Comparing the Harrod-Domar, Solow and Ramsey growth models and their implications for economic policies

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Abstract

Purpose – The principal aim of this paper is to review three basic theoretical growth models, namely the Harrod-Domar model, the Solow model and the Ramsey model, and examine their implications for economic policies.

Design/methodology/approach – The paper utilizes a positivist research framework that emphasizes the causal relationships between the variables in each of the three models. Mathematical methods are employed to formulate and examine the three models under study. Since the paper is theoretical, it does not use any empirical data although numerical illustrations are provided whenever they are appropriate.

Findings – The Harrod-Domar model explains why countries with high rates of saving may also enjoy high rate of economic growth. Both the Solow and Ramsey models can be used to explain the medium-income trap. The paper examines the impact of Covid shocks on the macroeconomy. While the growth rate can be recovered, it may not always possible to recover the output level.

Research limitations/implications – For the Harrod-Domar model, the public spending decreases the private consumption at the period 1, but there is no change in the capital stock and hence the production in subsequent periods. For the Ramsey model with AK production function, both the private consumption and the outputs will be lowered. In both the Harrod-Domar and Ramsey models with Cobb-Douglas production function, if the debt is not high and the interest rate is sufficiently low, it is better to use public debt for production rather than for consumption. If the country borrows to recover the Total Factor Productivity after the Covid pandemic, both the Harrod-Domar and Ramsey models with Cobb-Douglas production function show that the rate of growth is higher for the year just after the pandemic but is the same as before the pandemic.

Practical implications – The economy can recover the growth rate after a Covid shock, but the recovery process will generally take many periods.

Social implications – This paper focuses on economic implications and does not aim to examine social implications of policy changes or Covid-type shock.

Originality/value – The paper provides a comparison of three basic growth models with respect to public spending, public debts and repayments and Covid-type shocks.

Keywords Economic policies, Harrod-Domar model, Ramsey model, Solow model

Paper type Research paper

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1. Introduction

Economic growth can be considered to be the single most important long-term policy objective of any modern government, both at national and subnational levels. Not surprisingly, this topic has attracted the attention of many of the brightest economic theorists (see, for example, Ramsey, 1928; Harrod, 1939, 1948; Domar, 1946, 1947; Solow, 1956; Uzawa, 1961, 1963, 1965; Lucas, 1988; Romer, 1990). Economic growth was also a subject of interest to Ngo Van Long (see, for example, Long, 1982; Long & Wong, 1997; Long, Nishimura, & Shimomura, 1997; Long & Shimomura, 2004), to whom this paper is dedicated.

The principal aim of this paper is to review three basic theoretical growth models, namely the Harrod-Domar model, the Solow model and the Ramsey model and examine their implications for economic policies. We first introduce a closed economy which operates in a discrete infinite time horizon. In this context, the Harrod-Domar and Solow models are presented successively. We then consider a discrete-time version of Ramsey model (for continuous-time versions of the Ramsey model, refer to Cass, 1965; Koopmans, 1965). For simplicity and tractability, we assume the one-period utility function of the representative consumer in the Ramsey model has a very simple form $u(c) = \ln(c)$, $c > 0$.

For economic policies and economic shocks, we focus on:

(1) Public spending. In particular, with the Ramsey model, we show that the impacts of public spending change with the date of the announcement of the policy

(2) Public debt and its repayments

(3) Covid-type shock

The main results we obtain from these comparisons are as follows:

(1) In the Harrod-Domar model, the saving rate is given and positive. In the Ramsey model with AK production function, the optimal saving rate is constant over time but it can be negative.

(2) In the Harrod-Domar model, the change of the rate of growth with respect to the TFP A equals the saving rate, while in the Ramsey Model with AK production function, the change is higher than the saving rate.

(3) If both the Solow and Ramsey Models with Cobb-Douglas production function and with full depreciation of the capital, exhibit medium-income traps, a change of the TFP induces a larger change of these traps in the Ramsey Model than in the Solow Model.

(4) Another important difference is we can calculate the prices of the consumption goods and capital goods for the Ramsey Model, while only the prices of capital goods can be obtained with the Solow model.

(5) Impacts of the public spendings: For the Harrod-Domar Model, the public spending decreases the private consumption at the period 1, but there is no change in the capital stock and hence the production in subsequent periods. It is due to the saving rate that remains unchanged. In the Ramsey model with AK production function, since the saving rate is endogenous, both the private consumptions and the outputs will be lowered.

(6) In both the Harrod-Domar and Ramsey models with Cobb-Douglas production function, if the debt is not high and the interest rate is sufficiently low, it is better to use public debt for production rather than for consumption. We get more outputs.

(7) If the country borrows to recover the TFP after the Covid pandemic, both the Harrod-Domar and Ramsey models with Cobb-Douglas production function show that the

rate of growth is higher for the year just after the pandemic but is the same as before the pandemic.

