = w - c_1 - c_2^{1*}, c_3^{2*} = w + r - c_1 - c_2^{2*}, c_2^{1*} = \frac{w - c_1}{1+\beta\delta}, c_2^{2*} = \frac{w + r - c_1}{1+\beta\delta}. \quad (1)$$

These restrictions (1) correspond to the optimal choices calculated for later periods (thus the only choice variable left is c_1). Substituting the restrictions, the first period's problem is:

$$c_1^* = \text{argmax}_1^c\left\{\ln c_1 + p\left[\beta\delta\ln\frac{w-c_1}{1+\beta\,\delta} + \beta\delta^2\ln\left(w - c_1 - \frac{w-c_1}{1+\beta\delta}\right)\right]\right.$$
$$\left. + (1-p)\left[\beta\delta\ln\frac{w+r-c_1}{1+\beta\delta} + \beta\delta^2\ln\left(w + r - c_1 - \frac{w+r-c_1}{1+\beta\delta}\right)\right]\right\}$$

$$\text{with FOC } \frac{1}{c_1^*} = \beta\delta(1+\delta)\left[\frac{p}{w - c_1^*} + \frac{1-p}{w + r - c_1^*}\right],$$

which is the expanded form of the proposition's equation.

This is a quadratic function of c_1^*. Since, as discussed above, we require its solution to be in $(0, w)$, we discard the solution that is greater than w. Finally, the first period's consumption path is given by equation (2), a software-verified function of the parameters (β, δ, p, w, r):

$$c_1^* = \frac{2w + r + \gamma(w+pr) - \sqrt{r^2 + \gamma\left[2r(2pw - w + pr) + \gamma(w+pr)^2\right]}}{2(1+\gamma)} \quad (2)$$

where $\gamma := \beta\,\delta\,(1+\delta)$. In the case without uncertainty ($p=1$, $r=0$), when we know there will be no extra bonus (or its value is zero) and we simply maximize our initial endowment w throughout the three periods: $c_1^*|_{p=1,r=0} = \frac{w}{\beta\,\delta(1+\delta)}$. On the other extreme without uncertainty, where the extra bonus has no risk: $c_1^*|_{p=0} = \frac{w+r}{\beta\,\delta(1+\delta)}$. This also allows us to isolate the nature of the of the special case of simple exponential discounting, which happens when $\beta = 1$, with known solution.

After replacing c_1^* in (1), we get the consumption path of each $t = 2$ scenario. Iterating in $t = 3$, yields the final period consumption paths:

$$c_3^{1*} = \frac{\beta\,\delta}{1+\beta\,\delta}\left(w - c_1^*\right), c_3^{2*} = \frac{\beta\,\delta}{1+\beta\,\delta}\left(w + r - c_1^*\right). \tag{3}$$

The difference between both $t = 2$ consumptions is the discounted magnitude of the bonus: $r/(1 + \beta\,\delta)$. This difference is multiplied by $\beta\,\delta$ in the last period's consumptions. This is intuitive since future consumption positively depends on these individual discount factors.

3.2.1 Equilibrium analysis. Given the mathematical complexity of the equilibrium, to analyze the behaviour of the optimal choice variables, we assume the following parameterization: $\beta = 0.5, \delta = 0.8, p = 0.5, w = 3, r = 1$. When estimating, for instance, the effects of an increase of the initial endowment on consumption paths, the only free parameter is w. Like so, we verify that $\forall i, t: \partial c^i_t/\partial w > 0$.

As expected, an increase $dw > 0$ has a positive effect on all consumptions: the individual has more available resources to manage. Likewise, the reward r has a positive effect on all three periods. A potential additional income is smoothed through individual utility maximization. Below, we discuss estimation and simulation methods with more detail.

In contrast, under our parameterizations, the discount factor β (the source of the hyperbolic model's time inconsistencies) has a negative effect on the first period's consumption (c_1) and a positive effect on c_2 and c_3. An increase on β better describes a patient consumer, willing to sacrifice present-day consumption. Similarly, δ also has a negative effect on present consumption and an increasingly positive effect on future consumption. An increase on either discount factors makes present consumption relatively costly.

Lastly, the probability of receiving an additional income $(1 - p)$ has a positive impact on $t = 1$ consumption since our agent is concerned about expected wealth since the initial period. Interestingly, this implies we predict the opposite effect on the $t = 2$ and $t = 3$ consumption branches after the extra bonus r is actually *not* observed (c_2^1, c_3^1). Furthermore, if there are no concerns about temporal inconsistencies, as in the next model, the consumer only cares about expected wealth.

