Pricing reinsurance and determining optimal retention based on the criterion of maximizing social expected utility

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Abstract

Purpose – The authors consider the mutual benefits of the ceding company and reinsurance company in the design of reinsurance contracts. Two objective functions to maximize social expected utilities are established, which are to maximize the sum of the expected utilities of both the ceding company and reinsurance company, and to maximize their products. The first objective function, additive, emphasizes the total gains of both parties, while the second, multiplicative, accounts for the degree of substitution of gains of one party through the loss of the other party. The optimal price and retention of reinsurance are found by a grid search method, and numerical analysis is conducted. The results indicate that the optimal solutions for two objective functions are quite different. However, optimal solutions are sensitive to the change of the means and volatilities of the claim loss for both objective functions. The results are potentially valuable to insurance regulators and government entities acting as reinsurers of last resort.

Design/methodology/approach – In this paper, the authors apply relatively simple, but in the view significant, methods and models to discuss the optimization of excess loss reinsurance strategy. The authors only consider the influence of loss distribution on optimal retention and reinsurance price but neglect the investment factor. The authors also consider the benefits of both ceding company and reinsurance company to determine optimal premium and retention of reinsurance jointly based on maximizing social utility: the sum (or the product) of expected utilities of reinsurance company and ceding company. The authors solve for optimal solutions numerically, applying simulation.

Findings – This paper establishes two optimization models of excess-of-loss reinsurance contract against catastrophic losses to determine optimal premium and retention. One model considers the sum of the expected utilities of a ceding company and a reinsurance company's expected utility; another considers the product of them. With an example, the authors find the optimal solutions of premium and retention of excess loss reinsurance. Finally, the authors carry out the sensitivity analysis. The results show that increasing the means and the volatilities of claim loss will increase the optimal retention and premium. For objective function I, increasing the coefficients of risk aversion of or reducing the coefficients of risk aversion of will make the optimal retention reduced but the optimal premium increased, and vice versa. However, for objective function 2, the change of coefficient of risk aversion has no effect on optimal solutions.

Research limitations/implications – Utility of the two partners: The ceding company and the reinsurance company, may have different weights and different significance. The authors have not studied their relative significance. The simulation approach in numerical methods limits us to the probability distributions and stochastic processes the authors use, based on, generally speaking, lognormal models of rates of return. This may need to be generalized to other returns, including possible models of shocks through jump processes.

Practical implications – In the recent two decades, reinsurance companies have played a great role in hedging mega-catastrophic losses. For example, reinsurance companies (and special loss sharing arrangements) paid as much as two-thirds of the insured losses for the September 11, 2001 tragedy.

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Furthermore, large catastrophic events have increased the role of governments and regulators as reinsurers of last resort. The authors hope that the authors provide guidance for possible balancing of the needs of two counterparties to reinsurance contracts.
Social implications – Nearly all governments around the world are engaged in regulation of insurance and reinsurance, and some are reinsurers themselves. The authors provide guidance for them in these activities.
Originality/value – The authors believe this paper to be a completely new and original contribution in the area, by providing models for balancing the utility to the ceding insurance company and the reinsurance company.
Keywords Reinsurance, Optimization, Social expected utility, Retention and price of reinsurance, Mutual benefit
Paper type Research paper

1. Introduction

Two kinds of pricing models, mean-variance and expected utility models, are often used in research on reinsurance pricing. The former is focused on seeking optimal retention which minimizes the ceding company's risk, and the latter is focused on seeking optimal retention which maximizes the ceding company's utility. Kremer (2002) discusses the limit-determination for the excess-of-loss cover with a simple retrocession treaty. Gajek and Zagrodny (2004) derived optimal forms for stop-loss contracts when the insurer attempts to minimize the probability of ruin. Centeno (2005) studied the optimal excess-of-loss retention limits for two dependent risks. They established the expected utility of wealth with respect to the exponential utility function and adjustment coefficient of the retained aggregate claim amount and found optimal retention limits by means of the optimization objective function. Bai *et al.* (2010) studied the optimal strategies of excess-of-loss reinsurance and dividends by maximizing the expected total discounted dividends received by shareholders until the time of ruin. The problem was formulated as a stochastic impulse control problem and explicit solutions were obtained. The transaction costs and taxes were considered in calculation of dividends. Froot and O'Connell (2008) examined the equilibrium catastrophe reinsurance price when maximizing the values of primary insurers. They concluded that a reinsurance company should charge a price greater than required by a "fair" return. Cao and Xu (2010) assumed that investment funds follow the logarithm-normal distribution. They derived the proportional and excess-of-loss reinsurance contracts, and formulated the convex combination of the insurer and reinsurer's returns exceeding a constant value at a probability. The premium was determined based on the equivalence principle. Zhao *et al.* (2013) studied the optimal excess-of-loss reinsurance and investment problem for an insurer with a jump-diffusion risk model. They established the objective function for maximizing the expected exponential utility of the terminal wealth of a reinsurer. Liang and Bayraktar (2014) considered the optimal reinsurance and investment problem in an unobservable Markov modulated compound Poisson risk model, where the intensity of the jump size distribution is not known but must be inferred from observations of claim arrivals. They established an optimal model using stochastic control theory.

These existing models only consider the benefits of ceding companies but have not considered the mutual influence between ceding and reinsurance companies. Reinsurance companies play an important role in hedging catastrophic losses and reducing the disruption of insurance markets after a mega-catastrophic event (see Cummins, 2007). If their benefits have not been considered, the reinsurance company will be reluctant to provide catastrophe coverage. Baton and Lemaire (1981) analyzed a dynamic game in a multicriteria situation in which players attempt to maximize their payoffs but also try to enter a "stable coalition in the frame of discretion." Mao and Wen (2018) stated that, "In all of the existing systems, [an] appropriate amount of capital is required for insurance companies to hold, in order to remain financially sound with certain probability during a specified period. However, this does not necessarily guarantee maximal social welfare, and thus is not derived from an economically optimal design under social planning." This issue has already drawn some interest from academia. For example, Dasgupta and Nanda (1993) presented the asymmetric Nash bargaining outcomes based on the optimal capital structure, which maximize the product of the weighted exponent of the insurer's profit and consumer surplus (also note Thomson, 1981). Mao and Ostaszewski (2007) discussed pricing models for a deferred annuity, in which cooperative game theory is applied to formulate different pricing models according to customers' preferences about benefits and risks to maximize social welfare. Huang and Tzeng (2007) showed that the policymaker can select a tax deduction rate to maximize the weighted average of the insured's expected utility and the insurer's expected value. Zanjani (2010) derived prices that are consistent with a social optimum based on an insurance company's capital allocation and the consumer-level capital allocation. Nevertheless, all of the abovementioned studies only discuss one decision variable in the optimization problem, such as price, tax, or capital structure to maximize social benefit. Mao and Wen (2018) explored the optimal price, default ratio and capital for insurance companies under social welfare maximization from regulators' perspective. Traditional reinsurance pricing only considers the benefit to the ceding company. However, a firm's success depends not only on the price charged, but also on how a reinsurance company and its competitors respond.

