Equilibrium policy portfolios when some investors are restricted from holding certain assets

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Abstract

Purpose – The authors analyze the equilibrium effects of non-tradable assets on optimal policy portfolios. They study how the existence of non-tradable assets impacts optimal asset allocation decisions of investors who own such assets and of investors who do not have access to non-tradable assets.

Design/methodology/approach – In this theoretical analysis, the authors analyze a model with tradable and non-tradable asset classes whose cash flows are jointly normally distributed. There are two types of investors, with and without access to non-tradable assets. All investors have constant absolute risk aversion preferences. The authors derive closed form solutions for optimal investor demand and equilibrium asset prices. They calibrated the model using *US data* for listed equity, bonds and private equity. Further, the authors illustrate the sensitivities of quantities and prices with respect to the main parameters.

Findings – The study finds that the existence of non-tradable assets has a large impact on optimal asset allocation. Investors with (without) access to non-tradable assets tilt their portfolios of tradable assets away from (toward) assets to which non-tradable assets exhibit positive betas.

Practical implications – The model provides important insights not only for investors holding non-tradable assets such as private equity but also for investors who do not have access to non-tradable assets. Investors who ignore the effect of non-tradable assets when reverse-engineering risk premia from asset covariances and market capitalizations might severely underestimate the equity risk premium.

Originality/value – The authors provide the first comprehensive analysis of the equilibrium effects of non-tradability of some assets on optimal policy portfolios. Thus, this paper goes beyond analyzing the effects of market imperfections on individual portfolio choices.

Keywords Equilibrium policy portfolios, Asset allocation, Non-tradable assets

Paper type Research paper

1. Introduction

Under its very restrictive assumptions, the Capital Asset Pricing Model (CAPM) prescribes all investors to hold the market portfolio. We study the equilibrium effects on the optimal policy

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portfolios when the assumption of perfect tradability of all assets is relaxed. This is important, because even one ignores human capital, there are large parts of the asset universe that are difficult to trade, such as non-listed equity. Due to hedging motives, investors endowed with non-tradable assets will tilt their portfolios away from positively correlated tradable assets. Purely financial investors without access to non-tradable assets, conversely, will overweight these positively correlated assets. In our baseline calibration, the financial investors hold about 66% of their policy portfolio in listed equity while it makes up only 51% of the tradable market portfolio.

According to the CAPM, the policy portfolio should be the value-weighted market portfolio of all assets. However, the practical implementation of such a policy portfolio is not straightforward. First, it is non-trivial to determine the market value weights of tradable assets. Second, even if one ignores human capital as a component of the market portfolio, there are large parts of the asset universe, which are difficult to trade, such as non-listed equity. While some investors are endowed with such illiquid assets, or are able to pay the (fixed) costs associated with accessing them, other investors are effectively precluded from holding such assets [1]. Figure 1, updated from Cejnek *et al.* (2014) illustrates how the policy portfolio of the Yale endowment has shifted from domestic (i.e. US) equity to asset classes that are generally considered non-tradable or difficult to trade for a large segment of investors.

While there exist numerous models about how to account for capital market imperfections in *individual* portfolio choices, ours is – to the best of our knowledge – the first to consider the *equilibrium* effects of non-tradability on optimal policy portfolios.

For simplicity we consider two types of investors: endowed investors, E, who own shares of illiquid assets and financial investors, F, who do not. We find that the existence of non-tradable assets implies that neither investors F nor investors E hold the market portfolio of tradable assets. The investors, F (E), tilt their portfolios toward (away from) assets to which non-tradable assets exhibit positive betas.

The equilibrium effects depend on risk aversion, asset covariances, relative asset class sizes and the relative risk-bearing capacities of the two investor types. In the limit, as F becomes risk neutral, E investors only hold a pure regression hedge portfolio of tradable risky assets. Making the non-tradable assets tradable does not affect the prices of the assets that were initially tradable but raises the valuation of the initially non-tradable assets.

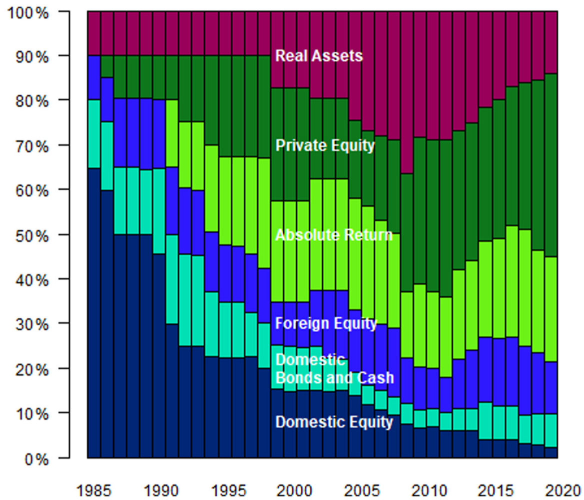


Figure 1.
Policy portfolio of the Yale endowment

The increase is largest if the tradable and non-tradable assets are uncorrelated.We use data on the market capitalization and covariances of returns for bonds, stocks and private equity to obtain a baseline calibration of our model. Using our parameter estimates, investors F will substantially tilt their portfolio holdings toward tradable equity in equilibrium. Due to the magnitude of the positive correlation between tradable equity and private equity, investors F may even hold more than 100% of the aggregate supply of public equity, while investors E hold short positions for hedging purposes.

Given the challenges in obtaining parameter estimates for non-tradable assets and the relative sizes of asset classes, we perform sensitivity analyses. The effects are more pronounced as the non-tradable assets make up a larger part of the investment universe, and their correlations with tradable assets are stronger. In order to facilitate the use of our model to gain insights for asset allocation decisions, we provide an implementation in a freely accessible app at https://assetallocation.shinyapps.io/policyportfolios/. The app allows users to change the parameters and obtain the resulting policy portfolios, with the equilibrium asset allocation weights for both types of investors and the market-wide expected returns.

The rest of the paper is structured as follows. Section 2 reviews the related literature. Section 3 introduces the basic model that is calibrated in the following section 4. The main results and its sensitivities are discussed in sections 5 and 6, respectively. Section 7 concludes.