3.3 Tied hands model: forward induction optimal paths

To compare typical solution methods with optimal behaviour, we now solve the problem of an individual able to *tie his own hands*; that is, when maximizing the present expected value of utility at the first period ($t = 1$), all future decision choices are determined and there is no possible deviation from these paths. When determining c_2^i and c_3^i, we do it from today's perspective (discounted consumption at $t = 1$) whereas, in the Untied Hands Model, the agent maximizes c_2^i from tomorrow's perspective. We can think of this as an agent with a perfect *commitment device* who in this benchmark model suffers no intertemporal inconsistencies and, thus, to predict his behaviour we do not need backward induction. This model is mathematically simpler than the previous one and the only restrictions are about total expected wealth. Therefore, the maximization problem is:

$$\max_{c_1, c_2^i, c_3^i}\left\{\ln c_1 + p\left[\beta\delta \ln {c^1}_2 + \beta\delta^2 \ln {c^1}_3\right] + (1 - p)\left[\beta\delta \ln {c^2}_2 + \beta\delta^2 \ln {c_3}^2\right]\right\}$$

$$s.t.: c_3^1 = w - c_1 - c_2^1, c_3^2 = w + r - c_1 - c_2^2.$$

After substituting the c_3^i restrictions in the objective function, the FOC solutions of the second period are:

$$c_2^{1*} = \frac{w - c_1^*}{1+\delta},\; c_2^{2*} = \frac{w + r + c_1^*}{1+\delta}.$$

Here we can identify the consequences of present bias and of the behavioural changes suffered because of intertemporal inconsistencies. As opposed to the first model's solutions, β gets cancelled and the difference between $t = 2$ and $t = 3$ consumptions no longer depends on the parameter representing present bias. This *Tied Hands* model is equal to the first model and to an exponential discount model in the particular and less realistic case of $\beta = 1$. This confirms a well-known result: with exponential discounting, there are no intertemporal inconsistencies so forward induction and backward induction yield the same results. In other words, if $\beta = 1$, even if the agent is allowed to re-evaluate her consumption schedule tomorrow, he will have no incentives to change today's planned behaviour.

The same thing happens when the discount factor δ tends to 0 and without having to assume intertemporal biases, we are already infinitely impatient: we do not care about future consumption and since first period decisions are independent of whether our hands are tied, consumption is the same in both cases. Note too how, as in the previous model, the difference $c_2^{1*} - c_2^{2*}$ does not depend on p. The risk related to the extra bonus r only impacts our optimal consumption schedule ${c_t^i}^*$ through its effect on c_1^*.

The third equation of this three-variable system is the FOC with respect to c_1:

$$\frac{1}{c_1^*} = \beta\delta^2\left(\frac{p}{w - c_1^* - c_2^{1*}} + \frac{1-p}{w + r - c_1^* - c_2^{2*}}\right),$$

whose solution after substituting c_2^{i*} coincides with the $c_1^* = c_1(\beta, \delta, w, p, r)$ solution (2) of the previous model: the maximization with respect to initial conditions is the same in both cases. So, although present biases only influence $t = 1$, their effects on relative consumption can only be observed from $t > 1$. After the comparison of c_2^i between models, this is the second most important intuition of the model. It is, perhaps, the most counterintuitive. After celebrating New Year's Eve, on 1 January I go shopping for sports clothing, food supplements, and gym memberships. Our models predict that investments related to New Year resolutions are exactly the same whether we have commitment devices at our disposal[2]. This is not related to lack of information: I am aware I'll stop working out in a month but today I act precisely as if I were going to train the whole year long, and this is a costly mistake. We numerically verify this result in the section on estimation and simulations. See Figures 3 and 4 as well as Appendix 2.

Consumptions c_3^{i*} are found by substituting in the restrictions. Main variable behaviour with respect to changes in the parameters (p, w, r, δ, β) is analogous to the results from the previous model. Particularly, in the extreme cases of perfect information ($p \in \{0,1\}$), the difference between consumption paths does not disappear and the *cost of untying your hands* is still real.

3.4 Contrast of equilibria: the cost of suffering behavioural biases

Untying your hands today allows yourself to fall into the temptation of over-consuming tomorrow compared to the optimal consumption levels, generating a future predictable scarcity which is costly. These intertemporal inconsistencies affect consumption starting in the second period (*tomorrow*) so we predict no over-consumption during the first one. This is what Odysseus (Ulysses) feared when approaching the Sirens: their singing would render him incapable of rational thought, so he devised the first known commitment device, known as the *Ulysses Pact*. He put wax in his men's ears and had them tie him to the mast so that he could not jump into the sea.

Formally, the indirect utilities of the *Untied Hands* and *Tied Hands* models are, respectively:

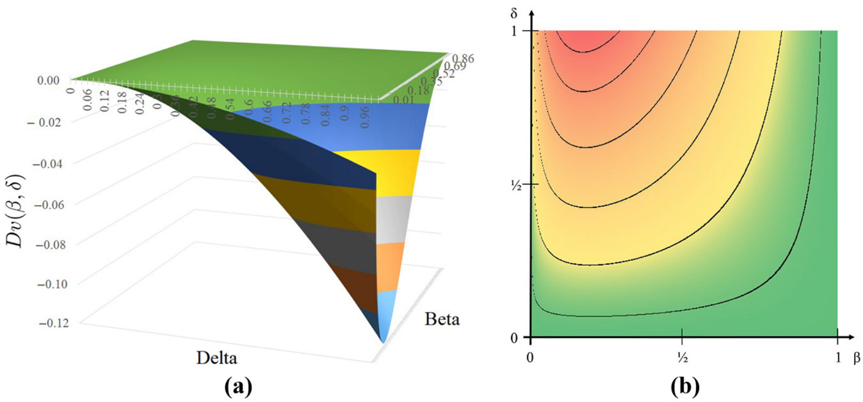
$$v_1(\beta,\delta,p,w,r) = \ln c_1^* + \beta\delta\left[p\ln\frac{w-c_1^*}{\beta\delta+1} + (1-p)\ln\frac{w-c_1^*+r}{\beta\delta+1}\right] + \beta\delta^2\left[p\ln\frac{\beta\delta\left(w-c_1^*\right)}{\beta\delta+1} + (1-p)\ln\frac{\beta\delta\left(w+r-c_1^*\right)}{\beta\delta+1}\right]$$

$$v_2(\beta,\delta,p,w,r) = \ln c_1^* + \beta\delta\left[p\ln\frac{w-c_1^*}{\delta+1} + (1-p)\ln\frac{w-c_1^*+r}{\delta+1}\right] + \beta\delta^2\left[p\ln\frac{\delta\left(w-c_1^*\right)}{\delta+1} + (1-p)\ln\frac{\delta\left(w+r-c_1^*\right)}{\delta+1}\right].$$

The *cost of untying your hands* is the difference between both, illustrated in Figure 2, where we can simultaneously observe the effects of β and δ:

$$Dv(\beta,\delta) := v_1 - v_2 = \beta\delta\left[(1+\delta)\ln\frac{\delta+1}{\beta\delta+1} + \delta\ln\beta\right] < 0.$$

This only depends on (β,δ) since v_1 and v_2 have similar structures so the other variables (p,w,r,c_1^*) get cancelled. Note v_2 is the objective function $\mathbb{E}U(\{c_t\}_{t=1}^3)$ evaluated at the optimum whereas v_1 is the same function evaluated on the more realistic consumptions from an optimization problem with more restrictions: budget constraints plus the restrictions that arise from backward induction. Therefore, *the cost of untying your hands* $(-Dv)$ is positive for all $\beta,\delta \in (0,1)$. The proof is in the Mathematical Appendix 1.



Notes: (a) 3D graph; (b) heat map
Source: The authors

Figure 2. $Dv(\beta,\delta) = \beta\delta[(1+\delta)\ln\frac{\delta+1}{\beta\delta+1} + \delta\ln\beta] < 0$

If $\beta \uparrow 1$ (when the hyperbolic model tends to an exponential model) or $\delta \downarrow 0$ (when the future does not have significant importance in the utility function), then DV tends to 0 as Figure 2 illustrates. It is also intuitive that the $\beta \downarrow 0$ case is parallel (see Mathematical Appendix 1): the consumer has no incentive to deviate from initial choices and has no need to *tie her hands*. In contrast, as δ tends to 1, the individual has a greater preference for the future, which implies that a deviation in her future choices entails a greater impact on utility. $Dv(\beta, \delta)$ has a global minimum at (1, 0.167) (Appendix 1).

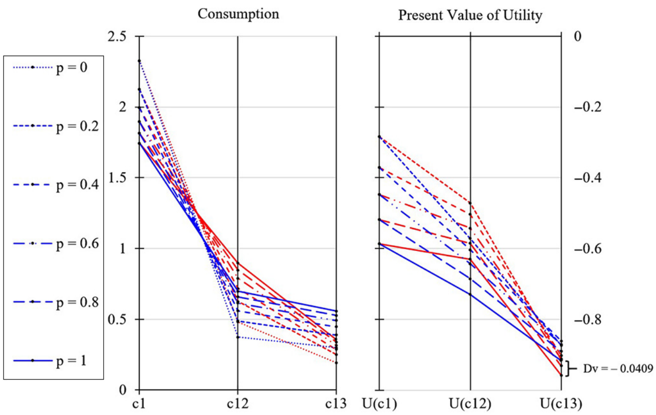
Table I summarizes and contrasts the results from both models where $\eta := \delta(1-\beta)/(1+\delta)(1+\beta\delta) \in (0,1)$. While the *Difference* column is at the core of our analysis, we add the *Ratio* column which we will use in the section on estimation. Expected consumption $\mathbb{E}c = c_1 + p\left(c_2^1 + c_3^1\right) + (1-p)\left(c_2^2 + c_3^2\right)$ is the wealth to be distributed between the three periods.