2. The economy

The economy we consider is a *closed economy* and lasts for an infinite number of time periods, denoted by t, with $t = 0, 1, 2, \ldots, +\infty$. It starts with a population, a production technology and an initial per capita capital stock $k_0 > 0$. It is assumed that the population is stationary. There is a produced aggregate good, which can be consumed and/or used as capital input.

At date t, we denote by

(1) c_t: total consumption per capita

(2) $c_{p,t}$: household consumption per capita

(3) S_t: saving per capita

(4) I_t: investment per capita

(5) k_t: capital stock per capita

(6) y_t: output per capita

(7) G_t: government expenditure per capita.

At any date t, the following two national accounting identities hold

$$c_t + S_t = y_t$$

$$c_t = c_{p,t} + G_t$$

The models will compute the total consumption c_t. The share between the households consumption and the public expenditure is exogenous. It can be considered as a tool of public policy.

The output market is described by the aggregate production function

$$y_t = F(k_t)$$

i.e., the per capita output y_t is obtained by using the per capita capital input k_t with a production technology F. Dynamic capital accumulation is dictated by

$$k_{t+1} = (1 - \delta)k_t + I_{t+1}$$

with $\delta \in [0, 1]$ denoting the capital depreciation rate and we assume that I_{t+1} equals S_t, i.e., all savings in the current period become investment in the next period.

3. The Harrod-Domar model

The main additional assumptions in this model specify the saving and production function as follows:

(1) $S_t = sY_t$ for any period t where $s(\in (0, 1))$ is exogenously given.

(2) $y_t = Ak_t$ where $A(>0)$ is exogenously given.The parameter A measures the efficiency of the production technology.

$v = 1/A$ is the usual capital coefficient. We present below the Harrod-Domar Model. For any period $t \geq 0$

$$c_t + I_{t+1} = y_t \Leftrightarrow c_t + k_{t+1} - (1-\delta)k_t = Ak_t \tag{1}$$

$$y_t = Ak_t \tag{2}$$

$$c_t = (1-s)y_t \tag{3}$$

$$I_{t+1} = sy_t \tag{4}$$

$$k_{t+1} = sAk_t + (1-\delta)k_t \tag{5}$$

Solving the Harrod-Domar model

Observe that equation (5) gives, for all $t \geq 0$,

$$k_{t+1} = (sA + 1 - \delta)^{t+1} k_0 \tag{6}$$

hence $$y_t = A(sA + 1 - \delta)^t k_0 \tag{7}$$

$$I_{t+1} = sA(sA + 1 - \delta)^t k_0 \tag{8}$$

$$c_t = (1-s)(sA + 1 - \delta)^t k_0 \tag{9}$$

We see in this model all the quantities grow with a constant growth rate g given by

$$g = sA - \delta \tag{10}$$

g increases when s or/and A increase. For instance, if $\delta = 0.05, A = 0.5$ then $g = 0.10$ if $s = 0.3$. We can now understand why some countries may have 0.1 as growth rate of the GDP.

Once we have determined the path (c_t, k_{t+1}, y_t), the planner shares the total consumption c_t between $c_{p,t}$ and G_t.

4. The Solow model

In the Solow model the production function is increasing and strictly concave. Here, we assume $F(k) = Ak^\alpha$, $0 < \alpha < 1$. We list below the equations of the Solow model. For any period $t \geq 0$, with the initial capital stock $k_0 > 0$

$$c_t + I_{t+1} = y_t \Leftrightarrow c_t + k_{t+1} - (1-\delta)k_t = Ak_t^\alpha \tag{11}$$

$$I_{t+1} = sy_t \tag{12}$$

$$y_t = Ak_t^\alpha \tag{13}$$

$$k_{t+1} = sAk_t^\alpha + (1-\delta)k_t \tag{14}$$

$$c_t = (1 - s)y_t \tag{15}$$

As in the Harrod-Domar model, the saving rate s is constant over time and exogenous, and we have saving = investment.

Solving the Solow model

As in the Harrod-Domar model, we first use the accumulation capital equation (14).

$$k_t = sAk_t^{\alpha} + (1 - \delta)k_t$$

Define $\phi(x) = (1 - \delta)x + Ax^{\alpha}$. We can rewrite as follows

$$k_{t+1} = \phi(k_t)$$

Starting with $k_0 >$ this equation gives an infinite sequence $\{k_0, k_1, \ldots, k_t, \ldots\}$ where $k_{t+1} = \phi(k_t)$ for every $t \geq 0$. It is easy to prove that there exists $\bar{k} = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$ such that:

(1) If $0 < k_0 < \bar{k}$ then $k_t < k_{t+1} < \bar{k}$ for any $t > 1$ and $\{k_t\}\uparrow\bar{k}$.

(2) If $k_0 > \bar{k}$ then $\bar{k} < k_{t+1} < k_t$ for any $t > 1$ and $\{k_t\}\downarrow\bar{k}$.