2. Literature

The literature on optimal asset allocation and investment decisions has long recognized that investors should not restrict themselves in investing into tradable securities like stocks and bonds. Instead, they should take into account their so-called background risk in solving their portfolio choice problem. Types of background risk that have been analyzed are labor income (e.g. Merton, 1971; Duffie *et al.*, 1997; Eiling, 2013), liabilities (Sharpe and Tint, 1990), real estate (Brueckner, 1997), business income (Heaton and Lucas, 2000a, b) and donation flows of university endowments (Cejnek *et al.*, 2022).

Among the background risks, the returns from private, non-tradable businesses represent major economic force. Moskowitz and Vissing-Jørgensen (2002) argue that the sector is at least as large as the traded equity one. Asker *et al.* (2015) report that private firms accounted for 69% of private-sector employment and 49% of aggregate pre-tax profits in 2010. Only 0.06% of all US firms were listed. Even among the firms with more than 500 employees, 86% were private. Heaton and Lucas (2000b) find that background risk from entrepreneurial proprietary income is an important driver of household portfolio choices, in that the share of stock holdings is negatively related to level and variability of proprietary income growth.

It is well recognized that the investment decisions of the non-tradable equity holders must have an equilibrium impact on other asset classes. Moskowitz and Vissing-Jørgensen (2002) call their inability to identify higher returns a “private equity premium puzzle”. Kartashova (2014) resolves the puzzle by updating and extending the analysis using data from 1989 to 2010. Using data from the Survey of Consumer Finances (SCF) and other sources, she finds that, on average, private business holdings significantly have outperformed public equity. Heaton and Lucas (2000a) find that adding an aggregate proprietary income factor to a conditional CAPM specification improves the overall model performance. But, to the best of our knowledge, our paper is the first that explicitly models and estimates the equilibrium effects of non-tradable assets on portfolio holdings and returns.

A related strand of literature analyzes restrictions on tradability and portfolio choice. Examples of broadly related issues analyzed include asset illiquidity (Ang *et al.*, 2014), environmental preferences (Heinkel *et al.*, 2001) and shareholder activism (Admati *et al.*, 1994). In a recent paper, Buss *et al.* (2018) show that making previously restricted assets accessible to inexperienced investors can actually increase the volatilities and risk premiums of these assets. This result obtains when inexperienced investors learn about the dynamics of the new assets only from their own experience and not from the market in aggregate.

Finally, our work is related to papers describing and explaining stylized facts on public and private equity markets. Section 4 explains in detail how we calibrate our model, but the main inputs relate to the relative sizes and return patterns of the public versus the private equity market. For traded equity, Mehra and Prescott (1985) raise the equity risk premium puzzle that average returns on equity are too high relative to reasonable levels of investors' risk aversion. Dimson *et al.* (2002) describe long time series of historical returns of major markets, while providing evidence on the size (i.e. market capitalizations) of various asset classes. Doeswijk *et al.* (2014) and Gadzinski *et al.* (2018) provide proxies for the global multi-asset market portfolio.

By definition, we cannot observe market prices of non-tradable asset classes. Most papers in this field rely on various government statistics and surveys, such as the Flow of Funds Accounts (FFA), SCF, the Internal Revenue Service (IRS) tax statistics or the census [2]. Especially the SCF data seems well suited to gauge the size of the overall US private equity market. But since it is conducted only every three years, it does not provide good data on returns. Some articles have used data on private equity funds, that make up a growing part of the non-listed equity sector for return data. Driessen *et al.* (2012) developed a panel method to impute the quarterly private equity returns from the cash flows of a sample of 958 private equity funds. They found that buyout funds have a beta of 1.3 relative to the S&P 500 index [3].

3. The model

We analyze a model with both tradable and non-tradable asset classes, whose cash flows are jointly normally distributed, and two types of investors with constant absolute risk aversion preferences. The setup is similar to the models analyzed in Heinkel *et al.* (2001) and Admati *et al.* (1994). We provide a Table of notations in the Appendix (Table A1).

3.1 Investment opportunities set

There are n risky assets. A subset n_T of these assets is tradable and the complementary set, n_N, is non-tradable. While there are no restrictions on the number of assets, it is intuitive to interpret each asset in the model as an entire asset class, consisting of sufficiently homogeneous securities. Without loss of generality, we normalize the supply of each asset to one, so that the supply vector for all assets is the vector of ones with length n. Finally, there is a riskless security which is in perfectly elastic supply and whose return is normalized to one.

At the end of the period, asset classes pay jointly normally distributed cash flows $CF \sim \mathcal{N}(\mu, \Sigma)$ to their holders, with expected payoffs μ and a positive definite covariance matrix Σ. We can split the vector of expected payoffs $\mu = \begin{pmatrix} \mu_T \\ \mu_N \end{pmatrix}$ into its components of tradable and non-tradable securities' expected cash flows. Similarly, the covariance matrix of cash flows, Σ, has the structure

$$\Sigma = \begin{pmatrix} \Sigma_{TT} & \Sigma_{TN} \\ \Sigma'_{TN} & \Sigma_{NN} \end{pmatrix},$$

where Σ_{TT} is the covariance matrix of tradable assets, Σ_{NN} is the covariance matrix of non-tradable assets and Σ_{TN} is the matrix of covariances between the tradable and non-tradable assets.

3.2 Investors and preferences

Investors differ with respect to their initial endowment of non-tradable assets. We assume that there are two investor types, $K \in \{E, F\}$, which we will refer to as *endowed* and *financial* investors, respectively. Investors of type E are endowed with non-tradable assets; yet they cannot dispose of or raise their holdings. They may own tradable assets in addition. In contrast, type F investors are restricted to holdings of zero in the non-tradable assets. These investors resemble financial investors who trade in liquid asset classes only. Note that the non-tradable assets in the economy have to be held entirely by investors E, while the proportions of tradable assets held by E and F investors will be determined in equilibrium. One interpretation of our assumptions is that assets N are more opaque and that investors face different information costs for accessing these assets. In this case there may exist an interior equilibrium where assets N are only held by a subset of investors with low information costs. The results generated by such a model would be qualitatively identical to the ones derived below. The initial endowments ω^E and ω^F are given by

$$\omega^E = \begin{pmatrix} \omega_T^E \\ \omega_N^E \end{pmatrix} = \begin{pmatrix} \omega_T^E \\ e_N \end{pmatrix} \text{ and } \omega^F = \begin{pmatrix} \omega_T^F \\ \mathbf{0} \end{pmatrix},$$