We can simulate various initial conditions as in Figures 3 and 4 where we control whether the consumer receives the extra bonus r (Appendix 2). On the left-hand-side graphs,

	Untied hands	Tied hands	Difference	Ratio
U	v_1	v_2	Dv	$v1/v2$
$\mathbf{c_1}$	c_1^*	c_1^*	0	1
$\mathbf{c_2^1}$	$\frac{1}{1+\beta\delta}\left(w-c_1^*\right)$	$\frac{1}{1+\delta}\left(w-c_1^*\right)$	$\eta\left(w-c_1^*\right)$	$\frac{1+\delta}{1+\beta\delta}$
$\mathbf{c_2^2}$	$\frac{1}{1+\beta\delta}\left(w+r-c_1^*\right)$	$\frac{1}{1+\delta}\left(w+r-c_1^*\right)$	$\eta\left(w+r-c_1^*\right)$	$\frac{1+\delta}{1+\beta\delta}$
$\mathbf{c_3^1}$	$\frac{\beta\delta}{1+\beta\delta}\left(w-c_1^*\right)$	$\frac{\delta}{1+\delta}\left(w-c_1^*\right)$	$-\eta\left(w-c_1^*\right)$	$\frac{\beta(1+\delta)}{1+\beta\delta}$
$\mathbf{c_3^2}$	$\frac{\beta\delta}{1+\beta\delta}\left(w+r-c_1^*\right)$	$\frac{\delta}{1+\delta}\left(w+r-c_1^*\right)$	$-\eta\left(w+r-c_1^*\right)$	$\frac{\beta(1+\delta)}{1+\beta\delta}$
Ec	$w+(1-p)r$	$w+(1-p)r$	0	1

Source: The authors

Table I.
Model comparison



Source: The authors

Figure 3.
Consumption and present value of utility with tied (blue) and untied hands (red), no extra income ($r = 0$)

we confirm how c_1 is always decreasing in p but all other consumptions c^t_j are increasing in p. The righthand-side graphs illustrate the evolution of the present value of utility as defined in Subsection 2.1.2. At $t = 2$, it is defined as the discounted total consumption during the first two periods: $U(c^2_j) := u(c_1) + \beta\,\delta\, u(c^2_j)$ whereas $U(c^3_j) := u(c_1) + \beta\,\delta\, u(c^2_j) + \beta\,\delta^2 u(c^3_j)$.

Interestingly, in both *Tied* and *Untied* models, the present value utility at the end ($U(c^3_j)$) is increasing on p when no extra income was realized but decreasing when it was. When calculating consumption from an optimistic scenario (determining c_1 given a low p), utility drops when the outcome is actually bad (when $r = 0$ is observed in $t = 2$). Also, we confirm $Dv(\beta, \delta)$ does not depend on (p, r, w).

3.5 Simulations, estimation and comparative statics

If experimental or field data about relative consumption is available with and without self-control mechanisms, it is possible to consistently estimate the population parameters ($\hat{\beta}$, $\hat{\delta}$). Under some reasonable assumptions (continuity, boundedness, finite variance), if we know $\bar{c}^1_{2,UH}$ and $\bar{c}^1_{2,TH}$, the average consumption levels at $t = 2$ after observing $r = 0$ from both models (Untied Hands and Tied Hands), Slutsky's Theorem tells us that their quotient is a consistent estimator (but biased, because of Jensen's Inequality) of the following expression:

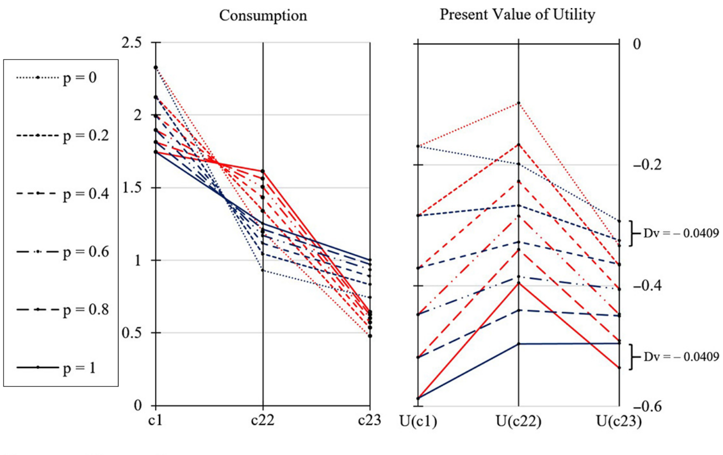
$$\hat{x} := \frac{\bar{c}^1_{2,SAM}}{\bar{c}^1_{2,AM}} = \frac{1 + \hat{\delta}}{1 + \hat{\beta}\hat{\delta}} \xrightarrow{d} \frac{1 + \delta}{1 + \beta\delta}.$$

Similarly, the quotient:

$$\hat{y} := \frac{\hat{\beta}\left(1 + \hat{\delta}\right)}{1 + \hat{\beta}\hat{\delta}}$$

can be estimated using $\bar{c}^1_{3,UH}/\bar{c}^1_{3,TH}$. See the *Ratio* column of Table I After rearranging these equations, we propose the following estimators:

$$\hat{\beta} = \frac{\hat{y}}{\hat{x}}, \hat{\delta} = \frac{1 - \hat{x}}{\hat{y} - 1}.$$



Source: The authors

Figure 4.
Consumption and present value of utility with tied (blue) and untied hands (red), with extra income ($r = 1$)

We use $c^1{}_t$ although consistent estimation can also be achieved with $c^2{}_t$, the consumption levels after observing *r = 1*. Efficient convex combinations of these estimators, as well as their asymptotic properties and a discussion on errors and hypothesis testing, goes beyond the scope of this article.