(3) If $k_0 = \bar{k}$ then $k_t = \bar{k}$ for all t.

(4) $\bar{k}$ can be viewed as the medium-income trap.

When $\delta = 1$ tedious computations give

$$k_{t+1} = (sA)^{\frac{1-\alpha^t}{1-\alpha}}(k_0)^{\alpha^t} \tag{16}$$

5. The Ramsey model

There exists a benevolent social planner who maximizes the intertemporal utility of the representative consumer under the constraints that at any period t the consumer's consumption plus her investment is less than or equal to the per capita output. Define $\mathcal{F}(k) \equiv F(k) + (1 - \delta)k$. Formally, with $k_0 > 0$

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t), 0 \leq \beta < 1 \tag{17}$$

$$\text{for any } t, \qquad c_t + (k_{t+1} - (1 - \delta)k_t) \leq y_t = F(k_t) \tag{18}$$

$$\Leftrightarrow \qquad c_t + k_{t+1} \leq \mathcal{F}(k_t) \tag{19}$$

The production function is concave, strictly increasing and differentiable. The utility function u is strictly concave and satisfies the "Inada condition" $u'(0) = +\infty$.

Existence of the solution is well-known (a simple proof: first show a solution exists at finite horizon T. This problem is easy: maximization under convex constraints of a concave function in finite dimension. And then, let T go to infinity. The limits of the quantities constitute the solution for the infinite-horizon model.

Some basic results: let $\{c_t^*, k_{t+1}^*\}_{t=0,\ldots,+\infty}$ be the solution. Then,

(1) if it satisfies the *Euler Equation*: for any t,

$$u'\left(c_t^*\right) = \beta u'\left(c_{t+1}^*\right)\left(F'\left(k_{t+1}^*\right) + 1 - \delta\right) \tag{20}$$

$$\Leftrightarrow u'\left(c_t^*\right) = \beta u'\left(c_{t+1}^*\right)\mathcal{F}'\left(k_{t+1}^*\right) \tag{21}$$

(2) and if it satisfies the *transversality condition*:

$$\lim_{T\uparrow+\infty}\left\{\beta^T u'\left(c_T^*\right)k_{T+1}^*\right\} = 0, \tag{22}$$

then it is optimal.

(1) The optimal sequence $\{k_{t+1}^*\}$is monotonic (increasing or decreasing)

(2) If the production function F is strictly concave and satisfies $F'(+\infty) = 0$ then the sequence $\{k_{t+1}^*\}$ converges to a value k^{*s} (steady state) which satisfies

$$\beta\left(F'\left(k^{*s}\right) + 1 - \delta\right) = 1 \Leftrightarrow \mathcal{F}'\left(k^{*s}\right) = 1 + r$$

where $\beta = \frac{1}{1+r}$.

In this case, if we define $1 + r_t^* = \mathcal{F}'(k_{t+1}^*)$, then

$$r_t^* \to r$$

That means the returns of the capital in the long run equals the real interest rate.

(3) If $0 < k_0 < k^{*s}$ then the optimal sequence $\{k_{t+1}^*\}$ is increasing and converges k^{*s}. And if $k_0 > k^{*s}$ then the optimal sequence $\{k_{t+1}^*\}$ is decreasing and converges k^{*s}.

Comments

1. Let r denote the real interest rate. We suppose it constant over time. Define $\beta = \frac{1}{1+r}$. Then

$$\sum_{t=0}^{\infty}\beta^t u(c_t) = \sum_{t=0}^{\infty}\frac{1}{(1+r)^t}u(c_t)$$

If we measure the utility function $u(c_t)$ at period t in money of period 0, then the sum $\sum_{t=0}^{\infty}\frac{1}{(1+r)^t}u(c_t)$ is the total value of the utilities measured in money of period 0.

2. About the *Euler equation*: We can define discounted prices p_t^* by $p_t^* = \beta^t u'(c_t^*)$. Define $1 + r_t^* = \mathcal{F}'(k_t^*)$. Then, the returns obtained by investing a unit of capital is r_t^* $(\mathcal{F}(k_t^*) - \mathcal{F}(0) = \mathcal{F}(k_t^*) - 0 \simeq \mathcal{F}'(k_t^*)k_t^*)$.

Since the capital good and the consumption good is assumed to be the same aggregate good, p_t^* is also the price of the capital at period t. Hence, the Euler equation is a no-arbitrage condition:

$$p_t^* = p_{t+1}^*\left(1 + r_t^*\right)$$

3. About the *transversality condition*, $\lim_{t\uparrow+\infty}\beta^t u'(c_t^*)k_{t+1}^* = 0$. We can write this condition $\lim_{t\uparrow+\infty}p_t^* k_{t+1}^* = 0$. The value of the capital vanishes at infinity. Observe since the capital k_{t+1}^* is bought at period t we value it with the price at period t, p_t^*.

