Letter to the Editor on HFF 32, 138 (2022); 32, 2282 (2022); 33, 965 (2023) and 34, 1189 (2024) for the recent shallow-water studies

Latterly, in *International Journal of Numerical Methods for Heat and Fluid Flow*, Wazwaz (2022), Li *et al.* (2022), Khuri (2023) and Ramos and Garcia Lopez (2024) have absorbingly studied the shallow water, i.e., on certain multiple soliton solutions and lump solutions for two integrable shallow water wave equations (Wazwaz, 2022), Gramian solutions and solitonic interactions for a (2+1)-dimensional Broer-Kaup-Kupershmidt system in the shallow water (Li *et al.*, 2022), soliton solutions for a (2+1)-dimensional Korteweg-de Vries equation describing the shallow water waves (Khuri, 2023) as well as blowup in finite time of the numerical solutions for a (1+1)-dimensional, bidirectional, nonlinear wave model for the small-amplitude waves in shallow water, depending upon the relaxation time, linear and nonlinear drift, power of the nonlinear advection flux, viscosity coefficient, viscous attenuation, etc. (Ramos and Garcia Lopez, 2024).

Enthusiasm for the shallow water has been warmed by Wazwaz (2022), Li *et al.* (2022), Khuri (2023) and Ramos and Garcia Lopez (2024), and consequently this shallow-water-oriented Letter aims to deal with a (2+1)-dimensional generalised modified dispersive water-wave system for the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth (Gao *et al.*, 2023a; Liu *et al.*, 2023):

$$u_{yt} + \alpha u_{xxy} - 2\alpha v_{xx} - \beta u u_{xy} - \beta u_x u_y = 0, \tag{1a}$$

$$v_t - \alpha v_{xx} - \beta (uv)_x = 0, \tag{1b}$$

in which the real differentiable functions $v(x, y, t)$ and $u(x, y, t)$, respectively, imply the horizontal velocity of the water wave and the height of the water surface, α and β are the real non-zero constants, and the subscripts represent the partial derivatives concerning the scaled space variables x, y and time variable t. With symbolic computation (Anderson and Farazmand, 2024; Kovacs *et al.*, 2024; Shen *et al.*, 2023a, 2023b, 2023e, 2023f; Gao *et al.*, 2023b; Wu and Gao, 2023; Wu *et al.*, 2023a, 2023c, 2023d; Zhou and Tian, 2022) in planning, Gao *et al.* (2023a) have given a set of the hetero-Bäcklund transformations, a set of the scaling transformations and four sets of the similarity reductions for system (1), whereas Liu *et al.* (2023) have investigated certain Lie point symmetry generators, Lie symmetry groups and symmetry reductions for system (1) with some analytic solutions. Shallow-water special cases of system (1) have been seen in Ying and Lou (2000), Li and Zhang (2004), Ma *et al.* (2015), Zhao and Han (2015), Kassem and Rashed (2019), Yamgoué *et al.* (2019), Liang and Wang (2019), Ren *et al.* (2019), Cao *et al.* (2020), Li *et al.*

The author expresses sincere thanks to the editors and reviewers for their valuable comments.
This work has been supported by the Scientific Research Foundation of North China University of Technology under Grant No. 11005136024XN147-87.

International Journal of Numerical Methods for Heat & Fluid Flow
Vol. 34 No. 6, 2024
pp. 2197-2204

0961-5539
DOI 10.1108/HFF-06-2024-944

(2022), Gao *et al.* (2023a) as well as Liu *et al.* (2023). Additionally, fluids from all over the Solar System have been observed and discussed (Lainey *et al.*, 2024; Neish *et al.*, 2024; Cheng *et al.*, 2022, 2023a, 2023b, 2024; Feng *et al.*, 2023; Gao, 2024a, 2024d; Gao *et al.*, 2023c; Shen *et al.*, 2023c, 2023d; Wu *et al.*, 2023b; Zhou *et al.*, 2023a, 2023b, 2024).