where e_N is a n_N-dimensional vector of ones and $\mathbf{0}$ a vector of zeros and the superscripts E and F denote the endowments of investors E and F, respectively. All investors maximize expected utility of wealth resulting from their portfolio payoffs and exhibit constant absolute risk aversion. Aggregate risk tolerance τ in the market is given by the sum of the aggregate risk tolerances of endowed investors, τ^E, and financial investors, τ^F. The aggregate trading behavior of each group of investors is modeled as that of a representative, price-taking investor with a risk tolerance of τ^E and τ^F, respectively. Under these assumptions, each representative investor of type $K \in \{E, F\}$ optimizes her utility by maximizing

$$U^K = x'^K \mu - \frac{x'^K \Sigma x^K}{2\tau^K} - \left(x^K - \omega^K\right)' p, \tag{1}$$

where x^K is the demand vector of the representative investor of type K and p the vector of asset prices. Note that for both types their demand x_N^K in the non-tradable asset is fixed with their initial endowment, so the demand vectors x^E and x^F enter Equation (1) in the form of

$$x^E = \begin{pmatrix} x_T^E \\ e_N \end{pmatrix} \text{ and } x^F = \begin{pmatrix} x_T^F \\ \mathbf{0} \end{pmatrix}.$$

3.3 Market equilibrium

Market equilibrium requires the first order condition to hold for both representative investors. As investors may freely set their demand for tradable assets only, E's and F's first order conditions are given by

$$\frac{\partial U^E}{\partial x_T^E} = \mu_T - \frac{1}{\tau^E}\left(\Sigma_{TT} x_T^E + \Sigma_{TN} e_N\right) - p_T = \mathbf{0} \tag{2}$$

$$\frac{\partial U^F}{\partial x_T^F} = \mu_T - \frac{1}{\tau^F}\Sigma_{TT} x_T^F - p_T = \mathbf{0} \tag{3}$$

Equations (2) and (3) differ with respect to one important component. Given that E holds non-tradable assets while F does not, covariances between tradable and non-tradable assets enter only the first order condition of investor E. Solving for x_T^E and x_T^F gives the aggregate demand functions

$$x_T^E = \Sigma_{TT}^{-1}\left(\tau^E(\mu_T - p_T) - \Sigma_{TN} e_N\right) \text{ and} \tag{4}$$

$$x_T^F = \Sigma_{TT}^{-1}\tau^F(\mu_T - p_T). \tag{5}$$

Imposing the market clearing condition $x_T^E + x_T^F = e_T$ allows to solve for the equilibrium price of tradable assets,

$$p_T^* = \mu_T - \frac{1}{\tau}(\Sigma_{TT}e_T + \Sigma_{TN}e_N). \tag{6}$$

Note from Equation (6) that the risk premium of a tradable asset is determined by the sum of its covariances with tradable and non-tradable assets, and scaled by total risk tolerance $\tau = \tau^E + \tau^F$ in the economy. Substituting for the equilibrium price in the demand function (Equation 4) and simplifying yields the equilibrium demand x_T^{E*} of investor E,

$$x_T^{E*} = \frac{\tau^E}{\tau}\left(e_T - \frac{\tau^F}{\tau^E}\mathbb{B}e_N\right), \tag{7}$$

where $\mathbb{B} = \Sigma_{TT}^{-1}\Sigma_{TN}$ is a matrix of betas. Each column of $\mathbb{B}$ represents the projection of a particular non-tradable asset's cash flow on the tradable assets' cash flows. Hence, positive betas of non-tradable assets toward a tradable asset i correspond to lower holdings of asset i by investor E. Similarly, equilibrium demand x_T^{F*} of investor F is given by

$$x_T^{F*} = \frac{\tau^F}{\tau}(e_T + \mathbb{B}e_N). \tag{8}$$

This specification of F's equilibrium holdings immediately allows the following comparison to the CAPM case of holding a fraction of the market portfolio.

Proposition 3.1. Unless $\mathbb{B}e_N$ is proportional to e_T, F does not hold the market portfolio of tradable assets; i.e. her equilibrium holdings are not proportional to e_T. The tradable component of investor E's portfolio is not proportional to the market portfolio of tradable assets e_T either.

To gain insight why F generally deviates from holding a proportion of the market portfolio of tradable assets although this investor is not exposed to any background risk from non-tradable assets that might require hedging, we have to consider the market equilibrium. The investor E takes into account her holdings of non-tradable assets when setting demand for tradable assets. This impacts equilibrium prices and therefore gets reflected in F's optimal portfolio. F tilts her portfolio toward (away from) the assets to which non-tradable assets exhibit positive (negative) betas. Thereby, F balances E's hedging needs. The magnitude of the adjustment relative to F's demand function in the absence of non-tradable assets is given by the term $\mathbb{B}e_N$. If non-tradable assets are uncorrelated with tradable assets, or the effects from the non-tradable assets offset each other such that $\mathbb{B}e_N = \mathbf{0}$, F will hold exactly the same portfolio as she would in the absence of non-tradable assets. For non-zero cases of proportional holdings ($\mathbf{0} \neq \mathbb{B}e_N \propto e_T$), intuitively F will hold a higher (lower) fraction of the market portfolio if there is an identical positive impact on each single tradable asset. This corresponds to observing the same positive (negative) row sum in all rows of $\mathbb{B}$. It is well established in the literature that an investor should take into account her shadow assets and liabilities when constructing an optimal portfolio. Our results emphasize that also investors who do not own shadow assets themselves have to consider non-tradable assets in their investment decision.

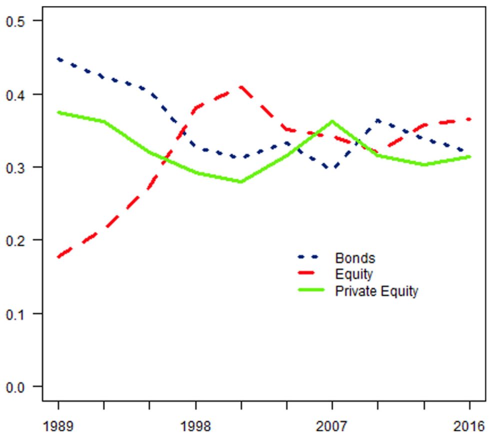
4. Calibration

For calibration, we turn to *US data*, and focus on traded equity, private equity and bonds [4]. We consider private equity to encompass all holdings of private businesses not traded on an organized marketplace. While private equity funds attract significant attention in both the academic literature and the popular press, they constitute only a subset of private business holdings. The raison-d-être of these funds is to give investors at least limited access to non-traded assets; yet given the special features of private equity funds that make it hard for investors to adjust holdings fast and cheaply, we still consider them as non-tradable assets in our setting. Further, private equity funds offer the unique possibility of obtaining price data instead of appraisal values for the estimation of non-traded assets' returns. As Kartashova (2014) notes, private equity funds represent the part of entrepreneurial equity that is closest to publicly traded equity. Hence, the resulting estimate for the public-private return correlation might in fact constitute an upper bound of the underlying sectors' correlation. Mainly driven by availability of reliable data for private equity, our main calibration is based on quarterly return estimates from the period from Q2/1996 to Q1/2015, while we use data from 2016 for the relative size of asset classes.