5.1 The Ramsey model with AK production function

We will assume $u(c) = \ln(c)$. The model can be expressed as

$$\max_{\{c_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \ln(c_t), 0 \le \beta < 1 \tag{23}$$

$$\text{for any } t, \qquad c_t + (k_{t+1} - (1-\delta)k_t) \le y_t = Ak_t \tag{24}$$

We claim that the sequence $\{k_{t+1}^*, c_t^*\}_{t\ge 0}$ which satisfies for any $t \ge 0$

$$k_{t+1}^* = [\beta(A+1-\delta)]^{t+1}k_0 \tag{25}$$

$$c_t^* = \beta^t(A+1-\delta)^{t+1}(1-\beta)k_0 \tag{26}$$

is *optimal*.

The claim is true if Euler Equation (20) and Transversality condition (22) are satisfied.

The optimal investment is

$$I_{t+1}^* = k_{t+1}^* - (1-\delta)k_t^* \tag{27}$$

$$I_{t+1}^* = [\beta(A+1-\delta)]^t[\beta(A+1-\delta) - (1-\delta)]k_0 \tag{28}$$

the optimal investment rate is

$$s_t^* = \frac{I_{t+1}^*}{Ak_t^*} \tag{29}$$

$$s_t^* = \frac{[\beta(A+1-\delta) - (1-\delta)]}{A} \tag{30}$$

A special case: at period 0, the constraint is

$$c_0 + k_1 \le B, B > 0 \tag{31}$$

The optimal solution is

$$k_{t+1}^* = \beta^{t+1}(A+1-\delta)^t B \tag{32}$$

$$c_t^* = [(A+1-\delta)\beta]^t(1-\beta)B \tag{33}$$

To verify that, define k_0 by $(A+1-\delta)k_0 = B$.

5.2 The Ramsey model with the production function $F(k) = Ak^{\alpha}$, $0 < \alpha < 1$

To make tractable the calculations, we assume full capital depreciation, $\delta = 1$.

We assume again $u(c) = \ln(c)$. The optimal solutions are: for any $t \geq 0$

$$k_t^* = (A\alpha\beta)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t} \tag{34}$$

$$c_t^* = A(1-\alpha\beta)\{k_t^*\}^{\alpha} \tag{35}$$

$$k_{t+1}^* = A\beta\alpha\{k_t^*\}^{\alpha} \tag{36}$$

$$\text{investment rate} \qquad s_t^* = \frac{k_{t+1}^*}{Ak^{*\alpha}} = \alpha\beta \tag{37}$$

To prove these claims, one uses Euler Equation (20) and Transversality condition (22).

Comments

(1) From (25), we see that the sequence $\{k_t^*\}$ converges when t goes to infinity, to a value $k^{*s} = (A\beta\alpha)^{\frac{1}{1-\alpha}}$.

(2) We can prove that if $0 < k_0 < k^{*s}$ then the optimal sequence $\{k_t^*\}$ is increasing and converges to k^{*s}, and if if $k_0 > k^{*s}$ then the optimal sequence $\{k_t^*\}$ is decreasing and converges to k^{*s}. This value k^{*s} can be viewed as the medium-income trap, as in Solow Model.

(3) The long term marginal productivity r^* verifies: $r^* = 1 + r$,(recall r is the real interest rate).

6. First comparisons of these models

6.1 The Harrod-Domar model and the Ramsey model with AK production function

(1) The Harrod-Domar model: The investment rate is exogenous. Investment is always positive. The quantities output y_t, capital stock k_t, consumption c_t, the investment I_t grow at the same rate $g = sA - \delta$. This rate g is positive if $sA > \delta$, negative if $sA < \delta$.

(2) The Ramsey model with AK production function: The quantities output, capital, consumption, investment grow also at constant rate $g^* = \beta(A + 1 - \delta)) - 1$. The investment equals

$$I_{t+1}^* = [\beta(A + 1 - \delta)]^t [\beta(A + 1 - \delta) - (1 - \delta)]k_0$$

If $\beta A > (1 - \beta)(1 - \delta)$ then the investment is positive. Otherwise, it is negative or zero.

The investment rate s^* equals $\frac{[\beta(A+1-\delta)-(1-\delta]}{A}$. It may be positive or negative. It increases with A and decreases when the real interest rate r decreases. We obtain that $g^* = s^*A - \delta$ as in the Harrod-Domar Model. However the impact of A on the growth rate differs. Indeed.

(1) In the Harrod-Domar model, $\frac{\partial g}{\partial A} = s$

(2) In the Ramsey model, $\frac{\partial g^*}{\partial A} = s^* + A\frac{\partial s^*}{\partial A} > s^*$.

6.2 The Solow model and the Ramsey model with $F(k) = Ak^{\alpha}$ *and* $\delta = 1$

In the Solow model there exists a medium-income trap $\bar{k} = (sA)^{\frac{1}{1-\alpha}}$. If we increase A then we enlarge the trap.