Key point: In accordance with Wazwaz (2022), Li *et al.* (2022), Khuri (2023) and Ramos and Garcia Lopez (2024), and using symbolic computation, this shallow-water-oriented Letter tries to construct two branches of the similarity reductions for system (1), different from those reported in Gao *et al.* (2023a) and Liu *et al.* (2023).

Towards system (1), similar to those in Clarkson and Kruskal (1989), Gao and Tian (2022, 2024) as well as Gao (2023, 2024b, 2024c), the assumptions

$$u(x,y,t) = \theta(x,y,t) + \omega(x,y,t)p[z(y,t)], \tag{2a}$$

$$v(x,y,t) = \gamma(x,y,t) + \kappa(x,y,t)q[z(y,t)], \tag{2b}$$

and choice of $z_y z_t \neq 0$ develop into

$$\begin{aligned}
&\omega z_y z_t p'' - 2\beta\omega\omega_x z_y p p' - \beta(\omega_x\omega_y + \omega\omega_{xy})p^2 - 2\alpha\kappa_{xx}q \\
&\quad + \left[(\omega_y z_t + \omega_t z_y + \omega z_{yt}) - \beta(\omega\theta_x + \omega_x\theta)z_y + \alpha\omega_{xx}z_y\right]p' \\
&\quad + [\omega_{yt} - \beta(\omega_y\theta_x + \omega_x\theta_y + \omega\theta_{xy} + \omega_{xy}\theta) + \alpha\omega_{xxy}]p \\
&\quad + \left[\theta_{yt} - \beta(\theta_x\theta_y + \theta\theta_{xy}) - 2\alpha\gamma_{xx} + \alpha\theta_{xxy}\right] = 0,
\end{aligned} \tag{3a}$$

$$\begin{aligned}
&\kappa z_t q' - \beta(\omega\kappa_x + \omega_x\kappa)pq + [\kappa_t - \beta(\theta\kappa_x + \theta_x\kappa) - \alpha\kappa_{xx}]q - \beta(\omega\gamma_x + \omega_x\gamma)p \\
&\quad + [\gamma_t - \beta(\theta\gamma_x + \theta_x\gamma) - \alpha\gamma_{xx}] = 0,
\end{aligned} \tag{3b}$$

with $\theta(x, y, t)$, $\omega(x, y, t)$, $\neq 0$, $\gamma(x, y, t)$, $\kappa(x, y, t) \neq 0$ and $z(y, t) \neq 0$ meaning certain real to-be-determined differentiable functions, $p(z)$ and $q(z)$ denoting two real differentiable functions while the "$'$" sign representing d/dz.

We prefer to consider the following ordinary differential equations (ODEs): By reason that equations (3) are designed as a set of the ODEs with respect to $p(z)$ and $q(z)$, we make the ratios of different derivatives and powers of $p(z)$ and $q(z)$ become the functions of z only. Consequently,

$$\Omega_1(z)\omega z_y z_t = -2\beta\omega\omega_x z_y, \tag{4a}$$

$$\Omega_2(z)\omega z_y z_t = -\beta(\omega_x\omega_y + \omega\omega_{xy}), \tag{4b}$$

$$\Omega_3(z)\omega z_y z_t = -2\alpha\kappa_{xx}, \tag{4c}$$

$$\Omega_4(z)\omega z_y z_t = (\omega_y z_t + \omega_t z_y + \omega z_{yt}) - \beta(\omega\theta_x + \omega_x\theta)z_y + \alpha\omega_{xx}z_y, \tag{4d}$$

$$\Omega_5(z)\omega z_y z_t = \omega_{yt} - \beta(\omega_y\theta_x + \omega_x\theta_y + \omega\theta_{xy} + \omega_{xy}\theta) + \alpha\omega_{xxy}, \tag{4e}$$