4.1 Relative size of the asset classes

We obtain the average holdings of US households in private businesses, public equity and aggregate financial assets for years from 1989 to 2016 from the triennial US SCF. The values we obtain for private and public equities closely match those from Kartashova (2014) for the time span where our data overlap. We calculate bond holdings as the difference between financial holdings and equity holdings. In our base case calibration, we use the most current values of 32.0% for bonds, 36.5% for listed equity and 31.5% for private equity. Figure 2 illustrates the almost equal importance of the three asset classes over the last two decades. Only at the beginning of the observation period the value of private businesses was about twice that of listed equity [5].

Alternative data sources give a wide range of the relative sizes of private and public equity. The main alternative to SCF data would be the use of US FFA data. Kartashova (2014) compares SCF and FFA data and finds a much lower size of the private share using the latter source. With respect to SCF, she documents that possible under- and overreporting of the value of private businesses by survey respondents does not seem to be an important issue in



Source(s): Survey of Consumer Finances,1989 to 2016

Figure 2. Relative size of asset classes

the long run. Anderson (2009) uses reported taxable earnings to derive an estimate for the size of the private business sector, which exceeds both the value of listed equity and the estimates implied by the SCF.

4.2 Returns

While SCF data allow us to estimate the relative size of each asset class we are interested in, given their publication every third year only, they do not allow for the calculation of returns at a frequency suitable to estimate covariances. We therefore resort to asset-class specific data sources to obtain returns at quarterly frequency from which we then estimate standard deviations and correlations.

4.2.1 Listed equity. We use the US Total Market Index from the Center for Research in Securiy Prices (CRSP) and download the value-weighted return including dividends.

4.2.2 Private equity. Estimating returns for non-traded assets is notoriously difficult due to the very fact that their market prices cannot be observed when assets are not traded on a market. However, there are several approaches to obtain estimates. We use data on buyout funds provided by Ang *et al.* (2018) for the period from Q2/1996 to Q1/2015. They estimate returns for various types of private equity funds from cash flows, thus avoiding the implausibly high autocorrelation from which most industry indices suffer. Buyout funds are by far the largest component of the market of private equity funds (see, for instance, McKinsey and Company, 2018). Consistent with other sources, buyout funds are found to deliver relatively high returns, yet they are also riskier in terms of standard deviation of returns and beta to listed equity. This is consistent with their high leverage. Axelson *et al.* (2013) provide evidence that private equity funds have about twice the leverage of a matched sample of private firms; a factor consistent with Stafford (2021), whose approach mimics private equity returns using investments in the listed equity market. The high leverage of companies in buyout funds is similar to the leverage of private businesses in general. Asker *et al.* (2015) compare a sample of 4,360 public and 99,040 private firms from the US over the period from 2002 to 2011. They find that private firms are twice as levered as public firms: The average book leverage ratio is 0.446 for private and 0.204 for public firms; again a ratio of about two.

Kartashova (2014) uses SCF data not only to estimate the size of private businesses but also to estimate returns; she validates the survey-based results using data from the US FFA and the National Income and Product Accounts (NIPA) and uses several other data sources to improve and validate her results. While this approach allows for sufficiently precise estimation of average returns, the data quality worsens for higher frequency and therefore does not allow us to estimate standard deviations and correlations with other asset classes. For SCF data, the main reason is that this survey is published only every third year. FFA data have other issues. First, it is challenging to split the time series for the size of an asset class into its components of in- and outflows of cash (such as dividend payments) and returns (such as a change in valuation) at a quarterly level. Second, the value of private businesses has several components. One is proprietors' income, which is almost uncorrelated to the stock market. This might be due to reporting issues, such as delayed valuations or imprecise reporting of the labor component. Another component, the value of C and S corporations, is almost perfectly correlated with listed equity as its value in the FFA is estimated using a method that relies on stock market multiples. The resulting time series of private equity return has an implausibly low, bond-like, standard deviation and positive autocorrelation. Similarly, using NIPA data, Heaton and Lucas (2000a) find an implausibly low correlation between the quarterly growth rate of real non-farm proprietary income and the CRSP value-weighted of a mere 0.14. This probably also reflects the staleness of officially reported private equity data. Therefore we consider the time series by Ang *et al.* (2018) better suited for our purpose.

4.2.3 Bonds. We calculate a weighted average of the returns of the USA 10-Year Government and the Dow Jones Corporate Bond Return Index, obtained from Global

Financial Data. We obtain the weights used to calculate the aggregated bond returns from estimates for their size from FFA [6].

4.3 Estimates

Tables 1 and 2 summarize the data that we use as baseline estimates. The first striking observation is that the three asset classes have broadly similar size. The mean excess return is 5.5% higher for private equity than listed equity, which in turn exceeds bonds by 3%. The return difference between private and public equity – the private equity premium – is close to the estimate by Kartashova (2014) who finds for the period from 1990 to 2010 average returns of 16.5 and 9.2% for private and public equity, respectively. The return standard deviations of bonds (6%) and public equity (18%) are in line with typical estimates, while the estimate for private equity might appear surprisingly high with about 27% p.a. Similarly, the stock market beta of private equity is above one. We discussed in section 4.2 that the typical private firm has higher leverage than its listed counterpart, consistent with higher risk measures. Using FFA or industry private equity indices as data source gives the appearance of lower risk, but estimates returns are highly autocorrelated. Ang *et al.* (2018) discuss that autocorrelation of industry private equity indices is likely to stem from reporting issues.