In an experimental design sketch to estimate $\vec{c}_t^{\,i}$, participants are divided into control and treatment groups. Each one attends three sessions. At the start of the first session, they observe the result of a random variable that determines how many minutes they can perform some pleasant activity (or avoid an activity that entails disutility or effort; for instance, some repetitive task, Neckermann *et al.*, 2014, or Augenblick and Rabin, 2018). During the first two sessions, they choose between spending the minutes they won or *saving* them for next session; this allows estimation of the individual utility parameters β and δ which in turn can be used to estimate *Dv*, the cost of suffering a present bias without taking care of it. Further tests could help identify whether the source of time inconsistencies is either present bias or systematic misevaluation of outcome probabilities.

The first experimental hypothesis is that in a treatment group were participants are given the chance to use commitment devices that facilitate self-control (the simplest is a binding contract in which they describe their strategy to the researcher), the use of pleasant time during the first session (c_1) will be equal to its use in a control group without access to these devices. The second hypothesis is that the use of these devices will depend positively on $\hat{\delta}$ and negatively on $\hat{\beta}$, the estimated values.

As an example and to illustrate the comparisons of Table I, in Table II we assign values in (0,1) to the parameters (p, β, δ). Following above methodology and assuming the population averages we observe coincide with the optimistic scenario below: $\left(\bar{c}^1_{2,UH}, \bar{c}^1_{2,TH}, \bar{c}^1_{3,UH}, \bar{c}^1_{3,TH}\right) = (0.565, 0.440, 2.26, 0.352)$. The estimated coefficients are $(\hat{x}, \hat{y}) = (1.286, 0.643)$ so in this case $(\hat{\beta}, \hat{\delta}) = (0.500, 0.801)$ is very close to the real values.

When contrasting the parameterizations that we call *optimistic* and *pessimistic scenarios*, we verify utilities and initial consumptions are decreasing in p, showing that the behavioural biases we analyze do not break the general tendency to soften intertemporal consumption. Also, in line with our predictions, *the cost of untying your hands*, the decrease of equilibrium utility between both models, does not depend on p. We keep the row E**c** to verify these predictions add up and to check the extent of rounding errors.

These equilibrium conditions can also serve as fundamentals of ABM simulations. ABM software can quickly run varying simulation sequences to get phenotypic characterizations after observing the interactions of individual agents given their genotypes, parameterizations or initial conditions. Consider a generalization of our models where

Table II. Contrast of equilibria, $\beta = 0.5, \delta = 0.8, w = 3, r = 1$

	UH	TH	Diff.	Ratio	UH	TH	Diff.	Ratio
U	0.618	0.659	*−0.041*	0.938	0.227	0.268	*−0.040*	0.848
$\mathbf{c_1}$	2.209	2.209	–	1.000	1.777	1.777	–	1.000
$\mathbf{c_1^2}$	0.565	0.440	0.126	*1.286*	0.873	0.679	0.194	*1.286*
$\mathbf{c_2^2}$	1.279	0.995	0.284	1.286	1.587	1.234	0.353	1.286
$\mathbf{c_1^3}$	0.226	0.352	−0.126	*0.643*	0.349	0.543	−0.194	*0.643*
$\mathbf{c_2^3}$	0.511	0.796	−0.284	0.643	0.635	0.987	−0.353	0.643
Ec	3.899	3.900	−0.001	1.000	3.099	3.099	–	1.000

Notes: Optimist Scenario, p = 0.1; Pessimist Scenario, p = 0.9
Source: The authors

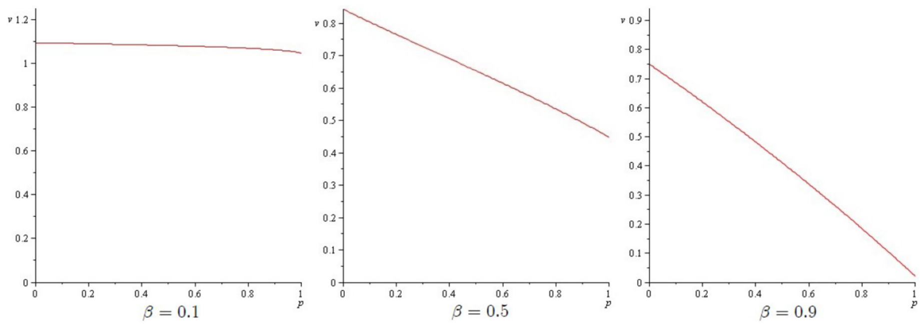
agents can soften their consumption by selling their current wealth to more impatient agents. ABM simulations could answer questions like what happens when agents are allowed to interact according to simple negotiation models? What would be the dynamic behaviour of the resulting endogenous interest rate? How does the cost of untying your hands depend on these dynamic relations? A potential result is that slow agents – those with artificial rigidities in their simulated behaviour – achieve higher profits in some cases, depending on the strength of their biases. This result would be counterintuitive as, in general, dynamic rigidities are costly.