$$\Omega_6(z)\omega z_y z_t = \theta_{yt} - \beta(\theta_x\theta_y + \theta\theta_{xy}) - 2\alpha\gamma_{xx} + \alpha\theta_{xxy}, \tag{4f}$$

$$\Gamma_1(z)\kappa z_t = -\beta(\omega\kappa_x + \omega_x\kappa), \tag{4g}$$

$$\Gamma_2(z)\kappa z_t = \kappa_t - \beta(\theta\kappa_x + \theta_x\kappa) - \alpha\kappa_{xx}, \tag{4h}$$

$$\Gamma_3(z)\kappa z_t = -\beta(\omega\gamma_x + \omega_x\gamma), \tag{4i}$$

$$\Gamma_4(z)\kappa z_t = \gamma_t - \beta(\theta\gamma_x + \theta_x\gamma) - \alpha\gamma_{xx}, \tag{4j}$$

with $\Omega_i(z)$'s ($i = 1, \ldots, 6$) and $\Gamma_j(z)$'s ($j = 1, \ldots, 4$) standing for the real to-be-determined functions as for z only.

Accordingly, any solutions for $\theta(x, y, t)$, $\omega(x, y, t) \neq 0$, $\gamma(x, y, t)$, $\kappa(x, y, t) \neq 0$ and $z(y, t) \neq 0$ can bring about, at least, a similarity reduction.

Let us keep the similarity-reduction issue in mind and turn to Clarkson and Kruskal (1989).

On the score of the second freedom in remark 3 in Clarkson and Kruskal (1989), equations (4a) and (4c) bring about

$$\omega(x,y,t) = -\frac{z_t}{2\beta}x, \qquad \kappa(x,y,t) = \frac{z_y z_t^2}{24\alpha\beta}x^3, \qquad \Omega_1(z) = \Omega_3(z) = 1. \tag{5}$$

In the light of the first freedom in remark 3 in Clarkson and Kruskal (1989), equation (4b) comes to

$$z(y,t) = \lambda_1 y + \lambda_2 t + \lambda_3, \qquad \Omega_2(z) = 0, \tag{6}$$

as a result that equation (4g) turns into

$$\Gamma_1(z) = 2, \tag{7}$$

with λ_1 and λ_2 implying two real non-zero constants, whereas λ_3 meaning a real constant.

So far, two branches of $\theta(x, y, t)$ can help us simplify $\Gamma_2(z)$ to a constant:

Branch 1:

In this branch, based on the first freedom in remark 3 in Clarkson and Kruskal (1989), equations (4h) and (4i) give rise to

$$\theta(x,y,t) = -\frac{3\alpha}{\beta}x^{-1}, \qquad \gamma(x,y,t) = 0, \qquad \Gamma_2(z) = \Gamma_3(z) = 0. \tag{8}$$

Then, equations (4d), (4e), (4f) and (4j) bloom into

$$\Omega_4(z) = \Omega_5(z) = \Omega_6(z) = \Gamma_4(z) = 0. \tag{9}$$

We are in a position to transform system (1) into two ODEs, i.e.,

$$p'' + pp' + q = 0, \tag{10a}$$

$$q' + 2pq = 0, \tag{10b}$$

and then to simplify ODEs (10) to a single ODE, i.e.,

$$p''' + 3pp'' + p'^2 + 2p^2p' = 0, \tag{11}$$

with the choice of

$$q = -(p'' + pp'). \tag{12}$$

In short, we build up the following branch of the similarity reductions for system (1):

$$u(x, y, t) = -\frac{3\alpha}{\beta}x^{-1} - \frac{\lambda_2}{2\beta}xp[z(y, t)], \tag{13a}$$

$$v(x, y, t) = -\frac{\lambda_1\lambda_2^2}{24\alpha\beta}x^3\{p''[z(y, t)] + p[z(y, t)]p'[z(y, t)]\}, \tag{13b}$$

$$z(y, t) = \lambda_1 y + \lambda_2 t + \lambda_3, \tag{13c}$$

$$p''' + 3pp'' + p'^2 + 2p^2p' = 0. \tag{13d}$$

ODE (13d) means a known ODE, the information of which has been given in Zwillinger and Dobrushkin (2022).