4.3.1 Risk tolerances. Finally, we have to select values for the risk tolerances of both investor types. Specific numbers for these parameters are hard to interpret, but it turns out that, all other parameters given, risk tolerance is closely tied to the market risk premium. To avoid the effect of market structure to impact our choice of a value for aggregate risk tolerance τ, we derive its value implied by a model of full tradability of all asset classes. Denote *MRP* as the aggregate risk premium measured in percent of the sum of expected cash flows over all assets. Under full tradability, Equation (6) extends to all assets and simplifies to $p = \mu - \frac{1}{\tau}\Sigma e$. The risk premium as a percentage of expected cash flows, *MRP*, can be expressed as $MRP = \frac{e'(\mu - p)}{e'\mu} = \frac{e'\Sigma e}{\tau e'\mu}$. Using here a normalization of the cash flows such that the sum of their expected cash flows equals 1, i.e. $e'\mu = 1$, we obtain the aggregate risk tolerance as the sum of all assets' covariances, scaled by the market risk premium:

$$\tau = \frac{e'\Sigma e}{MRP} \tag{9}$$

We assume a value of $MRP = 2.5\%$ for the aggregate market risk premium when all assets were tradable. To proxy for the covariance matrix of cash flows Σ, we use standard deviations of returns times the relative sizes of the asset classes from Table 1 and return

	Private business	Public equity	Bonds
Share in total, in %	31.48	36.47	32.05
Mean excess return, in % p.a.	12.89	7.35	4.34
Standard deviation, in % p.a.	26.90	18.20	5.85
Equity market beta	1.20	1.00	−0.09
Bond market beta	−0.88	−0.83	1.00

Table 1. Baseline parameters: size, returns, standard deviations, betas

	Private business	Public equity	Bonds
Private Business	1	0.830	−0.192
Public Equity		1	−0.266
Bonds			1.00

Table 2. Baseline parameters: correlations

correlations reported in Table 2 [7]. Given that we were unable to find evidence for the split of aggregate risk tolerances among endowed and financial investors, we start from the uninformative prior that $\tau^E = \tau^F = \tau/2$.

4.3.2 Alternative values: online app. We acknowledge that the choice of the above parameters incorporates a certain degree of judgment that users of our model might exert in a different way. We therefore provide an app freely accessible at https://assetallocation.shinyapps.io/policyportfolios/that allows users to change the parameters and obtain the resulting policy portfolios, with the equilibrium asset allocation weights for both types of investors and the market-wide expected returns [8].

5. Results

In this section, we report results based on the parameters derived from the calibration in section 4. Table 3 summarizes the input parameters and main results. Given the negative correlations of bonds to both listed and unlisted equity, the diversification benefit leads to a slightly negative risk premium for bonds. The risk premium for equity is positive and its magnitude reflects the hedging pressure from the endowed investor. Figure 3 illustrates the equilibrium asset allocation of both investor types. The chart on the left side refers to the market portfolio, in the middle there is the policy portfolio of investor F and to the right the policy portfolio of investor E. In equilibrium, none of the investors holds financial assets in the same proportion as the market. Note that investor E holds private equity, in addition, which is not reported in this diagram. It is the positive correlation between listed equity and private equity that leads investor E to underweight listed equity relative to the market. In the base calibration, we even obtain a short position in listed equity. Investor F takes a corresponding overweight position in listed equity. This is in contrast to the hypothetical case

Table 3.
Parameters and main results for the base case

Input parameters		
Risk premium on portfolio of all risky assets in percent	2.50	
Correlations		
Stocks – bonds	−0.266	
Stocks – private business	0.830	
Bonds – private business	−0.192	
Normalized cash flow standard deviations (in percent)		
Stocks	6.63	
Bonds	1.87	
Private business	8.47	
	F	E
Investors' relative share in total risk tolerance (in percent)	50.00	50.00
Results		
Expected excess returns (in percent)		
Stocks	3.10	
Bonds	−0.11	
Private business (based on shadow price)	5.92	
Percentage of market capitalization held	F	E
Stocks	104.25	−4.25
Bonds	57.77	42.23
Private business	0.00	100.00
Portfolio composition (in percent)	F	E
Stocks	65.60	−3.55
Bonds	34.40	33.42
Private business	0.00	70.13

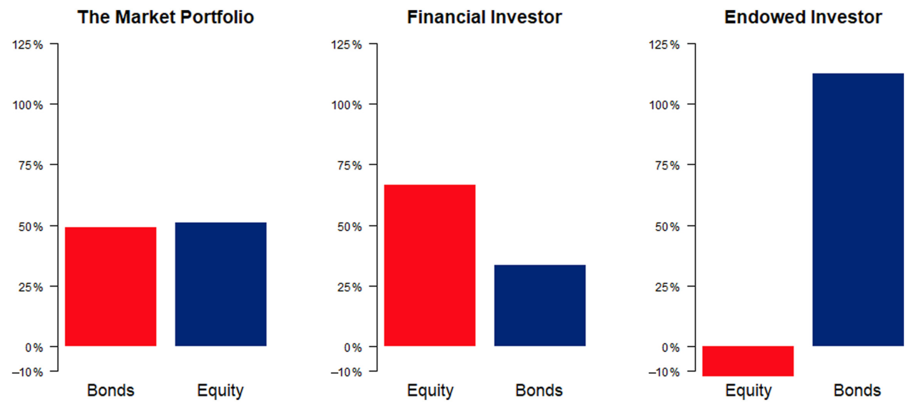


Figure 3.
Policy portfolios

if all assets were uncorrelated: Now both investors E and F would hold half of the supply of both stocks and bonds, corresponding to their shares in aggregate risk tolerance.

An analyst aiming at estimating expected returns of tradable assets from the observed market equilibrium therefore must consider covariances with non-tradable assets. This is usually not part of the recommended toolbox, e.g. the popular Black-Litterman approach to asset allocation [9]. At the heart of this method is the consistent combination of individual forecasts (or views) with equilibrium expected returns. To obtain the latter, it is common to re-engineer observed market capitalizations at market prices and estimated covariances [10]. In our notation, expected cash flows can correctly be imputed from Equation (6) as

$$\mu_T = p_T^* + \frac{1}{\tau}(\Sigma_{TT} e_T + \Sigma_{TN} e_N). \tag{10}$$

Ignoring covariances between tradable and non-tradable assets in the formation of implied cash flow expectations,

$$\mu_T^{imp} = p_T^* + \frac{1}{\tau}\Sigma_{TT} e_T, \tag{11}$$

gives rise to errors of magnitude $\frac{1}{\tau}\Sigma_{TN} e_N$. For each asset, i, the error is therefore the sum of its covariances with all non-tradable assets. Expected cash flows, and equivalently expected returns, under the true model will therefore be higher (lower) than the corresponding naively implied numbers for assets that are positively (negatively) correlated with non-tradable assets. Naively implied returns will only be correct for assets uncorrelated with the portfolio of non-tradable assets. Ignoring this fact could lead to severe mis-allocations.