Li *et al.* (2014) assumed that the claim process is described by a Brownian motion with drift, the insurer can purchase proportional reinsurance, and that both the insurer and reinsurer can invest in risk-free and risky assets. By taking both the insurer and reinsurer into account, they aim to maximize the expected product of the insurer and the reinsurer's exponential utilities of terminal wealth. Ya *et al.* (2018) studied a robust optimal reinsurance-investment problem for a general insurance company that holds the shares of insurance and reinsurance companies. That work utilized assumptions similar to those of Li *et al.* (2014) regarding the claim and investment processes. Moreover, the general insurance company's manager is an ambiguity-averse manager who worries about model uncertainty in model parameters. The ambiguity-averse manager's objective is to maximize the minimal expected product of the insurer and reinsurer's exponential utilities. Zhao *et al.* (2017) studied time-consistent solutions to an investment-reinsurance problem under a mean-variance framework. They considered the weighted average of the interests of both an insurer and reinsurer jointly in the decision-making process. The claim process of the insurer that they utilized was governed by a Brownian motion with drift. A proportional reinsurance treaty was considered, and the premium was calculated using the expected value (equivalence principle for net premiums). Both the insurer and reinsurer were assumed to invest in a risky asset, driven by a constant elasticity of variance model. Li *et al.* (2017) considered an equilibrium excess-of-loss reinsurance and investment strategy. They assumed that the surplus process follows the classical Cramér-Lundberg model and that there is both a risk-free and risky asset available for investment. Under the framework of mean-variance and game theory, the equilibrium solutions were obtained.

Existing studies on optimal excess-of-loss strategy are focused on the determination of optimal retention assuming the reinsurance premium is given. The reinsurance premium is generally determined based on the historical data. However, in a situation where there is no historical data on reinsurance claims, it is difficult to determine the reinsurance premium. Since the retention and reinsurance premiums mutually affect one another, the optimization problem must take both the retention and reinsurance premiums into account simultaneously; it is necessary to consider both the reinsurance premium and retention as decision variables. Moreover, analysis of reinsurance and investment strategy established based on stochastic control theory or stochastic differential game theory is generally quite complicated, and optimal solutions are only obtained by approximate numerical method. In our opinion, the value-added effect of investment and the dynamic consideration of random variables for short-term catastrophic, property and liability insurance are very limited, especially in China; however, some nonlife insurance companies have purchased investment-linked nonlife products in recent years, but the maturities of most of these products are limited to intermediate periods, generally, three to five years (Liu and Zhang, 2007). Furthermore, as reported by Sina Finance of Sina Web (2017) [1], the Banking and Insurance Supervision Committee of China declared the suspension of investment-linked products for nonlife insurance companies.

In this study, we apply relatively simple, but in our view significant, methods and models to discuss the optimization of excess-of-loss reinsurance strategies. We only consider the influence of loss distribution on optimal retention and reinsurance prices but neglect the investment factor at this time, leaving this issue for later studies. We also consider the benefits of both the ceding and reinsurance company to determine the optimal premium and retention of reinsurance based on the maximization of social utility, by considering the sum or product of the expected utilities of the reinsurance and ceding companies. The approach of considering benefits to the insurer and reinsurer in combination is relative rare in the existing literature. Syuhada *et al.* (2021) consider only specific types of contracts: combined stop-loss and quota-share, reinsurance and present conditional tail expectation (CTE)-based optimization from the joint perspective of the insurer and reinsurer. Chen (2021) studied the optimal reinsurance contracts that minimize the convex combination of the conditional value-at-risk (CVaR) of the insurer and reinsurer's losses over the class of ceded loss functions such that the retained loss function is increasing, and the ceded loss function satisfies the Vajda condition. Moreno *et al.* (2022) provided an interesting perspective on studying the soundness of insurance firms. We also note the work of Gong *et al.* (2021) and Marinakis and White (2022) providing the perspective of the trading markets and price discovery, as well as sustainability.

We believe our approach in this study is more general than those recent papers and can shed new light on the practice of reinsurance, as well as the relatively new approach of considering the benefits of the contract in aggregation with the perspectives of the two contract parties.

During periods of significant financial or economic stress, regulators and governments often assume new duties in their supervision of insurance markets, which often amount to the provision of special kinds of reinsurance for unexpectedly large losses due to shocks such as credit crises or pandemics. Such interventions are typically enacted as ad-hoc mechanisms designed to prevent a large systemic financial crisis but not with the perspective of maximizing welfare or social benefits or all sides of these transactions. We hope that our model can serve as guidance for regulators finding themselves in such stressful situations seeking not only to provide relief from current stress, but also longer-term benefits, such as financial stability, thus reducing the need for sudden reinsurance-like interventions.

We believe our work brings about a new contribution in the field, while building on the existing body of literature.

The remainder of the paper is organized as follows:

(1) Section 1 presents a model for determining the optimal price of reinsurance and optimal retention in the reinsurance contract. This section begins with a subsection discussing the main hypothesis.

(2) Section 2 presents model applications for specific utility functions, some theoretical results that supplement that application, as well as the results of numerical analyses, including graphical representation.

(3) The following section is a discussion.

(4) The paper ends with conclusions, summarizing the results and discussing possible future research.

2. Model for determining optimal price and retention of excess-of-loss reinsurance

Main hypothesis of the research: Our main hypothesis proposes that when the combined benefit of the ceding insurance company (i.e., an insurance company purchasing an insurance contract) and the reinsurance company (i.e., the insurance company selling that insurance contract) are considered, optimal pricing and contract structure vary, and sometimes vary significantly, when compared with pricing and structure when only the perspective of the ceding company is adopted (the dominant perspective in the existing literature). We verify this hypothesis in this research. It should be noted that our results are numerical and based on optimization in MATLAB, not closed form solutions, due to the complexity of the problem. We hope that this research will be furthered, improved upon or even perfected in future work.

2.1 Maximizing the sum of the expected utilities of a reinsurance company and ceding company (objective function I)

To simplify the analysis, we assume that there is only one ceding company and one reinsurance company. Let X be the claim of the catastrophic loss. By bargaining between the ceding and reinsurance company, they reach the following reinsurance contract: If the claim X is less than or equal to the retention M, the ceding company will pay the claim; if the claim X is larger than the retention M, the ceding company will pay M and the reinsurance company will pay the excess part of the claim. We assume that both the ceding and reinsurance company are risk averse.

Let the ceding and reinsurance companies sign their reinsurance contract at time $t = 0$; the claim loss is paid at the end of the year and the discounting factor is approximately offset by the investment factor. In this way, we can simplify our analysis. We assume that the information is complete and that there is no problem of moral hazard or adverse selection caused by private information. Let the cumulative distribution function of the claim X be $F(x)$, and its density function be $f(x)$, with $0 < x < \infty$. Let the contract premium of reinsurance be P. The net benefit obtained by the reinsurance company is

$$Y_1 = \begin{cases} P & \text{when } 0 \le X \le M, \\ P - (X - M) & \text{otherwise.} \end{cases} \tag{1}$$

The expected utility of the reinsurance company is

$$E(U_1(Y_1)) = \int_0^M U_1(P)f(x)dx + \int_M^{+\infty} U_1(P - (x - M))f(x)dx. \tag{2}$$

For the ceding company, its benefit is

$$Y_2 = \begin{cases} E(X) - X - P & \text{when } 0 \le X \le M, \\ E(X) - M - P & \text{otherwise.} \end{cases} \tag{3}$$

The assumption is that its own pricing of the product is based on the equivalence principle so that $E(X)$ is the premium collected by the ceding company. Its expected utility is

$$E(U_2(Y_2)) = \int_0^M U_2(E(X) - x - P)f(x)dx + \int_M^{+\infty} U_2(E(X) - P - M)f(x)dx \tag{4}$$

where $E(X)$ is the expected value of the claim loss.