In relation to the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, as for $u(x, y, t)$, the height of the water surface, and $v(x, y, t)$, the horizontal velocity of the water wave, similarity reductions (13) are dependent on α and β, the shallow-water coefficients in system (1).

Branch 2:

In this branch, on account of the second freedom in remark 3 in Clarkson and Kruskal (1989), equation (4h) becomes

$$\theta(x, y, t) = -\frac{\lambda_2}{4\beta}x - \frac{3\alpha}{\beta}x^{-1}, \qquad \Gamma_2(z) = 1. \tag{14}$$

On the ground of the first freedom in remark 3 in Clarkson and Kruskal (1989), equation (4i) makes for

$$\gamma(x, y, t) = 0, \qquad \Gamma_3(z) = 0, \tag{15}$$

so that equations (4d), (4e), (4f) and (4j) issue in

$$\Omega_4(z) = \frac{1}{2}, \qquad \Omega_5(z) = \Omega_6(z) = \Gamma_4(z) = 0. \tag{16}$$

We are now capable of transforming system (1) into the following two ODEs:

$$p'' + pp' + q + \frac{1}{2}p' = 0, \tag{17a}$$

$$q' + 2pq + q = 0, \tag{17b}$$

and then of simplifying ODEs (17) into the following single ODE:

$$p''' + 3pp'' + \frac{3}{2}p'' + p'^2 + 2p^2p' + 2pp' + \frac{1}{2}p' = 0, \tag{18}$$

with the choice of

$$q = -\left(p'' + pp' + \frac{1}{2}p'\right). \tag{19}$$

In brief, we construct another branch of the similarity reductions for system (1), i.e.,

$$u(x,y,t) = -\left(\frac{\lambda_2}{4\beta}x + \frac{3\alpha}{\beta}x^{-1}\right) - \frac{\lambda_2}{2\beta}xp[z(y,t)], \tag{20a}$$

$$v(x,y,t) = -\frac{\lambda_1\lambda_2^2}{24\alpha\beta}x^3\left\{p''[z(y,t)] + p[z(y,t)]p'[z(y,t)] + \frac{1}{2}p'[z(y,t)]\right\}, \tag{20b}$$

$$z(y,t) = \lambda_1 y + \lambda_2 t + \lambda_3, \tag{20c}$$

$$p''' + 3pp'' + \frac{3}{2}p'' + p^{'2} + 2p^2p' + 2pp' + \frac{1}{2}p' = 0. \tag{20d}$$

ODE (20d) implies a known ODE, the information of which has been presented in Zwillinger and Dobrushkin (2022).

In relation to the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, as for $u(x, y, t)$, the height of the water surface, and $v(x, y, t)$, the horizontal velocity of the water wave, similarity reductions (20) depend on α and β, the shallow-water coefficients in system (1).

Xin-Yi Gao
College of Science, and Beijing Key Laboratory on Integration and Analysis of Large-Scale Stream Data, North China University of Technology, Beijing, China

References

Anderson, W. and Farazmand, M. (2024), "Fast and scalable computation of shape-morphing nonlinear solutions with application to evolutional neural networks", *Journal of Computational Physics*, Vol. 498, p. 112649, doi: 10.1016/j.jcp.2023.112649.

Cao, X.Q., Guo, Y.N., Hou, S.C., Zhang, C.Z. and Peng, K.C. (2020), "Variational principles for two kinds of coupled nonlinear equations in shallow water", *Symmetry*, Vol. 12, p. 850, doi: 10.3390/sym12050850.

Cheng, C.D., Tian, B., Zhou, T.Y. and Shen, Y. (2023a), "Wronskian solutions and pfaffianization for a (3 + 1)-dimensional generalized variable-coefficient Kadomtsev-Petviashvili equation in a fluid or plasma", *Physics of Fluids*, Vol. 35 No. 3, p. 37101, doi: 10.1063/5.0141559.