Using the base calibration, the error from ignoring private equity, when reverse-engineering expected returns, amounts to an underestimation of the risk premium for listed equity of 1.34% and a slight overestimation of the risk premium for bonds of 9 basis points. Using such biased equilibrium risk premiums as a starting point in the Black-Litterman procedure would in turn lead to a suboptimally low weight of equities in a traditional stock-bond portfolio.

6. Comparative statics

In section 5 we have presented our main results using our baseline calibration from section 4. To gain further insights, we derive expressions for and illustrate the sensitivities of quantities and prices with respect to the main parameters.

6.1 Sensitivities to relative risk tolerances

In a world where all assets are accessible by all investors, shifting a larger share of aggregate risk tolerance toward a single investor will also lead to a larger share of the market portfolio held by this investor. In the limit, a single investor with infinite risk tolerance (i.e. zero risk aversion) holds the entire portfolio of risky assets. The situation is more complex in our model: In the limit, if only investor E has infinite risk tolerance, she will hold the entire portfolio of tradable securities in addition to her holdings of non-tradable assets. F will hold no securities. Note that increasing E's share of aggregate risk tolerance corresponds to a situation where more and more investors obtain access to private equity. However, if it is investor F who has infinite risk tolerance, E will generally not exit the market for tradable securities but will set up a regression hedge minimizing the variability that stems from her positions in non-tradable assets. F therefore will hold the remainder part of tradable assets. The comparative statics are immediately seen from taking partial derivatives,

$$\frac{\partial x_T^{F*}}{\partial \tau^F}\Big|_{\tau\,\text{fixed}} = \frac{\partial x_T^{E*}}{\partial \tau^E}\Big|_{\tau\,\text{fixed}} = \frac{1}{\tau}(e_T + \mathbb{B}e_N) = \frac{1}{\tau^F} x_T^{F*}. \tag{12}$$

Note that F will amplify her positions if her risk tolerance, τ^F, increases in a way that is proportional to her holdings. Hence, F will buy additional assets where she already has long positions and sell short where she already has short positions. In the following proposition we formulate the limiting cases for the holdings of E and F when one of these two representative investors becomes risk neutral.

Proposition 6.1. As the type E investor becomes risk neutral, i.e. $\tau^E \to \infty$, she holds the entire portfolio of tradable assets, i.e. $x^E* = e_T$, and the type F investor holds zero, i.e. $x^F* = \mathbf{0}$. If investor F becomes risk neutral, i.e. $\tau^F \to \infty$, endowed investor E holds tradable assets corresponding to a regression hedge, i.e. $x^{E*} = -\mathbb{B}e_N$. Investor F holds the remaining tradable assets, i.e. $x^{F*} = e_T\mathbb{B}e_N$.

Proof. The proof is straightforward, taking Equations (7) and (8) to the limits $\tau^E \to \infty$ and $\tau^F \to \infty$.

$$\lim_{\tau^E\to\infty} x_T^{E*} = \lim_{\tau^E\to\infty} \frac{\tau^E}{\tau^E+\tau^F} e_T - \frac{\tau^F}{\tau^E+\tau^F}\mathbb{B}e_N = e_T \tag{13}$$

$$\lim_{\tau^E\to\infty} x_T^{F*} = \lim_{\tau^E\to\infty} \frac{\tau^F}{\tau^E+\tau^F}(e_T + \mathbb{B}e_N) = \mathbf{0} \tag{14}$$

$$\lim_{\tau^F\to\infty} x_T^{E*} = \lim_{\tau^F\to\infty} \frac{\tau^E}{\tau^E+\tau^F} e_T - \frac{\tau^F}{\tau^E+\tau^F}\mathbb{B}e_N = -\mathbb{B}e_N \tag{15}$$

$$\lim_{\tau^F\to\infty} x_T^{F*} = \lim_{\tau^F\to\infty} \frac{\tau^F}{\tau^E+\tau^F}(e_T + \mathbb{B}e_N) = e_T + \mathbb{B}e_N \tag{16}$$

■

A shift in relative risk tolerances between investors E and F therefore changes the size and the composition of the portfolios of tradable assets held by these investors. However, such a shift does not affect the prices of tradable assets as will be shown in the next proposition.

Proposition 6.2. Holding total risk tolerance, τ, constant, a change in risk tolerance τ^E of investor E does not lead to a change in the market value of tradable assets.

Proof. This is seen directly from Equation (6), where the price of tradable assets only depends on aggregate risk tolerance τ and not on the relative contributions of τ^F and τ^F. ■

6.1.1 Sensitivities to a shift in investors' relative risk tolerances. Figure 4 illustrates the sensitivity to variation in the relative risk tolerances. The endowed investor E always holds the non-tradable asset (not shown in the chart). If E has the entire risk-bearing capacity of the market, she will own in addition the entire supply of equity and bonds. If it is F who has the entire risk tolerance, E will use a regression hedge to minimize her risks from the private equity holdings, leading to a pronounced short position in stocks and a minor short position in bonds. F will own levered positions in these assets.

6.2 Sensitivity to tradability

To analyze the impact of non-tradability on asset prices we first establish benchmark valuations that would prevail in the standard case – an economy without trading restrictions on some assets but otherwise identical. Maximization of aggregate utility of both E and F over aggregate holdings $x^E + x^F$ gives the benchmark asset prices p^{ac} (full accessibility):

$$p^{ac} = \mu - \frac{1}{\tau}\Sigma e. \tag{17}$$

To allow comparison of the benchmark asset prices with the results from the model with tradable and non-tradable assets, we split the pricing vector p^{ac} into its components,

$$p^{ac} = \begin{pmatrix} p_T^{ac} \\ p_N^{ac} \end{pmatrix} = \begin{pmatrix} \mu_T \\ \mu_N \end{pmatrix} - \frac{1}{\tau}\begin{pmatrix} \Sigma_{TT} & \Sigma_{TN} \\ \Sigma'_{TN} & \Sigma_{NN} \end{pmatrix}\begin{pmatrix} e_T \\ e_N \end{pmatrix}. \tag{18}$$

The following results compare asset values in our model and the benchmark case for both tradable (proposition 6.3) and non-tradable (proposition 6.4) assets.