The sequence of capital stocks (k) is

$$k_t = (sA)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}$$

(1) In the Ramsey model, the investment rate s^* equals $\alpha\beta$. It increases with the elasticity α of the production function. It decreases if the real interest rate r increases. The medium-income trap is $k^{*s} = (A\alpha\beta)^{\frac{1}{1-\alpha}} = (As^*)^{\frac{1}{1-\alpha}}$. Its expression is quite similar to the one in Solow model. It increases when A increases and increases with s^*. The sequence of optimal capital stocks is

$$k_t^* = (s^* A)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}$$

Consider the impact of the TFP, A.

We have

(2) $\frac{\partial \ln(\bar{k})}{\partial A} = \frac{1}{1-\alpha}\frac{1}{A}$

(3) $\frac{\partial \ln(\bar{k}^{*s})}{\partial A} = \frac{1}{1-\alpha}\frac{1}{A} + \frac{1}{1-\alpha}\frac{1}{s^*}\frac{\partial s^*}{\partial A} > \frac{1}{1-\alpha}\frac{1}{A}$

Another important difference.

6.3 Prices

(1) In the Solow model, we can obtain only the marginal production cost $r_t^* = F'(k_t^*)$

(2) In the Ramsey model (see Le Van & Dana, 2003): If we define the discounted prices p_t^* by $p_t^* = \beta^t u'(c_t^*)$ then

- the sequence (c_t^*) maximizes the intertemporal utility of the representative consumer $\sum_{t=0}^{\infty} \beta^t \ln(c_t)$ under the budget constraint

$$\sum_{t=0}^{\infty} p_t^* c_t \leq F'(k_0)k_0 + \pi$$

where π denotes the maximal profit of firm,

- $\pi = \max_{\{(k_1,k_2,\ldots,k_t,\ldots\}} \sum_{t=0}^{\infty} p_t^* (Ak_t^\alpha - k_{t+1}) - F'(k_0)k_0$.

We have in particular the no-arbitrage condition

$$p_t^* = p_{t+1}^* \left(1 + r_t^*\right)$$

where $1 + r_t^* = \mathcal{F}'(k_{t+1}^*) = F'(k_{t+1}^*)$, since we assume $\delta = 1$.

In the Ramsey model, one can define consumption prices because the social preferences are specified while in the Solow model, the demand side is missing.

7. Second comparisons of these models: economic policies

7.1 With the Harrod-Domar model

1. Public spending

a. We suppose the government announces in period -1 it will increase its spending by $\gamma > 0$ in period 0.

The constraints will be

$$\begin{aligned} & c_0 + \gamma + I_1 = Ak_0 \\ & I_1 = sy_0 = sAk_0 \\ \text{for } t \geq 1, \quad & c_t + I_{t+1} = Ak_t \\ & I_{t+1} = sy_t = sAk_t \end{aligned}$$

Let $(k^*_{t+1}, c^*_t)_t$ be the sequences of initial capitals and consumptions (without public spending).

It is easy to check that

$$\begin{aligned} & k_t = k^*_t, y_t = y^*_t, \text{ for all} t \\ \text{but} \quad & c_0 = c^*_0 - \gamma, c_t = c^*_t, \text{ for all } t \geq 1 \end{aligned}$$

If the government announces at the beginning of period 0, there will be an increase of public expenditures in period 1 then

$$\begin{aligned} & k_t = k^*_t, y_t = y^*_t, \text{for all } t \\ \text{but} \quad & c_0 = c^*_0, c_t = c^*_t \text{ for } t \geq 2, c_1 = c^*_1 - \gamma \end{aligned}$$

2. Public debt

The government borrows an amount D in period 0 and reimburses in period 1 the amount $D(1+\rho)$, ρ is the real interest rate. The constraints are

$$\begin{aligned} & c_0 + I_0 = Ak_0 + D \\ & I_0 = sy_0 = sAk_0 \\ & c_1 + D(1+\rho) + I_1 = Ak_1 \\ & I_1 = sy_1 = sAk_1 \\ & c_2 + I_2 = Ak_2 \\ & I_2 = sy_2 = sAk_2 \\ & \ldots \end{aligned}$$

2.1. The government uses the debt only for consumption

In this case:

$$\begin{aligned} & k_t = k^*_t, y_t = y^*_t, \text{ for all } t \\ & c_0 = c^*_0 + D, c_t = c^*_t, \text{ for } t \geq 2 \\ & c_1 = c^*_1 - D(1+\rho) \end{aligned}$$

D and ρ must not be too large in order to ensure that $c_1 > 0$.