4. Real-world implications and extensions

Behavioural economics, in conjunction with psychology and other social sciences, has focused on explaining the way individuals act in response to different concerns. This is why our model contrasting becomes relevant after we introduce concepts from the social sciences such as optimism, pessimism, procrastination, and anxiety. We can identify how consumer choices depend on their behavioural characteristics: consumption paths differ when facing optimistic (p close to 0), neutral or pessimistic scenarios (p close to 1).

The graphs from Figure 5 are only functions of the discount parameters. We can see how an increase in the probability of not obtaining an additional income causes a decrease in the equilibrium utility. The leftmost depicts the utility of a consumer with a severe risk bias that cares little for future consumption. As such, changes in p will almost not affect her. For these anxious agents, future events have small impacts on utility. Thus, their behaviour is inelastic to changes of p. In the middle graph, we observe how a neutral agent is more affected by changes in p. This is exacerbated in the right graph where those consumers who value all periods of their lives almost equally and are almost not affected by a present bias suffer the most when receiving bad news about the future.

These findings can describe different sectors of the economy. The results of this intertemporal consumption model can easily be extrapolated to the labour market if it is assumed that consumption is strongly correlated with wealth and, as in our model, with salary. Thus, in the labour market, the identification of risk characteristics can determine the incentives needed to capture and retain human capital. With empirical or experimental data, an employer could develop contracts to take advantage of this bias since, on the one hand, this model is evidence that workers may be vulnerable to renegotiations of contracts that harm them (the cost of untying their hands), as well as evidence on the value of a potential government intervention to restore ex-ante efficiency. Our analysis also justifies



Source: The authors

Figure 5. Present value of expected utility (v_1) under varying risk scenarios

the need for protection against contact with addictive substances or behaviours such as the use of some drugs and participation in gambling games.

Likewise, the private sector, through the recognition of these consumption profiles and the parametric estimates described above, can focus attention on people who are anxious, overly optimistic or susceptible to addictions since they will allocate a greater part of their current income to consumption. A policy-maker aware of the proportion of the population not too interested in their future could develop educational campaigns in which self-control mechanisms and the importance of saving for individual and community benefit are highlighted.

In general, the mechanism design literature requires modelling that allows the inclusion of intertemporal inconsistencies. To predict behaviour across time, the typical *individual rationality* (IR) and *incentive compatibility* (IC) restrictions generate biased predictions as long as they do not include hyperbolic discounting elements. For instance, to implement a high-effort level in a dynamic moral hazard model, it would be incorrect to assume discount is simply exponential ($\beta = 1$) in an IR restriction where reserve utility is smaller than the present value of effort. Otherwise, since in reality $\beta < 1$, the IR restriction will not hold, and the agent will exert no effort. Moreover, even if it holds, the principal is losing rent as long as renegotiation temptations are not taken advantage of.

5. Conclusions

In this paper, we focus on the present bias model of hyperbolic discounting which, according to the empirical evidence, has a better adjustment than the more restrictive exponential discounting as it allows for intertemporal inconsistencies. From this starting point, we construct a three-period scenario that incorporates risk on future income. This allows us to model varying risk profiles and to analyze the different choices individuals make. Using this information, we can measure how these intertemporal inconsistencies are costly across different starting behavioural characteristics.

Our main contribution is related to the analysis of the cost of deviating from the optimal decisions chosen at the beginning of a sequence of payoffs. That is, when an individual maximizes discounted expected utility but later deviates (*acting with untied hands*), she will face a cost that will only depend on the intertemporal discount factor (δ) and the present bias measure (β). We show this cost is the difference between our *Tied Hands* and *Untied Hands* models and we also compare both with the exponential discounting ($\beta = 1$) case. In addition, our model predicts that an agent affected by a present bias will have the *same* initial consumption levels whether she has access to commitment devices (*Ulysses Pacts*). However, these devices help control *future* consumption paths, generating present-value utility gains, implying that, given the chance, a rational consumer would choose to take advantage of these devices, to limit future choices[3].

We discuss how the abuse of these biases can allow employers to identify incentives to obtain and retain human resources by providing evidence of how workers are vulnerable to contract renegotiations. Similarly, some consumers are defenceless against firms that recognize addictive or procrastinating tendencies. This, depending on the costs we estimate, can be interpreted as a justification for a policymaker intervention that restores ex ante efficiency. We then address how our equilibrium conditions can be used as the foundation of ABM simulations. Finally, given that the theoretical validity of this model can be further evidenced by experimental results, we propose estimation and calibration methods. Thus, the refinements we postulate have the potential to improve the analysis of commitment devices, their abuse and their regulation.