Proposition 6.3. Making the non-tradable assets accessible to all investors does not affect risk premiums of traded assets.

Proof. Matrix multiplication gives the upper part of Equation (18) as

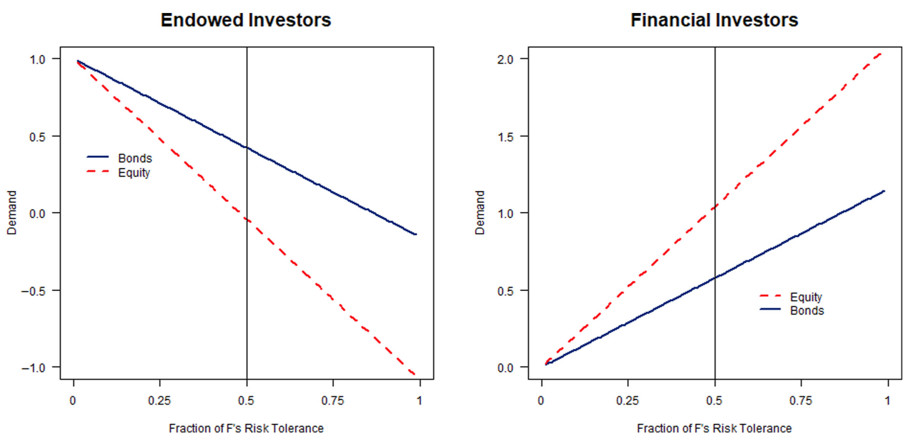


Figure 4.
Sensitivity of investors' demand to a shift in relative risk tolerances

$$p_T^{ac} = \mu_T - \frac{1}{\tau}(\Sigma_{TT}e_T + \Sigma_{TN}e_N),$$

which is identical to p_T in Equation (6). ■

Proposition 6.3 shows that the prices of tradable assets fully incorporate covariances with non-tradable assets, to the same extent as if those assets actually were traded. Therefore, a private firm going public or even an abundant Initial Public Offering (IPO) wave, making previously illiquid assets accessible to a broader investor base, will not impact the prices of those assets that were already tradable before. Yet the valuation of these formerly illiquid assets themselves will change systematically, as shown in the following proposition 6.4. As non-tradable assets do not have a market price by assumption, we have to establish a valuation measure. We define the shadow price p_N^S of non-tradable assets as the price vector that would make investors E optimally hold the entire supply e_N of non-tradable assets, given their holdings in tradable assets derived in Equation (7).

Proposition 6.4. Making the non-tradable assets accessible to all investors will lead to lower risk premiums of the non-tradable assets incorporated in market prices compared to the risk premiums incorporated in the corresponding shadow prices.

Proof. To derive shadow prices p_N^S, we express the quantities of non-tradable assets that maximize E's utility from her FOC:

$$\frac{\partial U^E}{\partial x_N^E} = \mu_N - \frac{1}{\tau^E}\left(\Sigma_{NN}x_N^E + \Sigma'_{TN}x_T^E\right) - p_N^S = \mathbf{0}. \tag{19}$$

This gives the demand function:

$$x_N^E = \Sigma_{NN}^{-1}\left(\tau^E\left(\mu_N - p_N^S\right) - \Sigma'_{TN}x_T^E\right). \tag{20}$$

Knowing that E holds the optimal amount of tradable assets $x_T^E = x_T^{E*} = \frac{\tau^E}{\tau}\left(e_T - \frac{\tau^F}{\tau^E}\mathbb{B}e_N\right)$ and all non-tradable assets, i.e. $x_N^E = e_N$, we obtain the shadow price as

$$p_N^S = \mu_N - \frac{1}{\tau}\left(\Sigma'_{TN}e_T + \frac{\tau}{\tau^E}\Sigma_{NN}e_N\right) + \frac{\tau^F}{\tau^E \cdot \tau}\Sigma'_{TN}\mathbb{B}e_N. \tag{21}$$

The valuation difference of non-tradable assets under full accessibility versus non-tradability therefore reads

$$p_N^{ac} - p_N^S = \left(\frac{1}{\tau^E} - \frac{1}{\tau}\right)\cdot\left(\Sigma_{NN}e_N - \Sigma'_{TN}\mathbb{B}e_N\right). \tag{22}$$

From positive definiteness of the covariance matrix Σ, the valuation difference in Equation (22) is positive if $\tau^F > 0$. ■

To interpret proposition 6.4, first note that the shadow price is lower than the price under full tradability. The difference is the maximum amount investor E would be willing to pay for brokerage services to make a non-tradable asset tradable. This difference will be higher the larger the risk tolerance of investor F, τ^F. If F adds little to aggregate risk tolerance, there is not much to be gained from making assets accessible to her. The better the cash flows of non-tradable assets can be projected onto the space spanned by tradable assets, the smaller the price difference. Intuitively, high correlations between tradable and non-tradable assets allow investor E to hedge her holdings of non-tradable assets anyway, thereby reducing the need for an additional risk-premium related to imperfect risk-sharing. It is illustrative to aggregate non-tradable assets into a single non-tradable portfolio PN. Equation (22) then reads $p_N^{ac} - p_{PN}^S = \left(\frac{1}{\tau^E} - \frac{1}{\tau}\right)\sigma_{PN}^2(1 - R^2)$, with R^2 from a multiple regression of the cash flow from PN on the tradable assets. The valuation difference depends on the difference in E's and aggregate risk aversion, the volatility of the portfolio of non-tradable assets PN and the fit of the projection of PN on tradable assets. Only the part of the variance of PN that cannot be explained using tradable assets warrants an additional risk premium.