We establish the objective function to maximize the social utility or the sum of the expected utilities of the reinsurance and ceding companies as follows:

$$\begin{aligned} \text{Max}\, E(U(Y)) &= E(U_1(Y_1)) + E(U_2(Y_2)) - \\ &= \int_0^M U_1(P)f(x)dx + \int_M^{+\infty} U_1(P - (x - M))f(x)dx \\ &+ \int_0^M U_2(E(X) - x - P)f(x)dx + \int_M^{+\infty} U_2(E(X) - M - P)f(x)dx \end{aligned} \tag{5}$$

2.2 Maximizing the product of the expected utilities of a reinsurance company and ceding company (objective function II)

One of most important characteristics of reinsurance is co-fatality, that is, the fatality of the reinsurance and ceding companies is bound together for good or ill. It is especially incisive when using the product rather than the sum of the expected utilities of the reinsurance and ceding companies, since the expected utility of any of them equals zero; the total expected utility of social expected utility equals zero. Moreover, the expanding or shrinking function of the expected social utility is much more obvious because of the effect of the multiplier.

Using the same definitions of the benefits of the reinsurance and ceding companies, Y_1 and Y_2, the objective function to maximize the product of the reinsurance and ceding companies can be written as

$$\begin{aligned} &\text{Max}\, E(U(Y)) = E(U_1(Y_1)U_2(Y_2)) = \\ &= \int_0^M U_1(P)U_2(E(X) - x - P)f(x)dx + \int_M^{+\infty} U_1(P - x + M)U_2(E(X) - P - M)f(x)dx \end{aligned} \tag{6}$$

3. Model applications, analysis supporting those applications and results of numerical optimization

We assume that the ceding and reinsurance companies are risk averse. Let

$$U_1(Y_1) = Y_1 - \frac{Y_1{}^2}{2\gamma_1} \tag{7}$$

and

$$U_2(Y_2) = Y_2 - \frac{Y_2{}^2}{2\gamma_2} \tag{8}$$

where Y_1 and Y_2, satisfying equations (1) and (3), respectively, and γ_1 and γ_2 are the risk aversion coefficients of the reinsurance and ceding companies, respectively. We use the example in Mao *et al.* (2016) to illustrate its application. In a manner similar to Example 1 in Mao *et al.* (2016), we consider a lognormal loss distribution with parameters $\mu = 9.294$ and $\sigma = 1.627$. In this case, the loss density is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, x > 0. \tag{9}$$

Using equations (7), (8), (9), and the objective function (5), we have the following:

$$\begin{aligned}
&\operatorname{Max} E(U(Y)) = E(U_1(Y_1)) + E(U_2(Y_2)) = \\
&= \int_0^M U_1(P) f(x)dx + \int_M^{+\infty} U_1(P - x + M) f(x)dx \\
&+ \int_0^M U_2(E(x) - x - P) f(x)dx + \int_M^{+\infty} U_2(E(x) - P - M) f(x)dx \\
&= \int_0^M \left(-\frac{P^2}{2\gamma_1} + e^{\mu+\frac{1}{2}\sigma^2} - x - \frac{(\mu - x - P)^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_M^{+\infty} \left(e^{\mu+\frac{1}{2}\sigma^2} - x - \frac{(P - x + M)^2}{2\gamma_1} - \frac{(\mu - P - M)^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&= -\frac{P^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{1}{2}\sigma^2} - P\right)^2}{2\gamma_2} \\
&+ \frac{\left(e^{\mu+\frac{1}{2}\sigma^2} - P\right) e^{\mu+\frac{1}{2}\sigma^2}}{\gamma_2} \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right) - \frac{e^{2\left(\mu+\sigma^2\right)}}{2\gamma_2} \Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right) \\
&+ \frac{P e^{\mu+\frac{1}{2}\sigma^2}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)\right)}{\gamma_1} - M\left(\frac{P}{\gamma_1} - \frac{e^{\mu+\frac{1}{2}\sigma^2} - P}{\gamma_2}\right)\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma}\right)\right) \\
&- \frac{e^{2\left(\mu+\sigma^2\right)}}{2\gamma_1}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right)\right) + \frac{M e^{\mu+\frac{1}{2}\sigma^2}}{\gamma_1}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)\right) \\
&- M^2\left(\frac{1}{2\gamma_1} + \frac{1}{2\gamma_2}\right)\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma}\right)\right)
\end{aligned} \tag{10}$$

Similarly, applying equations of (7), (8), (9), and the objective function (6), we obtain the objective function as follows:

$$Max\,E(U(Y)) = E(U_1(Y_1)U_2(Y_2)) =$$
$$= \int_0^M U_1(P)U_2(E(X) - x - P)f(x)dx$$
$$+ \int_M^{+\infty} U_1(P - x + M)U_2(E(X) - P - M)f(x)dx = \tag{11}$$
$$A\Phi\left(\frac{\ln M - \mu}{\sigma}\right) + Be^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)$$
$$+Ce^{2\left(\mu+\sigma^2\right)}\Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right) + D\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma}\right)\right)$$
$$-Ee^{\mu+\frac{\sigma^2}{2}}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)\right) - Ge^{2\left(\mu+\sigma^2\right)}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right)\right),$$

where

$$A = P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P}{2\gamma_2} - \frac{P}{2\gamma_1} + \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{4\gamma_1\gamma_2}\right),$$

$$B = P\left(-1 + \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} + \frac{P}{2\gamma_1} - \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{2\gamma_1\gamma_2}\right), C = -P\left(\frac{1}{2\gamma_2} - \frac{P}{4\gamma_1\gamma_2}\right),$$

$$D = (P + M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_2} - \frac{P + M}{2\gamma_1}\right.$$
$$\left. + \frac{(P + M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{4\gamma_1\gamma_2}\right),$$

$$E = e^{\mu+\frac{\sigma^2}{2}} - P - M - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_2} - \frac{(P + M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{2\gamma_1}$$
$$+ \frac{(P + M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_1\gamma_2},$$

and $G = \frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{4\gamma_1\gamma_2}$.

Derivation of the above results is presented in the Appendix.

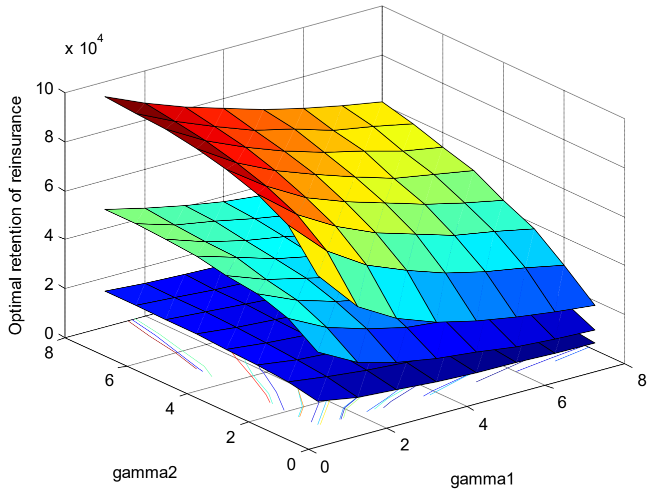
We obtained optimal solutions of (M^*, P^*) by solving objective functions (10) and (11) numerically with the help of MatLab software. The first objective function, additive, emphasizes the total gains of both parties, while the second, multiplicative, accounts for the degree of substitution of gains of one party by the loss of the other party.