2.2. The government uses debt for production. It buys $\tilde{k}_0$, s.t. $D = A\tilde{k}_0$. Define $k'_0 = k_0 + \tilde{k}_0$. The constraints will be:

$$
\begin{aligned}
&c_0 + I_1 = Ak_0' \\
&I_1 = sAk_0' \\
&c_1 + A\tilde{k}_0(1+\rho) + I_2 = Ak_1 \\
&I_2 = sAk_1 \\
&c_2 + I_3 = Ak_2 \\
&I_3 = sAk_3 \\
&\ldots
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
&c_0 = (1-s)Ak_0' > c_0^* \\
&c_1 = (1-s)A(sA+1-\delta)k_0' - A\tilde{k}_0(1+\rho) \\
&c_1 = (1-s)A(sA+1-\delta)k_0 + AD\zeta \\
\text{with} \quad &\zeta = (1-s)(sA+1-\delta) - (1+\rho)
\end{aligned}
$$

We have to make sure that $c_1 > 0$.
If $\zeta > 0$ then. $c_1 > c_1^* > 0$
If $\zeta < 0$ then we must have the condition

$$
D < \frac{(1-s)A(sA+1-\delta)k_0}{-\zeta}
$$

Under this condition $c_1 > 0$ but $c_1 < c_1^*$.

Summing up, If D is not too large and ρ small enough then

$$
\begin{aligned}
\text{for all } t, \quad &y_t > y_t^* \\
&c_t > c_t^*, \text{for all } t \neq 1,\ c_1 \text{ may be larger or smaller than } c_1^*
\end{aligned}
$$

3. Covid shock [1]

Question: what is the cost to recover from Covid-19 pandemic?

At $t = 0$, because of Covid pandemic, the TFP A passes to $A' < A$.

$$
c_0 + I_1 = A'k_0 < c_0^*,\ k_1 = \left(sA' + 1 - \delta\right)k_0 < k_1^*
$$

The country borrows in period 0 an amount D to recover A in period 1. But it has to reimburse $D(1+\rho)$.

$$
\begin{aligned}
&c_1 + D(1+\rho) + I_2 = Ak_1 \\
&I_2 = sAk_1 \\
&c_2 + I_3 = Ak_2 \\
&I_3 = sAk_2 \\
&\ldots
\end{aligned}
$$

We have the following results

$$\text{for all } t \geq 1, \quad k_{t+1} = (sA + 1 - \delta)^t (sA' + 1 - \delta) k_0$$

$$y_{t+1} = A(sA + 1 - \delta)^t (sA' + 1 - \delta) k_0 \quad t \geq 1$$

$$\frac{y_1}{y_0} = A\left(s + \frac{1-\delta}{A'}\right) > \frac{y_1^*}{y_0^*}$$

$$\frac{y_{t+1}}{y_t} = \frac{y_{t+1}^*}{y_t^*}, t \geq 1$$

At date 1, the output growth rate is higher than the one before Covid. It becomes the same after period 1. However

$$y_t < y_t^*, t \geq 1$$

We can recover the growth rate but not the level of the output.

But we have to ensure that $c_1 > 0$. It will be true if

$$(1 - s)A(sA + 1 - \delta)\frac{sA' + 1 - \delta}{sA + 1 - \delta}k_0 - D(1 + \rho) > 0$$

If the cost D is very big, the TFP cannot be recovered in period 1. The recovering process will last for many periods.

7.2 With the Ramsey model and AK production function

1. Public spending

We will show the importance of the date of the announcement of the government.

- The government announces at period -1 that it will spend $\gamma > 0$ in period 1.

The constraints of the economy will be

$$\begin{aligned}
&c_0 + (k_1 - (1-\delta)k_0) = Ak_0 \\
&c_1 + (k_2 - (1-\delta)k_1) = Ak_1 - \gamma \\
&c_2 + k(K_3 - (1-\delta)k_2) = Ak_2 \\
&\ldots
\end{aligned}$$

Define $B_1 = (A + 1 - \delta)k_1 - \gamma$. We suppose $(A + 1 - \delta)k_1 - \gamma > 0$. We will show actually the new optimal sequence of capitals will satisfy this condition.

How do we process to solve the model? Observe the constraint at period 1 becomes

$$c_1 + k_2 = B_1$$

We solve first

$$\begin{aligned}
&\max \sum_{t \geq 1} \beta^t \ln(c_t) \\
&c_1 + k_2 = B_1 \\
&c_t + (k_{t+1} - (1-\delta)k_t) = Ak_t, \; t \geq 2
\end{aligned}$$

Go back to (31). The solution is given by

$$\tilde{k}_t = (A+1-\delta)^{t-2}\beta^{t-1}B_1$$
$$\tilde{c}_t = (A+1-\delta)^{t-1}\beta^{t-1}B_1$$

Define $S_1 = \sum_{t\geq 1}\beta^t \ln\tilde{c}_t$. We find

$$S_1 = \frac{\beta}{1-\beta}\ln B_1 = \frac{\beta}{1-\beta}\ln((A+1-\delta)k_1-\gamma)$$

To find the optimal $\tilde{k}_1$, we solve

$$\max_{k_1} \ln((A+1-\delta)k_0-k_1) + \frac{\beta}{1-\beta}\ln((A+1-\delta)k_1-\gamma)$$

We find

$$\tilde{k}_1 = \frac{1-\beta)\gamma+\beta(A+1-\delta)^2k_0}{A+1-\delta} > k_1^*$$

$$\tilde{k}_t = [(A+1-\delta)\beta]^{t-2}\left((A+1-\delta)\tilde{k}_1-\gamma\right) < k_t^*,\ t\geq 2$$

We show that $(A+1-\delta)\tilde{k}_1-\gamma>0$ if, and only if $(A+1-\delta)^2k_0>\gamma$. We have to assume that γ must be lower to $(A+1-\delta)^2k_0$.