Notes

1. We thank Héctor Galindo-Silva, Alexander Gotthard and the participants at Universidad Javeriana's Economics Seminar for comments and suggestions. All errors are our own.
2. Like a Saint Augustine-like agent who urges the model-maker to grant him access to a control device that will govern his consumption, *starting next period.*
3. Much like the young Saint Augustine (*Confessions*, Book VIII, Chapter VII) from the epigraph who "Feared lest Thou [O Lord] shouldest hear me soon, and soon cure me of the disease of concupiscence, which I wished to have satisfied, rather than extinguished".
4. The implicit function theorem for two variables states that if $f(x,y)$ is C^1 in an open set $\mathbf{A}$ containing (x_0,y_0), with $f(x_0,y_0) = 0$ and $f_2(x_0,y_0) \neq 0$, then there exists an interval $\mathbf{I}_1 = (x_0 - \delta, x_0 + \delta)$ and an interval $\mathbf{I}_2 = (y_0 - \varepsilon, y_0 + \varepsilon)$ (with $\delta > 0$ and $\varepsilon > 0$) such that $\mathbf{I}_1 \times \mathbf{I}_2 \subseteq \mathbf{A}$ and $\forall x \in \mathbf{I}_1$, the equation $f(x,y) = 0$ has a unique solution in $\mathbf{I}_2$ which defines y as a function $y = \phi(x)$ in $\mathbf{I}_1$ (Sydsæter *et al.*, 2005).

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Appendix 1

1. Mathematical appendices

1.1 Proof of Dv < 0

We want to show that $\forall \beta, \delta \in (0,1)$: $Dv(\beta, \delta) < 0$. Given Dv's domain, this is equivalent to showing that $\psi(\beta, \delta) > 0$, where $\psi(\beta, \delta) := \frac{\beta\delta}{\delta+1} + \frac{1}{\delta+1} - \beta^{\frac{\delta}{\delta+1}}$. Just like dv, ψ is not defined if $\beta = \delta = 0$, because the optimization problem is not well defined in this case. However since it is differentiable, it is a continuous function in the domain's open intervals[4]. Since $\forall \beta, \delta \in (0,1)$: $\psi(\beta,0) = 0, \psi(0,\delta) > 0$, $\psi\ (\beta,1) > 0$ and $\psi(1,\delta) = 0$, it suffices that $\forall \beta, \delta \in (0,1) : \psi_\beta = \frac{\delta}{\delta+1}\left(1 - \beta^{\frac{-1}{\delta+1}}\right) < 0$ and $\psi_\delta = \frac{1}{(\delta+1)^2}\left[\beta\left(1 - e^{\frac{\delta}{\delta+1}}\right) - 1\right] < 0$.

1.2 Asymptotic properties

Since $\lim_{\beta\downarrow 0}\beta\,\hat{}\,\beta = \lim_{\beta\downarrow 0} e^{\beta \ln\beta} = \lim_{\beta\downarrow 0}\exp\left(\frac{d\ln\beta/d\beta}{d\beta^{-1}/d\beta}\right) = 1$, then $\lim_{\beta\downarrow 0} Dv = \lim_{\beta\downarrow 0}\,[\beta\delta\ (1+\delta)\ \ln\frac{\delta+1}{\beta\delta+1} + \delta^2 \ln\beta^\beta] = 0$. The rest of the boundary values of $Dv(\beta,\delta)$'s domain can be calculated directly: $\forall \beta, \delta \in (0,1) : Dv(1,\delta) = Dv(\beta,0) = 0, Dv(\beta,1) = \beta\left(2\ln\frac{2}{\beta+1} + \ln\beta\right) < 0$.

1.3 Global maxima and minima

The $(\beta,\ \delta)$ combination that maximizes *the cost of untying your hands* is $\text{argmin}_{(\beta,\delta)}\ \{Dv\ (\beta,\delta)\} = \text{argmax}_{(\beta,\delta)}\ \{-Dv\ (\beta,\delta)\} = (1, \beta^*)$ where $\beta^* \cong 0.167$ since Dv is increasing in δ (see Appendix 1.1) and the solution to the FOC $\frac{\partial Dv}{\partial\beta}\big|_{\delta=1} = 0$ is the solution to $\ln\frac{4\beta^*}{(\beta^*+1)^2} = \frac{\beta^*-1}{\beta^*+1}$, which we approximate numerically. See Figure 2.