The objective functions (10) and (11) show that there are no optimal solutions if $M = 0$ or $M \to +\infty$. Otherwise, we can have optimal solutions if γ_1 and γ_2 take the values, satisfying both the first and second order conditions.

From the objective functions of (10) and (11), we find that both have explicit expressions. Although we cannot directly obtain the optimal explicit solutions by analytical method, we can find approximate optimal solutions with the help of the grid search method in MatLab software.

Figures 1 through 4 display the change patterns of optimal retention and premium when the mean and volatility of claim loss change by $\pm 10\%$ with objective function I. Figures 5 through 8 describe the change patterns of optimal retention and premium when the mean and volatility of claim loss change by $\pm 10\%$ with objective function II. Figures 1 through 4 indicate that an increase (decrease) of the means and the volatilities of claim loss will increase (decrease) the optimal retention and optimal premium. The sensitivities of optimal solutions to the change of the mean are much greater than those to the change of the volatility of claim loss, especially in situations where the mean of claim loss increases by 10%. Figures 1 through 4 also show that increasing the coefficients of risk aversion of γ_1 or reducing the coefficients of risk aversion of γ_2 reduce the optimal retention but increase the optimal premium. Figures 5 through 8 illustrate that the increase (decrease) of the mean and the volatility of claim loss increase (decrease) the optimal retention and optimal premium. Unlike in the case of objective function I, the case with objective function II shows that the coefficients of risk aversion have no effect on the optimal solutions, regardless of the changes to the mean and volatility of the claim loss. It is important to note that both the optimal retention and premium are sensitive to the change of the mean and the volatility of claim loss, unlike in the case of objective function I.

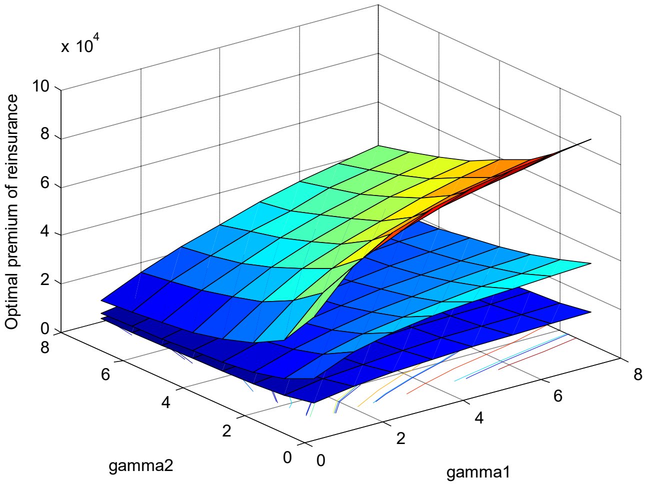
The curve (flat) surfaces on the top and bottom of Figures 1 through 8 correspond to situations where the mean or volatility of claim loss increase or decrease by 10%, respectively. The curve (flat) surfaces in the middle of the figure correspond to situations where the mean or volatility of claim loss are at the levels obtained by estimating them from empirical data.



Note(s): Objective Function I

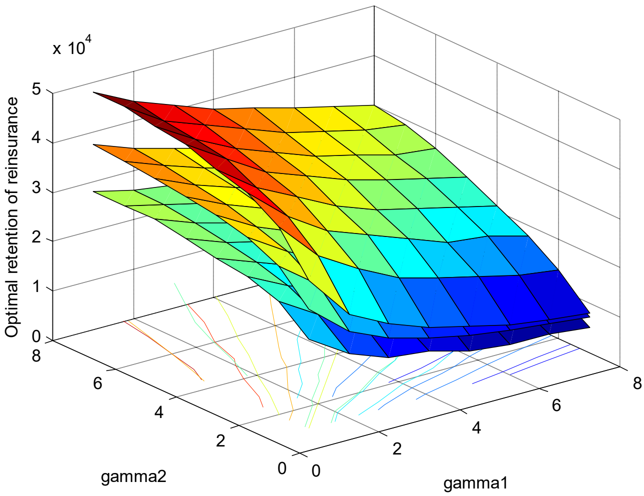
Source(s): Authors' calculation

Figure 1. The patterns of change of optimal retention when $\mu_1 = \mu(1 \pm 10\%)$



Note(s): Objective Function I
Source(s): Authors' calculation

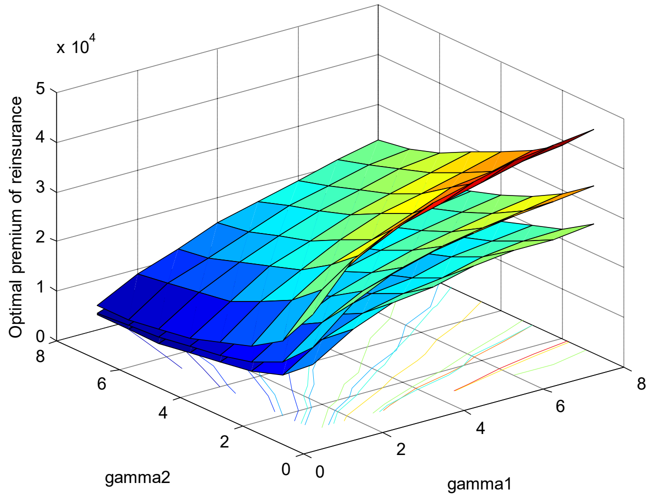
Figure 2.
The patterns of change of optimal premium when $\mu_1 = \mu(1 \pm 10\%)$



Note(s): Objective Function I
Source(s): Authors' calculation

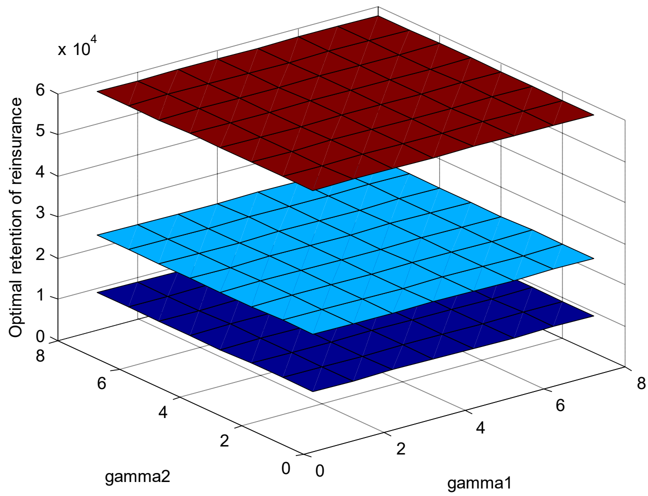
Figure 3.
The patterns of change of optimal retention when $\sigma_1 = \sigma(1 \pm 10\%)$

Table 1 presents the results of optimal solutions with two different situations: one is that the reinsurance premium is given, and the other is that the reinsurance premium (not given) is obtained by optimization. The results in Table 1 indicate that the optimal expected social benefits are greater if the reinsurance premium is determined by optimization rather than by being given for both objective function I and II. The increased optimal expected social benefits are greater for objective function II than for objective function I.