Cheng, C.D., Tian, B., Shen, Y. and Zhou, T.Y. (2023b), "Bilinear form, auto-Bäcklund transformations, Pfaffian, soliton, and breather solutions for a (3+1)-dimensional extended shallow water wave equation", *Physics of Fluids*, Vol. 35, p. 087123, doi: 10.1063/5.0160723.

Cheng, C.D., Tian, B., Zhou, T.Y. and Shen, Y. (2024), "Nonlinear localized waves and their interactions for a (2+1)-dimensional extended Bogoyavlenskii-Kadomtsev-Petviashvili equation in a fluid", *Wave Motion*, Vol. 125, p. 103246, doi: 10.1016/j.wavemoti.2023.103246.

Cheng, C.D., Tian, B., Ma, Y.X., Zhou, T.Y. and Shen, Y. (2022), "Pfaffian, breather, and hybrid solutions for a (2 + 1)-dimensional generalized nonlinear system in fluid mechanics and plasma physics", *Physics of Fluids*, Vol. 34 No. 11, p. 115132, doi: 10.1063/5.0119516.

Clarkson, P.A. and Kruskal, M.D. (1989), "New similarity reductions of the Boussinesq equation", *Journal of Mathematical Physics*, Vol. 30 No. 10, pp. 2201-2213, doi: 10.1063/1.528613.

Feng, C.H., Tian, B., Yang, D.Y. and Gao, X.T. (2023), "Lump and hybrid solutions for a (3 + 1)-dimensional Boussinesq-type equation for the gravity waves over a water surface", *Chinese Journal of Physics*, Vol. 83, pp. 515-526, doi: 10.1016/j.cjph.2023.03.023.

Gao, X.Y. (2023), "Considering the wave processes in oceanography, acoustics and hydrodynamics by means of an extended coupled (2+1)-dimensional Burgers system", *Chinese Journal of Physics*, Vol. 86, pp. 572-577, doi: 10.1016/j.cjph.2023.10.051.

Gao, X.Y. (2024a), "In the shallow water: auto-Bäcklund, hetero-Bäcklund and scaling transformations via a (2 + 1)-dimensional generalized Broer-Kaup system", *Qualitative Theory of Dynamical Systems*, Vol. 23 No. 4, p. 184, doi: 10.1007/s12346-024-01025-9.

Gao, X.Y. (2024b), "Two-layer-liquid and lattice considerations through a (3 + 1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system", *Applied Mathematics Letters*, Vol. 152, p. 109018, doi: 10.1016/j.aml.2024.109018.

Gao, X.Y. (2024c), "Auto-Bäcklund transformation with the solitons and similarity reductions for a generalized nonlinear shallow water wave equation", *Qualitative Theory of Dynamical Systems*, Vol. 23 No. 4, p. 181, doi: 10.1007/s12346-024-01034-8.

Gao, X.Y. (2024d), "Symbolic computation on a (2 + 1)-dimensional generalized nonlinear evolution system in fluid dynamics, plasma physics, nonlinear optics and quantum mechanics", *Qualitative Theory of Dynamical Systems*, Vol. 23 No. 5, p. 202, doi: 10.1007/s12346-024-01045-5.

Gao, X.T. and Tian, B. (2022), "Water-wave studies on a (2 + 1)-dimensional generalized variable-coefficient Boiti-Leon-Pempinelli system", *Applied Mathematics Letters*, Vol. 128, p. 107858, doi: 10.1016/j.aml.2021.107858.

Gao, X.T. and Tian, B. (2024), "Similarity reductions on a (2 + 1)-dimensional variable-coefficient modified Kadomtsev-Petviashvili system describing certain electromagnetic waves in a thin film", *International Journal of Theoretical Physics*, Vol. 63 No. 4, p. 99, doi: 10.1007/s10773-024-05629-4.