Appendix 2

Table AI. Benchmark simulation values

	No extra income (r) observed																		
	Hands untied									Hands tied									
	Consumption ($)			Instant utility			PV utility			Consumption ($)			Instant utility			PV utility			
p	c1	c12	c13	u(c1)	u(c12)	u(c13)	U(c1)	U(c12)	U(c13)	c1	c12	c13	u(c1)	u(c12)	u(c13)	U(c1)	U(c12)	U(c13)	Dv
0	2.326	0.482	0.193	0.844	−0.730	−1.647	−0.170	−0.462	−0.989	2.326	0.375	0.300	0.844	−0.982	−1.205	−0.170	−0.562	−0.948	−0.0409
0.1	2.209	0.565	0.226	0.792	−0.571	−1.487	−0.233	−0.461	−0.937	2.209	0.440	0.352	0.792	−0.822	−1.045	−0.233	−0.561	−0.896	−0.0409
0.2	2.123	0.626	0.251	0.753	−0.468	−1.384	−0.284	−0.471	−0.914	2.123	0.487	0.390	0.753	−0.719	−0.942	−0.284	−0.572	−0.873	−0.0409
0.3	2.053	0.676	0.271	0.719	−0.391	−1.307	−0.329	−0.486	−0.904	2.053	0.526	0.421	0.719	−0.642	−0.866	−0.329	−0.586	−0.863	−0.0409
0.4	1.994	0.719	0.287	0.690	−0.330	−1.247	−0.371	−0.503	−0.902	1.994	0.559	0.447	0.690	−0.582	−0.805	−0.371	−0.604	−0.861	−0.0409
0.5	1.942	0.756	0.302	0.664	−0.280	−1.196	−0.410	−0.522	−0.905	1.942	0.588	0.470	0.664	−0.531	−0.754	−0.410	−0.623	−0.864	−0.0409
0.6	1.895	0.789	0.316	0.639	−0.237	−1.153	−0.448	−0.542	−0.911	1.895	0.614	0.491	0.639	−0.488	−0.711	−0.448	−0.643	−0.870	−0.0409
0.7	1.853	0.820	0.328	0.617	−0.199	−1.115	−0.484	−0.563	−0.920	1.853	0.637	0.510	0.617	−0.450	−0.673	−0.484	−0.664	−0.879	−0.0409
0.8	1.814	0.847	0.339	0.595	−0.166	−1.082	−0.519	−0.585	−0.931	1.814	0.659	0.527	0.595	−0.417	−0.640	−0.519	−0.685	−0.890	−0.0409
0.9	1.778	0.873	0.349	0.575	−0.136	−1.052	−0.553	−0.607	−0.944	1.778	0.679	0.543	0.575	−0.387	−0.610	−0.553	−0.708	−0.903	−0.0409
1.0	1.744	0.897	0.359	0.556	−0.109	−1.025	−0.586	−0.630	−0.958	1.744	0.698	0.558	0.556	−0.360	−0.583	−0.586	−0.730	−0.917	−0.0409
	With extra income (r) observed																		
	Hands untied									Hands tied									
	Consumption ($)			Instant utility			PV utility			Consumption ($)			Instant utility			PV utility			
p	c1	c22	c23	u(c1)	u(c22)	u(c23)	U(c1)	U(c22)	U(c23)	c1	c22	c23	u(c1)	u(c22)	u(c23)	U(c1)	U(c22)	U(c23)	Dv
0	2.326	1.196	0.478	0.844	0.179	−0.737	−0.170	−0.098	−0.334	2.326	0.930	0.744	0.844	−0.072	−0.295	−0.170	−0.199	−0.293	−0.0409
0.1	2.209	1.279	0.512	0.792	0.246	−0.670	−0.233	−0.134	−0.348	2.209	0.995	0.796	0.792	−0.005	−0.228	−0.233	−0.235	−0.308	−0.0409
0.2	2.123	1.341	0.536	0.753	0.293	−0.623	−0.284	−0.167	−0.366	2.123	1.043	0.834	0.753	0.042	−0.181	−0.284	−0.267	−0.325	−0.0409
0.3	2.053	1.391	0.556	0.719	0.330	−0.587	−0.329	−0.197	−0.385	2.053	1.082	0.865	0.719	0.078	−0.145	−0.329	−0.298	−0.344	−0.0409
0.4	1.994	1.433	0.573	0.690	0.360	−0.557	−0.371	−0.227	−0.405	1.994	1.115	0.892	0.690	0.108	−0.115	−0.371	−0.328	−0.364	−0.0409
0.5	1.942	1.470	0.588	0.664	0.385	−0.531	−0.410	−0.256	−0.426	1.942	1.144	0.915	0.664	0.134	−0.089	−0.410	−0.357	−0.385	−0.0409
0.6	1.895	1.504	0.601	0.639	0.408	−0.508	−0.448	−0.284	−0.447	1.895	1.169	0.936	0.639	0.157	−0.067	−0.448	−0.385	−0.406	−0.0409
0.7	1.853	1.534	0.614	0.617	0.428	−0.488	−0.484	−0.312	−0.469	1.853	1.193	0.954	0.617	0.177	−0.047	−0.484	−0.413	−0.428	−0.0409
0.8	1.814	1.562	0.625	0.595	0.446	−0.471	−0.519	−0.340	−0.491	1.814	1.215	0.972	0.595	0.194	−0.029	−0.519	−0.441	−0.450	−0.0409
0.9	1.778	1.587	0.635	0.575	0.462	−0.454	−0.553	−0.368	−0.513	1.778	1.235	0.988	0.575	0.211	−0.012	−0.553	−0.469	−0.473	−0.0409
1.0	1.744	1.611	0.645	0.556	0.477	−0.439	−0.586	−0.396	−0.536	1.744	1.253	1.003	0.556	0.226	0.003	−0.586	−0.496	−0.495	−0.0409

Note: Consumption and present value of utility with tied and untied hands
Source: The authors