Note(s): Objective Function I
Source(s): Authors' calculation

Figure 4. The patterns of change of optimal premium when $\sigma_1 = \sigma(1 \pm 10\%)$

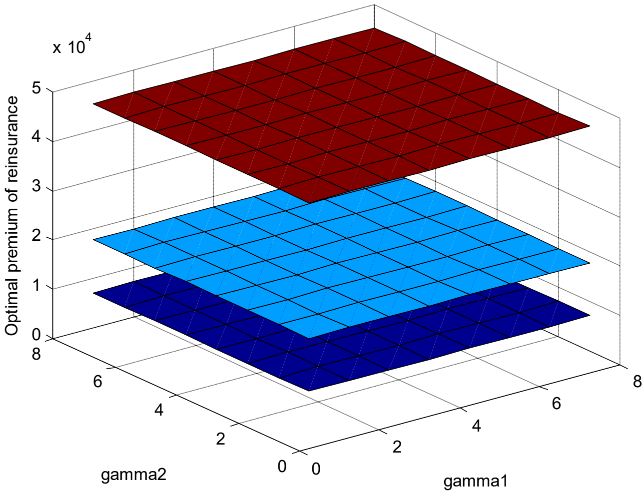


Note(s): Objective Function II
Source(s): Authors' calculation

Figure 5. The patterns of change of optimal retention when $\mu_1 = \mu(1 \pm 10\%)$

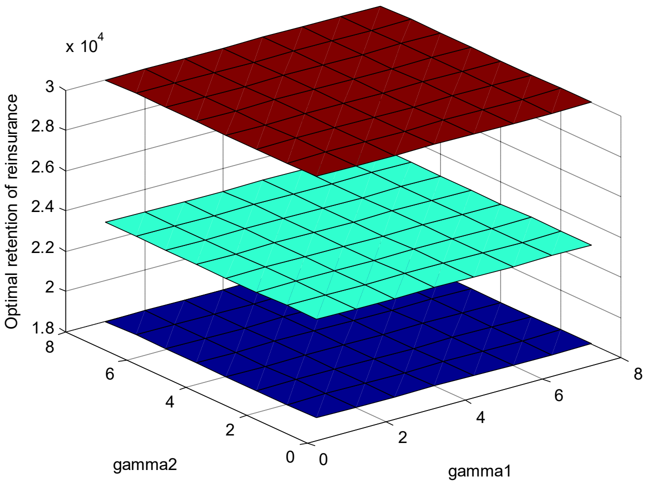
4. Discussion

In excess-of-loss reinsurance practice, it is commonly necessary to set the upper limit of covered claim loss. In this section, we discuss the joint optimization of the retention, premium and upper limit of claim loss. Let the upper limit of claim loss in the reinsurance contract be M_1. The net benefit obtained by the reinsurance company is



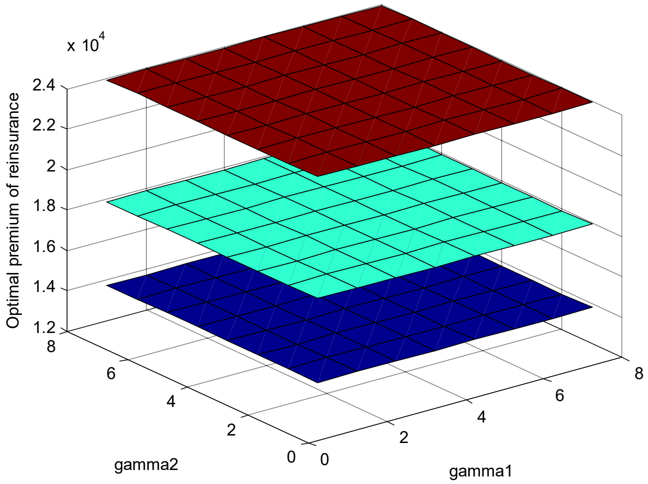
Note(s): Objective Function II
Source(s): Authors' calculation

Figure 6.
The patterns of change of optimal premium when $\mu_1 = \mu(1 \pm 10\%)$



Note(s): Objective Function II
Source(s): Authors' calculation

Figure 7.
The patterns of change of optimal retention when $\sigma_1 = \sigma(1 \pm 10\%)$



Note(s): Objective Function II
Source(s): Authors' calculation

Figure 8. The patterns of change of optimal premium when $\sigma_1 = \sigma(1 \pm 10\%)$

$$Y_1 = \begin{cases} P & \text{when } 0 \leq X \leq M, \\ P - X + M & \text{when } M \leq X \leq M_1, \\ P - X + M_1 & \text{otherwise.} \end{cases} \tag{16}$$

The expected utility of the reinsurance company is

$$E(U_1(Y_1)) = U_1(P)f(x)dx + \int_M^{M_1} U_1((P - x + M))f(x)dx$$
$$+ \int_{M_1}^{+\infty} U_1((P - x + M_1))f(x)dx. \tag{17}$$

For the ceding company, its benefit is

$$Y_2 = \begin{cases} E(X) - X - P \text{ when } 0 \leq X \leq M, \\ E(X) - P - M \text{ otherwise.} \end{cases} \tag{18}$$

where $E(X) - P - M = 0$ based on the equivalent principle.

The expected utility of the ceding company is

$$E(U_2(Y_2)) = \int_0^M U_2(E(X) - x - P)f(x)dx + \int_M^{+\infty} U_2(E(X) - P - M)f(x)dx \tag{19}$$

where $E(X)$ is the premium income of the ceding company based on the equivalence principle.

We establish the objective function to maximize the social utility or sum of expected utilities of the reinsurance and ceding companies as follows:

Maximizing the sum of the benefits of the insurer and reinsurer $\mu = 9.294, \sigma = 1.627, \gamma_2 = 2$				
γ_1	2	4	6	8
P^*	16,600	23,200	26,800	29,100
M^*	24,200	17,600	14,000	11,700
$E^*(Y)$	-0.5375×10^{10}	-0.2706×10^{10}	-0.1810×10^{10}	-0.1360×10^{10}
P	10,000	10,000	10,000	10,000
M^*	30,846	30,846	30,846	30,846
$E^*(Y)$	-0.5389×10^{10}	-0.2735×10^{10}	-0.1823×10^{10}	-0.1367×10^{10}
Maximizing the sum of the benefits of the insurer and reinsurer $\mu = 9.294, \sigma = 1.627, \gamma_2 = 4$				
γ_1	2	4	6	8
P^*	10,700	16,600	20,500	23,200
M^*	30,100	24,200	20,300	17,600
$E^*(Y)$	-0.5339×10^{10}	-0.2688×10^{10}	-0.1799×10^{10}	-0.1353×10^{10}
P	10,000	10,000	10,000	10,000
M^*	30,846	30,846	30,846	30,846
$E^*(Y)$	-0.5339×10^{10}	-0.2694×10^{10}	-0.1813×10^{10}	-0.1367×10^{10}
Maximizing the product of the benefits of the insurer and reinsurer $\mu = 9.294, \sigma = 1.627, \gamma_2 = 2$				
γ_1	2	4	6	8
P^*	22,900	22,900	22,900	22,900
M^*	17,946	17,946	17,946	17,946
$E^*(Y)$	3.8017×10^{15}	1.9004×10^{15}	1.2667×10^{15}	0.9498×10^{15}
P	10,000	10,000	10,000	10,000
M^*	30,846	30,846	30,846	30,846
$E^*(Y)$	2.7892×10^{15}	1.3941×10^{15}	0.9290×10^{15}	0.6965×10^{15}
Maximizing the product of the benefits of the insurer and reinsurer $\mu = 9.294, \sigma = 1.627, \gamma_2 = 4$				
γ_1	2	4	6	8
P^*	22,900	22,900	22,900	22,900
M^*	17,946	17,946	17,946	17,946
$E^*(Y)$	1.9004×10^{15}	0.95×10^{15}	0.6332×10^{15}	0.4748×10^{15}
P	10,000	10,000	10,000	10,000
M^*	30,846	30,846	30,846	30,846
$E^*(Y)$	1.3944×10^{15}	0.6969×10^{15}	0.4644×10^{15}	0.3482×10^{15}