Gao, X.Y., Guo, Y.J. and Shan, W.R. (2023a), "Symbolically computing the shallow water via a (2+1)-dimensional generalized modified dispersive water-wave system: similarity reductions, scaling and hetero-Bäcklund transformations", *Qualitative Theory of Dynamical Systems*, Vol. 22 No. 1, p. 17, doi: 10.1007/s12346-022-00684-w.

Gao, X.Y., Guo, Y.J. and Shan, W.R. (2023b), "Ultra-short optical pulses in a birefringent fiber via a generalized coupled Hirota system with the singular manifold and symbolic computation", *Applied Mathematics Letters*, Vol. 140, p. 108546, doi: 10.1016/j.aml.2022.108546.

Gao, X.Y., Guo, Y.J. and Shan, W.R. (2023c), "Ocean shallow-water studies on a generalized Boussinesq-Broer-Kaup-Whitham system: Painlevé analysis and similarity reductions", *Chaos, Solitons and Fractals*, Vol. 169, p. 113214, doi: 10.1016/j.chaos.2023.113214.

Kassem, M.M. and Rashed, A.S. (2019), "N-solitons and cuspon waves solutions of (2+ 1)-dimensional Broer-Kaup-Kupershmidt equations via hidden symmetries of Lie optimal system", *Chinese Journal of Physics*, Vol. 57, pp. 90-104, doi: 10.1016/j.cjph.2018.12.007.

Khuri, S. (2023), "New approach for soliton solutions for the (2 + 1)-dimensional KdV equation describing shallow water wave", *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 33 No. 3, pp. 965-973, doi: 10.1108/HFF-08-2022-0498.

Kovacs, Z., Brown, C., Recio, T. and Vajda, R. (2024), "Computing with Tarski formulas and semi-algebraic sets in a web browser", *Journal of Symbolic Computation*, Vol. 120, p. 102235, doi: 10.1016/j.jsc.2023.102235.

Lainey, V., Rambaux, N., Tobie, G., Cooper, N., Zhang, Q., Noyelles, B. and Baillie, K. (2024), "A recently formed ocean inside Saturn's moon Mimas", *Nature*, Vol. 626 No. 7998, pp. 280-282, doi: 10.1038/s41586-023-06975-9.

Li, D.S. and Zhang, H.Q. (2004), "New families of non-travelling wave solutions to the (2+1)-dimensional modified dispersive water-wave system", *Chinese Physics*, Vol. 13, pp. 1377-1381, doi: 10.1088/1009-1963/13/9/001.

Li, L.Q., Gao, Y.T., Yu, X., Deng, G.F. and Ding, C.C. (2022), "Gramian solutions and solitonic interactions of a (2 + 1)-dimensional Broer-Kaup-Kupershmidt system for the shallow water", *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 32 No. 7, pp. 2282-2298, doi: 10.1108/HFF-07-2021-0441.

Liang, J. and Wang, X. (2019), "Consistent Riccati expansion for finding interaction solutions of (2+1)-dimensional modified dispersive water-wave system", *Mathematical Methods in the Applied Sciences*, Vol. 42, pp. 6131-6138, doi: 10.1002/mma.5709.

Liu, F.Y., Gao, Y.T., Yu, X., Ding, C.C. and Li, L.Q. (2023), "Lie group analysis for a (2 + 1)-dimensional generalized modified dispersive water-wave system for the shallow water waves", *Qualitative Theory of Dynamical Systems*, Vol. 22 No. 4, p. 129, doi: 10.1007/s12346-023-00792-1.

Ma, Z.Y., Fei, J.X. and Du, X.Y. (2015), "Symmetry reduction of the (2+1)-dimensional modified dispersive water-wave system", *Communications in Theoretical Physics*, Vol. 64, pp. 127-132, doi: 10.1088/0253-6102/64/2/127.