- Now suppose the government announces at the end of period 0 that it will spend $\gamma>0$ in the next period, period 1. It is a surprise for the consumer. This one has already planned the optimal investments, hence the optimal capital k_1^* by solving

$$\max_{\{c_t\}}\sum_{t\geq 0}\beta^t\ln(c_t)$$
$$c_t+k_{t+1}-(1-\delta)k_t = Ak_t,\ t\geq 0$$

The consumer, at the beginning of period 1 will solve a new program

$$\max_{\{c_t\}}\sum_{t\geq 1}\beta^t\ln(c_t)$$
$$c_1+k_2 = (A+1-\delta)k_1^*-\gamma$$
$$c_t+k_{t+1}-(1-\delta)k_t = Ak_t,\ t\geq 2$$

We have to assume now $(A+1-\delta)^2\beta k_0>\gamma$ to have $(A+1-\delta)k_1^*-\gamma>0$.

We have the results for the new optimal sequences $\{\widehat{k}_t\}$, for $t\geq 2$

$$\widehat{k}_2 = \beta\Big[(1+1-\delta)^2\beta k_0-\gamma<\tilde{k}_2$$

and

$$\widehat{k}_t<\tilde{k}_t<k_t^*, t\geq 3$$

and $\widehat{k}_1 = k_1^* < \tilde{k}_1$. In particular

$$\sum_{t\geq 0}\beta^t\ln\left(\widehat{c}_t\right)<\sum_{t\geq 0}\beta^t\ln\left(\tilde{c}_t\right)<\sum_{t\geq 0}\beta^t\ln\left(c_t^*\right)$$

2. Public debt

We suppose the government borrows in order to increase the initial capital. The constraints are

$$\begin{aligned}
&c_0 + k_1 = (A+1-\delta)k_0 + D \\
&c_1 + D(1+\rho) + k_2 = (A+1-\delta)k_1 \\
&c_2 + k_3 = (A+1-\delta)k_2 \\
&\dots
\end{aligned}$$

Define $\tilde{k}_0$ by $D = (A+1-\delta)\tilde{k}_0$.
The constraints become

$$\begin{aligned}
&c_0 + k_1 = (A+1-\delta)k'_0, k'_0 \equiv k_0 + \tilde{k}_0 \\
&c_1 + k_2 = B_2 \equiv (A+1-\delta)\left(k_1 - (1+\rho)\tilde{k}_0\right) \\
&c_2 + k_3 = (A+1-\delta)k_2 \\
&\dots
\end{aligned}$$

Use the same technics as in the previous section to obtain

$$\begin{aligned}
&\tilde{k}_t = [(A+1-\delta)\beta]^{t-2}\beta^{t-1}B_2 \\
&\tilde{c}_t = [(A+1-\delta)\beta]^{t-1}\beta^{t-1}(1-\beta)B_2
\end{aligned}$$

We have to make sure $B_2 > 0$. The necessary and sufficient condition is

$$(A+1-\delta) + \tilde{k}_0(A-\delta-\rho) > 0$$

This condition is satisfied if.

(1) either $A > \delta + \rho$ (ρ is not large)

(2) or, if $A < \delta + \rho$, then $\tilde{k}_0 < \frac{(A+1-\delta)k_0}{\delta+\rho-A}$ ($\tilde{k}_0$, hence D, not too large)

We then have $\tilde{c}_0 > 0, \tilde{c}_1 > 0$:

$$\begin{aligned}
\tilde{k}_1 \quad &= \beta(A+1-\delta)k_0 + \tilde{k}_0[\beta(A+1-\delta) + (1-\beta)(1+\rho) \\
&= k_1^* + \tilde{k}_0[\beta(A+1-\delta) + (1-\beta)(1+\rho) > 0 \\
\tilde{c}_0 \quad &= (1-\beta)[(A+1-\delta)k_0 + \tilde{k}_0(A-\delta-\rho) \\
&= c_0^* + (1-\beta)\tilde{k}_0(A-\delta-\rho) > 0
\end{aligned}$$

$\tilde{c}_0$ may be higher or smaller than c_0^*.