Source(s): Authors' calculation

Table 1. Optimal solutions with two different situations

$$
\begin{aligned}
&\operatorname{Max} E(U(Y)) = E(U_1(Y_1)) + E(U_2(Y_2)) \\
&\int_0^M U_1(P)f(x)dx + \int_M^{M_1} U_1(P - x + M)f(x)dx + \int_{M_1}^{+\infty} U_1(P - x + M_1)f(x)dx \\
&+ \int_0^M U_2(E(X) - x - P)f(x)dx + \int_M^{+\infty} U_2(E(X) - P - M)f(x)dx
\end{aligned} \tag{20}
$$

The first order condition with respect to M_1 is:

$$\frac{\partial E(U(Y))}{\partial M_1} = \left(M - M_1 + \frac{P(M - M_1)}{\gamma_1} - \frac{(M - M_1)^2}{2\gamma_1} \right) f(M_1)$$
$$- \int_{M_1}^{+\infty} \frac{P - x + M_1}{\gamma_1} f(x)dx = 0 \tag{21}$$

If $f(M_1)$ is the density function of lognormal distribution, equation (21) holds when $M_1 \to +\infty$. Therefore, the optimal strategy of reinsurance is not to set an upper limit of reinsurance; this case reduces to the case of objective function 1.

5. Conclusions

In the last two decades, reinsurance companies have played a significant role in hedging mega-catastrophic losses. For example, reinsurance companies (and special loss sharing arrangements) paid as much as two-thirds of the insured losses for the September 11, 2001 tragedy. Furthermore, large catastrophic events have increased the role of governments and regulators as reinsurers of last resort.

This study establishes two optimization models of excess-of-loss reinsurance contracts against catastrophic losses to determine optimal premiums and retention. One model considers the sum of the expected utilities of a ceding company and reinsurance company's expected utility; another considers their products. Using an example, we find the optimal solutions of the premium and retention of excess loss reinsurance. Finally, we conduct a sensitivity analysis. The results show that increasing the means and volatilities of claim loss will increase the optimal retention and premium. For objective function I, increasing the coefficients of risk aversion of γ_1 or reducing the coefficients of risk aversion of γ_2 will reduce the optimal retention but increase the optimal premium, and vice versa. However, for objective function II, the change of the coefficient of risk aversion has no effect on optimal solutions.

While our research makes what we believe to be a valuable contribution to the field, there exists significant potential for future research in this area. For example, we consider a ceding company and reinsurer, but of course both companies operate in a market where the ceding company offers the original insurance product. An expanded model could consider the welfare of the customers of the ceding insurance company, in addition to the insurer/reinsurer pair. Furthermore, the previous three decades have seen substantial growth of insurance derivatives replacing traditional reinsurance, for example, catastrophe bonds, sidecars or exchange-traded option spreads. An interesting and very natural question is whether the insurance derivatives that replace reinsurance can benefit from designs based on the models proposed in our study, or models built based on it. However, we must admit that models including interactions of more than two entities, as well as large numbers of market participants, become challenging and complex. We hope that such research can be developed in the future.

Specifically, we hope that our work can serve as an inspiration for the following:

(1) The regulatory supervision of reinsurance. Individual firms pursue their own objectives, especially the profit objective and can scarcely be expected to optimize overall social welfare, but regulators can respond through the process of financial supervision.

(2) The systematic or ad hoc reinsurance activities of governments. Examples of such activities include financial bailouts or the restructuring of financial institutions. Those activities are often conducted out of political necessity, or for other policy reasons, but could, and in our view should, benefit from consideration of the welfare of all stakeholders in the process.

Note

1. http://finance.sina.com.cn/money/insurance/bxdt/2017-03-21/doc-ifycnpvh5134470.shtml

References

Bai, L., Guo, J. and Zhang, H. (2010), "Optimal excess-of-loss reinsurance and dividend payments with transaction costs and taxes", *Quantitative Finance*, Vol. 10, pp. 1163-1172.

Baton, B. and Lemaire, J. (1981), "The bargaining set of a reinsurance market", *Astin Bulletin*, Vol. 12, pp. 101-114.

Cao, Y. and Xu, J. (2010), "Proportional and excess-of-loss reinsurance under investment gain", *Applied Mathematics and Computation*, Vol. 217, pp. 2546-2550.

Centeno, M. (2005), "Dependent risks and excess of loss reinsurance", *Insurance: Mathematics and Economics*, Vol. 37, pp. 229-238.

Chen, Y. (2021), "Optimal reinsurance from the viewpoints of both an insurer and a reinsurer under the CVaR risk measure and Vajda condition", *ASTIN Bulletin*, Vol. 51 No. 2, pp. 631-659, doi: 10.1017/asb.2021.9.

Cummins, J. (2007), "Reinsurance for natural and man-made catastrophes in the United States: current state of the market and regulatory reforms", *Risk Management and Insurance*, Vol. 10, pp. 179-220.

Dasgupta, S. and Nanda, V. (1993), "Bargaining and brinkmanship: capital structure choice by regulated firms", *International Journal of Industrial Organization*, Vol. 11 No. 4, pp. 475-497, doi:10.1016/0167-7187(93)90021-4.

Froot, K. and O'Connell, P. (2008), "On the pricing of intermediated risks: theory and application to catastrophe reinsurance", *Journal of Banking and Finance*, Vol. 32, pp. 69-85.

Gajek, L. and Zagrodny, D. (2004), "Reinsurance arrangements maximizing insurer's survival probability", *Journal of Risk and Insurance*, Vol. 71, pp. 421-435.

Gong, Q., Tang, Z. and Xu, B. (2021), "Trading behaviors on knowledge of price discovery in futures markets", *Journal of Innovation & Knowledge*, Vol. 6 No. 3, pp. 191-195.

Huang, R. and Tzeng, L. (2007), "Optimal tax deduction for net losses under private insurance with upper limit", *Journal of Risk and Insurance*, Vol. 74, pp. 883-893.

Kremer, E. (2002), "Limit-determination for the excess-of-loss treaty in case of simple retrocession", *Blätter DGVFM*, Vol. 21, pp. 813-818.

Li, D., Rong, X. and Zhao, H. (2014), "Optimal reinsurance-investment problem for maximizing the product of the insurer's and reinsurer's utilities under a CFV model", *Journal of Computational and Applied Mathematics*, Vol. 255, pp. 671-683.

Li, D., Rong, X. and Zhao, H. (2017), "Equilibrium excess-of-loss reinsurance–investment strategy for a mean–variance insurer under stochastic volatility model", *Communication in Statistics: Theory and Methods*, Vol. 46, pp. 9459-9475.