Neish, C., Malaska, M.J., Sotin, C., Lopes, R.M., Nixon, C.A., Affholder, A., Chatain, A., Cockell, C., Farnsworth, K.K., Higgins, P.M., Miller, K.E. and Soderlund, K.M. (2024), "Organic input to Titan's subsurface ocean through impact cratering", *Astrobiology*, Vol. 24 No. 2, pp. 177-189, doi: 10.1089/ast.2023.005.

Ramos, J.I. and Garcia Lopez, C.M. (2024), "Effect of initial conditions on a one-dimensional model of small-amplitude wave propagation in shallow water. II: Blowup for nonsmooth conditions", *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 34, pp. 1189-1226, doi: 10.1108/HFF-07-2023-0413.

Ren, B., Ma, W.X. and Yu, J. (2019), "Rational solutions and their interaction solutions of the (2+1)-dimensional modified dispersive water wave equation", *Computers & Mathematics with Applications*, Vol. 77, pp. 2086-2095, doi: 10.1016/j.camwa.2018.12.010.

Shen, Y., Tian, B., Zhou, T.Y. and Cheng, C.D. (2023a), "Multi-pole solitons in an inhomogeneous multi-component nonlinear optical medium", *Chaos, Solitons and Fractals*, Vol. 171, p. 113497, doi: 10.1016/j.chaos.2023.113497.

Shen, Y., Tian, B., Cheng, C.D. and Zhou, T.Y. (2023b), "N-soliton, *M*th-order breather, *H*th-order lump, and hybrid solutions of an extended (3 + 1)-dimensional Kadomtsev-Petviashvili equation", *Nonlinear Dynamics*, Vol. 111 No. 11, pp. 10407-10424, doi: 10.1007/s11071-023-08369-y.

Shen, Y., Tian, B., Cheng, C.D. and Zhou, T.Y. (2023c), "Pfaffian solutions and nonlinear waves of a (3 + 1)-dimensional generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt system in fluid mechanics", *Physics of Fluids*, Vol. 35, p. 025103, doi: 10.1063/5.0135174.

Shen, Y., Tian, B., Zhou, T.Y. and Gao, X.T. (2023d), "Extended (2 + 1)-dimensional Kadomtsev-Petviashvili equation in fluid mechanics: solitons, breathers, lumps and interactions", *The European Physical Journal Plus*, Vol. 138 No. 4, p. 305, doi: 10.1140/epjp/s13360-023-03886-6.

Shen, Y., Tian, B., Zhou, T.Y. and Gao, X.T. (2023e), "N-fold Darboux transformation and solitonic interactions for the Kraenkel-Manna-Merle system in a saturated ferromagnetic material", *Nonlinear Dynamics*, Vol. 111 No. 3, pp. 2641-2649, doi: 10.1007/s11071-022-07959-6.

Shen, Y., Tian, B., Yang, D.Y. and Zhou, T.Y. (2023f), "Hybrid relativistic and modified Toda lattice-type system: equivalent form, N-fold Darboux transformation and analytic solutions", *European Physical Journal Plus*, Vol. 138, p. 744, doi: 10.1140/epjp/s13360-023-04331-4.

Wazwaz, A.M. (2022), "New integrable (2 + 1)- and (3 + 1)-dimensional shallow water wave equations: multiple soliton solutions and lump solutions", *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 32 No. 1, pp. 138-149, doi: 10.1108/HFF-01-2021-0019.

Wu, X.H. and Gao, Y.T. (2023), "Generalized Darboux transformation and solitons for the Ablowitz-Ladik equation in an electrical lattice", *Applied Mathematics Letters*, Vol. 137, p. 108476, doi: 10.1016/j.aml.2022.108476.

Wu, X.H., Gao, Y.T., Yu, X. and Liu, F.Y. (2023a), "Generalized Darboux transformation and solitons for a Kraenkel-Manna-Merle system in a ferromagnetic saturator", *Nonlinear Dynamics*, Vol. 111 No. 15, pp. 14421-14433, doi: 10.1007/s11071-023-08510-x.