3. Covid shock

At $t = 0$, because of Covid-pandemic, the TFP A passes to $A' < A$.
The country borrows in period 0 an amount D to recover A in period 1. But it has to reimburse $D(1+\rho)$.

$$\begin{aligned}
&c_0 + k_1 = \left(A' + 1 - \delta\right)k_0 \\
&c_1 + D(1+\rho) + k_2 = (A+1-\delta)k_1 \\
&c_2 + k_3 = (A+1-\delta)k_2 \\
&\dots
\end{aligned}$$

Define $B_3 = (A + 1 - \delta)k_1 - D(1 + \rho)$. Then

$$\begin{aligned} &c_1 + k_2 = B_3 \\ &c_2 + k_3 = (A + 1 - \delta)k_2 \\ &\dots \end{aligned}$$

We obtain the solutions

$$\begin{aligned} \widehat{k}_t &= [(A + 1 - \delta)\beta]^{t-2}\beta^{t-1}B_3, \ t \geq 2 \\ \widehat{c}_t &= (A + 1 - \delta)^{t-1}\beta^{t-1}(1 - \beta)B_3, \ t \geq 1 \end{aligned}$$

Let $S_1 = \sum_{t \geq 1}\beta^t \ln c^t$. We find

$$S_1 = \frac{\beta}{1 - \beta}\ln B_3 + H$$

where H is a constant.

Since $B_3 = (A + 1 - \delta)k_1 - D(1 + \rho)$ to obtain the optimal value $\widehat{k}_1$ we solve

$$\max_{k_1}\{\ln(c_1) + S_1\}$$

$$\Leftrightarrow \max_{k_1}\left\{\ln\left(\left(A' + 1 - \delta\right)k_0 - k_1\right) + \frac{\beta}{1 - \beta}\ln((A + 1 - \delta)k_1 - D(1 + \rho))\right\}$$

Define $\widehat{k}_0$ by $D = (A + 1 - \delta)\widehat{k}_0$. We obtain

$$\widehat{k}_1 = (1 - \beta)(1 + \rho)\widehat{k}_0 + \beta\left(A' + 1 - \delta\right)k_0$$

We have to check that $B_3 > 0$. We find that $B_3 > 0 \Leftrightarrow \widehat{k}_0 < \frac{A' + 1 - \delta}{1 + \rho}k_0$. If this condition is satisfied then $\widehat{c}_0 > 0, \ \widehat{c}_1 > 0$.

If the cost to recover the productivity D, hence $\widehat{k}_0$, is too high, then the recovering process will last for more than one period.

Let us compute the growth rates of the output $\widehat{y}^t$. We find

$$\begin{aligned} 1 + \widehat{g}_1 \quad &= \frac{\widehat{y}_1}{\widehat{y}_0} = \frac{A\widehat{k}_1}{A'k_0} = \frac{A(1 - \beta)(1 + \rho)\widehat{k}_0 + \beta\left(A' + 1 - \delta\right)k_0}{A'k_0} \\ &> \frac{\beta\left(A' + 1 - \delta\right)}{A'} > \frac{\beta(A + 1 - \delta)}{A} = \frac{y_1^*}{y_0^*} = 1 + g_1^* \end{aligned}$$

But for $t \geq 2, \widehat{g}_t = g_t^*$.

8. Conclusion

The main results we obtain from the comparisons of the Harrod-Domar model, the Solow model and the Ramsey model can be summarized as follows:

(1) In the Harrod-Domar model, the saving rate is given and positive. In the Ramsey model with AK production function, the optimal saving rate is constant over time but it can be negative.

(2) In the Harrod-Domar model the change of the rate of growth with respect to the TFP A equals the saving rate, while in the Ramsey model with AK production function, the change is higher than the saving rate.

(3) If both the Solow and Ramsey models with Cobb-Douglas production function and with full depreciation of the capital, exhibit medium-income traps, a change of the TFP induces a larger change of these traps in the Ramsey model than in the Solow model.

(4) Another important difference is we can calculate the prices of the consumption goods and capital goods for the Ramsey model, while only the prices of capital goods can be obtained with the Solow model.

(5) Impacts of the public spendings: For the Harrod-Domar model, the public spending decreases the private consumption at the period 1, but there is no change in the capital stock and hence the production in subsequent periods. It is due to the saving rate which remains unchanged. In the Ramsey model with AK production function, since the saving rate is endogenous, both the private consumptions and the outputs will be lowered.

(6) In both the Harrod-Domar and Ramsey models with Cobb-Douglas production function, if the debt is not high and the interest rate is sufficiently low, it is better to use public debt for production rather than for consumption. We get more outputs.

(7) If the country borrows to recover the TFP after the Covid pandemic, both the Harrod-Domar and Ramsey models with Cobb-Douglas production function show that the rate of growth is higher for the year just after the pandemic but is the same as before the pandemic.

Note

1. We suppose due to Covid, the TFP decreases for the first period, because Covid deteriorates the health of the workers and hence the TFP. The country will borrow to buy vaccin and medicine to help the workers recover their health and hence the TFP as well.

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