Liang, Z. and Bayraktar, E. (2014), "Optimal reinsurance and investment with unobservable claim size and intensity", *Insurance: Mathematics and Economics*, Vol. 55, pp. 156-166.

Liu, R. and Zhang, H. (2007), "Introduction to the development of investment-linked products of non-life insurance in China", *Shanghai Insurance* (In Chinese), No. 41, pp. 35-38.

Mao, H. and Wen, Z. (2018), "Optimization of price, default ratio and capital under regulatory criterion of maximizing social benefit", *Asia Pacific Journal of Risk and Insurance*, Vol. 12, pp. 1-15.

Mao, H. and Ostaszewski, K. (2007), "Application of game theory to pricing participating deferred annuity", *Journal of Insurance Issues*, Vol. 30, pp. 102-122.

Mao, H., Carson, J., Ostaszewski, K. and Wang, Y. (2016), "Determination of optimal cap on claim settlements based on the criterion of social benefit maximization", *Journal of Insurance Regulation*, Vol. 35 No. 3, pp. 1-23.

Marinakis, Y.D. and White, R. (2022), "Hyperinflation potential in commodity-currency trading systems: implications for sustainable development", *Sustainable Technology and Entrepreneurship*, Vol. 1 No. 1, 100003.

Moreno, I., Parrado-Martínez, P. and Trujillo-Ponce, A. (2022), "Using the Z-score to analyze the financial soundness of insurance firms", *European Journal of Management and Business Economics*, Vol. 31 No. 1, pp. 22-39, doi: 10.1108/EJMBE-09-2020-0261.

Syuhada, K., Hakim, A. and Sari, S. (2021), "The combined stop-loss and quota-share reinsurance: conditional tail expectation-based optimization from the joint perspective of insurer and reinsurer", *Risks*, Vol. 9, p. 125, doi: 10.3390/risks9070125.

Thomson, W. (1981), "Nash's bargaining solution and utilitarian choice rules", *Econometrica*, Vol. 49, pp. 535-538.

Ya, H., Yao, O. and Tang, L. (2018), "Robust optimal investment and reinsurance problem for the product of the insurer's and the reinsurer's utilities", *Journal of Computational and Applied Mathematics*, Vol. 344, pp. 532-552.

Zanjani, G. (2010), "An economic approach to capital allocation", *Journal of Risk and Insurance*, Vol. 77, pp. 1-27.

Zhao, H., Rong, X. and Zhao, Y. (2013), "Optimal excess-of-loss reinsurance and investment problem for an insurer with jump-diffusion risk process under the Heston model", *Insurance: Mathematics and Economics*, Vol. 53, pp. 504-514.

Zhao, H., Weng, C., Shen, Y. and Zeng, Y. (2017), "Time-consistent investment-reinsurance strategies towards joint interests of the insurer and the reinsurance under CEV models", *Science China Mathematics*, Vol. 60, pp. 317-344.

(The Appendix follows overleaf)

Appendix

This appendix presents certain proofs and examples. We begin by presenting a proof of equation (5):

$$
\begin{aligned}
E(U(Y)) &= E(U_1(Y_1)) + E(U_2(Y_2)) \\
&= \int_0^M \left(-\frac{P^2}{2\gamma_1} + e^{\mu+\frac{\sigma^2}{2}} - x - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - x - P\right)^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_M^{+\infty} \left(e^{\mu+\frac{\sigma^2}{2}} - x - \frac{(P - x + M)^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&= \int_0^{+\infty} \left(e^{\mu+\frac{\sigma^2}{2}} - x - \frac{P^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_0^M \left(\frac{x\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} - \frac{x^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_M^{+\infty} \left(\frac{(x-M)P}{\gamma_1} + \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)M}{\gamma_2} - \frac{(x-M)^2}{2\gamma_1} - \frac{M^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx
\end{aligned} \tag{A1}
$$

Let $y = \frac{\ln x - \mu}{\sigma}$, then $dy = \frac{dx}{x\sigma}$. The left-hand side of Equation (A1) can be written as:

$$
\begin{aligned}
E(U(Y)) &= \int_0^{+\infty} \left(e^{\mu+\frac{\sigma^2}{2}} - e^{y\sigma+\mu} - \frac{P^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)^2}{2\gamma_2} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\
&+ \int_{-\infty}^{\frac{\ln M-\mu}{\sigma}} \left(\frac{e^{y\sigma+\mu}\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} - \frac{e^{2(y\sigma+\mu)}}{2\gamma_2} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\
&+ \int_{\frac{\ln M-\mu}{\sigma}}^{+\infty} \left(\frac{(e^{y\sigma+\mu} - M)P}{\gamma_1} + \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)M}{\gamma_2} - \frac{(e^{y\sigma+\mu} - M)^2}{2\gamma_1} - \frac{M^2}{2\gamma_2} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy
\end{aligned} \tag{A2}
$$

Example: Let $\mu = 9.294, \sigma = 1.627, \gamma_1 = \gamma_2 = 2$. Then, optimal solutions are

$$P* = 16600, M* = 24200, E*(Y) = -0.5375 \cdot 10^{10}.$$

Here, we present a proof of equation (6):

$$
= \int_0^M \begin{pmatrix} P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P}{2\gamma_2} - \frac{P}{2\gamma_1} + \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{4\gamma_1\gamma_2}\right) \\ P\left(1 + \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} + \frac{P}{2\gamma_1} + \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{2\gamma_1\gamma_2}\right)x \\ +P\left(\frac{1}{2\gamma_2} - \frac{P}{4\gamma_1\gamma_2}\right)x^2 \end{pmatrix} \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x}dx
$$

$$
+ \int_M^{+\infty} \begin{pmatrix} (P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_2} - \frac{P+M}{2\gamma_1} + \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{4\gamma_1\gamma_2}\right) \\ -\left(e^{\mu+\frac{\sigma^2}{2}} - P - M - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_2} - \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{\gamma_1} + \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_1\gamma_2}\right)x \\ -\left(\frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{4\gamma_1\gamma_2}\right)x^2 \end{pmatrix} \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x}dx \tag{A3}
$$

Let $y = \frac{\ln x - \mu}{\sigma}$, then $dy = \frac{dx}{x\sigma}$.

The left-hand side of Equation (A3) can be written as:

$$
E(U_1(Y_1)U_2(Y_2)) =
$$

$$
= \int_{-\infty}^{\frac{\ln M-\mu}{\sigma}} \begin{pmatrix} P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P}{2\gamma_2} - \frac{P}{2\gamma_1} + \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{4\gamma_1\gamma_2}\right) \\ P\left(1 + \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} - \frac{P}{\gamma_1} + \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{2\gamma_1\gamma_2}\right)e^{y\sigma+\mu} \\ +P\left(\frac{1}{2\gamma_2} - \frac{P}{4\gamma_1\gamma_2}\right)e^{2(y\sigma+\mu)} \end{pmatrix} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy +
$$

$$
+ \int_{\frac{\ln M-\mu}{\sigma}}^{+\infty} \left(\begin{pmatrix} (P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_2} - \frac{P+M}{2\gamma_1} + \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{4\gamma_1\gamma_2}\right) \\ -\left(e^{\mu+\frac{\sigma^2}{2}} - P - M - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_2} - \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{\gamma_1} + \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_1\gamma_2}\right)e^{y\sigma+\mu} \\ -\left(\frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{4\gamma_1\gamma_2}\right)e^{2(y\sigma+\mu)} \end{pmatrix}\right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \tag{A4}
$$