Wu, X.H., Gao, Y.T., Yu, X. and Ding, C.C. (2023b), "N-fold generalized Darboux transformation and asymptotic analysis of the degenerate solitons for the Sasa-Satsuma equation in fluid dynamics and nonlinear optics", *Nonlinear Dynamics*, Vol. 111 No. 17, pp. 16339-16352, doi: 10.1007/s11071-023-08533-4.

Wu, X.H., Gao, Y.T., Yu, X. and Liu, F.Y. (2023d), "On a variable-coefficient AB system in a baroclinic flow: generalized Darboux transformation and non-autonomous localized waves", *Wave Motion*, Vol. 122, p. 103184, doi: 10.1016/j.wavemoti.2023.103184.

Wu, X.H., Gao, Y.T., Yu, X., Li, L.Q. and Ding, C.C. (2023c), "Vector breathers, rogue and breather-rogue waves for a coupled mixed derivative nonlinear Schrödinger system in an optical fiber", *Nonlinear Dynamics*, Vol. 111 No. 6, pp. 5641-5653, doi: 10.1007/s11071-022-08058-2.

Yamgoué, S.B., Deffo, G.R. and Pelap, F.B. (2019), "A new rational sine-Gordon expansion method and its application to nonlinear wave equations arising in mathematical physics", *The European Physical Journal Plus*, Vol. 134, p. 380, doi: 10.1140/epjp/i2019-12733-8.

Ying, J.P. and Lou, S.Y. (2000), "Abundant coherent structures of the (2+1)-dimensional Broer-Kaup-Kupershmidt equation", *Zeitschrift für Naturforschung A*, Vol. 56, pp. 619-625, doi: 10.1515/zna-2001-0903.

Zhao, Z. and Han, B. (2015), "On optimal system, exact solutions and conservation laws of the Broer-Kaup system", *The European Physical Journal Plus*, Vol. 130, p. 223, doi: 10.1140/epjp/i2015-15223-1.

Zhou, T.Y. and Tian, B. (2022), "Auto-Bäcklund transformations, Lax pair, bilinear forms and bright solitons for an extended (3 + 1)-dimensional nonlinear Schrödinger equation in an optical fiber", *Applied Mathematics Letters*, Vol. 133, p. 108280, doi: 10.1016/j.aml.2022.108280.

Zhou, T.Y., Tian, B., Shen, Y. and Cheng, C.D. (2023a), "Lie symmetry analysis, optimal system, symmetry reductions and analytic solutions for a (2 + 1)-dimensional generalized nonlinear evolution system in a fluid or a plasma", *Chinese Journal of Physics*, Vol. 84, pp. 343-356, doi: 10.1016/j.cjph.2023.05.017.

Zhou, T.Y., Tian, B., Shen, Y. and Gao, X.T. (2023b), "Auto-Bäcklund transformations and soliton solutions on the nonzero background for a (3 + 1)-dimensional Korteweg-de Vries-Calogero-Bogoyavlenskii-Schiff equation in a fluid", *Nonlinear Dynamics*, Vol. 111 No. 9, pp. 8647-8658, doi: 10.1007/s11071-023-08260-w.

Zhou, T.Y., Tian, B., Shen, Y. and Cheng, C.D. (2024), "Painlevé analysis, auto-Bäcklund transformations, bilinear form and analytic solutions on some nonzero backgrounds for a (2+1)-dimensional generalized nonlinear evolution system in fluid mechanics and plasma physics", *Nonlinear Dynamics*, Vol. 112, pp. 9355-9365, doi: 10.1007/s11071-024-09450-w.

Zwillinger, D. and Dobrushkin, V. (2022), *Handbook of Differential Equations*, 4th ed., Chapman and Hall/CRC, Boca Raton, FL, doi: 10.1201/9780429286834.

Corresponding author
Xin-Yi Gao can be contacted at: xin_yi_gao@163.com