$$
= A\Phi\left(\frac{\ln M - \mu}{\sigma}\right) + Be^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right) + Ce^{2(\mu+\sigma^2)}\Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right)
$$

$$
+D\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma}\right)\right) - Ee^{\mu+\frac{\sigma^2}{2}}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)\right) - Ge^{2(\mu+\sigma^2)}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right)\right),
$$

where

$$
A = P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P}{2\gamma_2} - \frac{P}{2\gamma_1} + \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{4\gamma_1\gamma_2}\right),
$$

$$B = P\left(-1 + \frac{e^{\mu+\frac{\sigma^2}{2}} - P}{\gamma_2} + \frac{P}{2\gamma_1} - \frac{P\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{2\gamma_1\gamma_2}\right),$$

$$C = -P\left(\frac{1}{2\gamma_2} - \frac{P}{4\gamma_1\gamma_2}\right),$$

$$D = (P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)\left(1 - \frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_2} - \frac{P+M}{2\gamma_1} + \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{4\gamma_1\gamma_2}\right),$$

$$E = e^{\mu+\frac{\sigma^2}{2}} - P - M - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_2} - $$

$$-\frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)}{2\gamma_1} + \frac{(P+M)\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_1\gamma_2},$$

and

$$G = \frac{e^{\mu+\frac{\sigma^2}{2}} - P - M}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{4\gamma_1\gamma_2}. \tag{A5}$$

Example: Let $\mu = 9.294, \sigma = 1.627, \gamma_1 = \gamma_2 = 2$. Then, optimal solutions are

$$P* = 22900, M* = 17946, E*(Y) = 3.8017 \times 10^{15}$$

Here, we present the proof of equation (20):

$$\begin{aligned} E(U(Y)) &= E(U_1(Y_1)) + E(U_2(Y_2)) \\ &= \int_0^M \left(-\frac{P^2}{2\gamma_1} + e^{\mu+\frac{\sigma^2}{2}} - x - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - x - P\right)^2}{2\gamma_2}\right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\ &+ \int_M^{M_1} \left(-x + M - \frac{(P - x - M)^2}{2\gamma_1}\right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\ &+ \int_{M_1}^{+\infty} \left(-x + M_1 - \frac{(P - x + M_1)^2}{2\gamma_1}\right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\ &+ \int_M^{+\infty} \left(e^{\mu+\frac{\sigma^2}{2}} - M - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P - M\right)^2}{2\gamma_2}\right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \end{aligned} \tag{A6}$$

$$
\begin{aligned}
&= \int_0^{+\infty} \left(e^{\mu+\frac{\sigma^2}{2}} - x - \frac{P^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_0^{M} \left(\frac{x\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} - \frac{x^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_M^{M_1} \left(M + \frac{P(x-M)}{\gamma_1} - \frac{M^2}{2\gamma_1} + \frac{xM}{\gamma_1} - \frac{x^2}{2\gamma_1} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_{M_1}^{+\infty} \left(M_1 + \frac{P(x-M_1)}{\gamma_1} - \frac{M_1^2}{2\gamma_1} + \frac{xM_1}{\gamma_1} - \frac{x^2}{2\gamma_1} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx \\
&+ \int_M^{+\infty} \left(-M + \frac{M\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} + \frac{M^2}{2\gamma_2} \right) \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} dx
\end{aligned}
$$

Let $y = \frac{\ln x - \mu}{\sigma}$, then $dy = \frac{dx}{x\sigma}$. The left-hand side of Equation (A1) can be written as:

$$
\begin{aligned}
E(U(Y)) &== \int_0^{+\infty} \left(e^{\mu+\frac{\sigma^2}{2}} - e^{y\sigma+\mu} - \frac{P^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)^2}{2\gamma_2} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dx \\
&+ \int_{-\infty}^{\frac{\ln M-\mu}{\sigma}} \left(\frac{e^{y\sigma+\mu}\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} - \frac{e^{2(y\sigma+\mu)}}{2\gamma_2} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dx \\
&+ \int_{\frac{\ln M-\mu}{\sigma}}^{\frac{\ln M_1-\mu}{\sigma}} \left(M + \frac{P(e^{y\sigma+\mu} - M)}{\gamma_1} - \frac{M^2}{2\gamma_1} + \frac{e^{y\sigma+\mu}M}{\gamma_1} - \frac{e^{2(y\sigma+\mu)}}{2\gamma_1} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dx \\
&+ \int_{\frac{\ln M_1-\mu}{\sigma}}^{+\infty} \left(M_1 + \frac{P(e^{y\sigma+\mu} - M_1)}{\gamma_1} - \frac{M_1^2}{2\gamma_1} + \frac{e^{y\sigma+\mu}M_1}{\gamma_1} - \frac{e^{2(y\sigma+\mu)}}{2\gamma_1} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dx \\
&+ \int_{\frac{\ln M-\mu}{\sigma}}^{+\infty} \left(-M + \frac{M\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)}{\gamma_2} - \frac{M^2}{2\gamma_2} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{P^2}{2\gamma_1} - \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)^2}{2\gamma_2} \\
&+ \frac{\left(e^{\mu+\frac{\sigma^2}{2}} - P\right)e^{\mu+\frac{1}{2}\sigma^2}}{\gamma_2}\Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right) - \frac{e^{2\left(\mu+\sigma^2\right)}}{2\gamma_2}\Phi\left(\frac{\ln M - \mu}{\sigma} - 2\sigma\right) \\
&+ M\left(1 - \frac{P}{\gamma_1} - \frac{M}{2\gamma_1}\right)\left(\Phi\left(\frac{\ln M_1 - \mu}{\sigma}\right) - \Phi\left(\frac{\ln M - \mu}{\sigma}\right)\right) \\
&+ \frac{Me^{\mu+\frac{1}{2}\sigma^2}}{\gamma_1}\left(\Phi\left(\frac{\ln M_1 - \mu}{\sigma} - \sigma\right) - \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)\right) \\
&- \frac{Pe^{\mu+\frac{1}{2}\sigma^2}\Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)}{\gamma_1} \\
&- \frac{e^{2\left(\mu+\sigma^2\right)}}{2\gamma_1}\left(1 - \Phi\left(\frac{\ln M_1 - \mu}{\sigma} - 2\sigma\right)\right) + \frac{M_1 e^{\mu+\frac{1}{2}\sigma^2}}{\gamma_1}\left(1 - \Phi\left(\frac{\ln M_1 - \mu}{\sigma} - \sigma\right)\right) \\
&+ \frac{Pe^{\mu+\frac{1}{2}\sigma^2}}{\gamma_1} + M_1\left(1 - \frac{P}{\gamma_1} - \frac{M_1}{2\gamma_1}\right)\left(1 - \Phi\left(\frac{\ln M_1 - \mu}{\sigma}\right)\right) \\
&+ \frac{Me^{\mu+\frac{1}{2}\sigma^2}}{\gamma_2}\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma} - \sigma\right)\right) - M\left(1 + \frac{P}{\gamma_2} + \frac{M}{2\gamma_2}\right)\left(1 - \Phi\left(\frac{\ln M - \mu}{\sigma}\right)\right)
\end{aligned} \quad . \quad \text{(A7)}
$$

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