6.2.1 Price impact of non-tradability. In the extreme case of $\tau^F = 0$, tradability does not matter for prices. Yet the shadow price still depends on covariances between tradable and non-tradable assets, due to changing diversification benefits of a given portfolio. It is instructive to analyze a measure that does not depend on the latter component of risk premiums, and therefore allows to extract the valuation impact that is due to non-tradability: the difference between the shadow price of an asset and the price that would prevail if this asset were tradable. From Equation (22) it is seen that the difference between the vector of fully accessible prices p_N^{ac} and shadow prices p_N^S is proportional to $\left(\Sigma_{NN} - \Sigma_{TN}'\mathbb{B}\right)e_N$ [11]. This is the part of the non-tradable assets' covariance matrix that is orthogonal to tradable assets. To gain intuition, consider the case $n_N = 1$. Remember that $p_N^{ac} - p_N^S$ reduces to $\left(\frac{1}{\tau^E} - \frac{1}{\tau}\right)\sigma_N^2(1 - R^2)$, where R^2 is the R-squared from the projection of the non-tradable asset's cash flow on the tradable assets' cash flows. The difference between the price of the non-tradable asset when it becomes tradable and the shadow price when it is non-tradable is therefore maximized when tradable assets cannot be used to hedge the cash flow from the non-tradable asset. It approaches zero when $R^2 \to 1$, irrespective of the sign of the correlation.

Figure 5 displays the risk premiums of the asset classes. Making private equity tradable does not affect the risk premiums of bonds and stocks. The risk premium of private equity calculated from its shadow price (restricted access) is 0.9% higher than the risk premium calculated from the full accessibility price. This moderate price difference is due to the gains from risk sharing given the high and positive correlation between private and public equity. Note that, in our setting, this difference can be interpreted as the maximum annual fee investor E would be willing to pay for brokerage services to make private equity tradable. We will show in section 6.4 that this difference gets significantly larger for a correlation closer to zero.

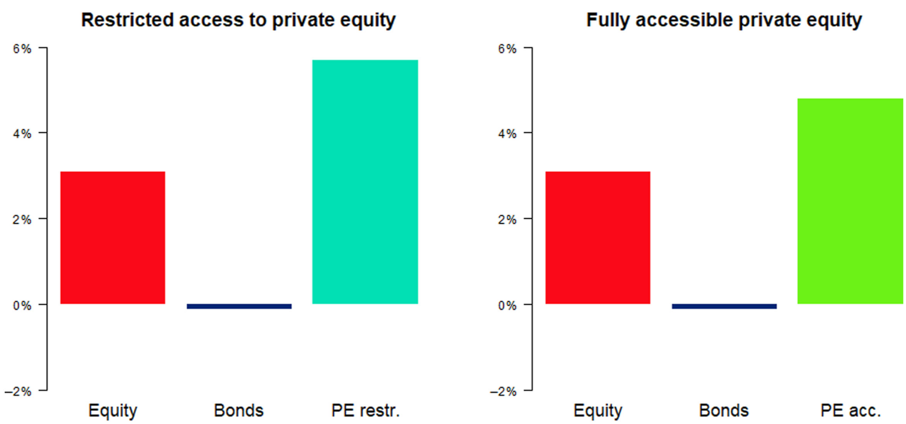


Figure 5.
Sensitivity of risk premiums to accessibility

6.3 Sensitivity to the size of non-tradable assets

We will now isolate the effects of a change in the size of a non-tradable asset class, measured as its expected cash flow.

Proposition 6.5. An increase in the size of a non-tradable asset will lead to a higher risk premium for tradable assets which are positively correlated with the non-tradable asset.

Proof. It is seen from Equation (6) that the first part of the risk premium of tradable assets does not depend on covariances with non-tradable assets. Thus, it can be ignored. The second part is $\frac{1}{\tau}\Sigma_{TN}e_N$. For each tradable asset i, we can write its covariance with the non-tradable asset j as $\sigma_{i,j} = \sigma_i\sigma_j\rho_{i,j}$. The relevant part of the risk premium of asset i is therefore

$$\frac{1}{\tau}\sigma_{i,N} = \frac{1}{\tau}\left(\sigma_i\rho_{i,j_1}\sigma_{j_1} + \sigma_i\rho_{i,j_2}\sigma_{j_2} + \ldots\right).$$

The effect of increasing the size of non-tradable asset j is therefore given by

$$\frac{\partial\frac{1}{\tau}\sigma_{i,N}}{\partial\sigma_j} = \frac{\sigma_i\rho_{i,j}}{\tau}.$$

This is positive if $\rho_{i,j} > 0$. ■

The above proof shows that the sensitivity of the risk premium for tradable assets to the size of non-tradable assets is highest when the correlation is ±1. Figure 6 shows the sensitivity of optimal demand of endowed and financial investors for tradable assets as the size of private equity varies relative to the size of public equity. Figure 7 shows the sensitivities of asset prices. The valuation of private equity as well as the price of listed equity decrease with the size of the asset class private equity, while the price of bonds slightly increases.

6.4 Sensitivities to covariances

6.4.1 Tradable assets. The results from section 3.3 show that in equilibrium the price of tradable assets fully incorporates the covariances between tradable and non-tradable assets. It is seen directly from Equation (6) that a tradable asset *i*'s risk premium increases with the sum of asset *i*'s covariances with the non-tradable assets.

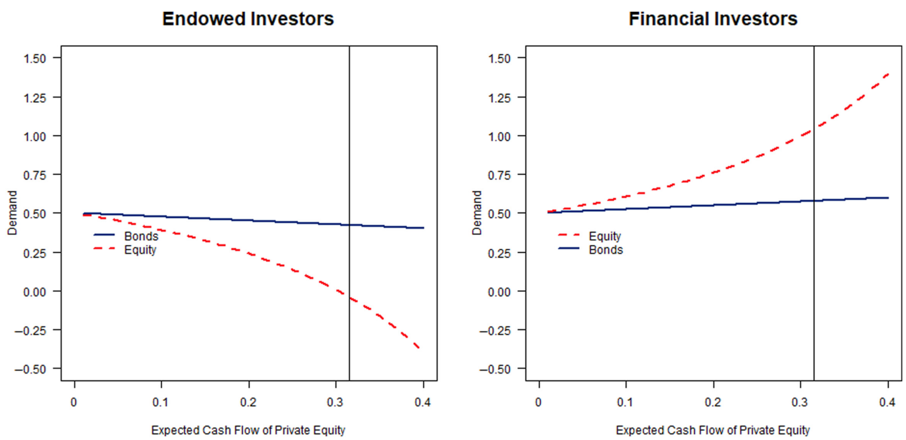


Figure 6.
Sensitivity of investor demand to the size of private equity

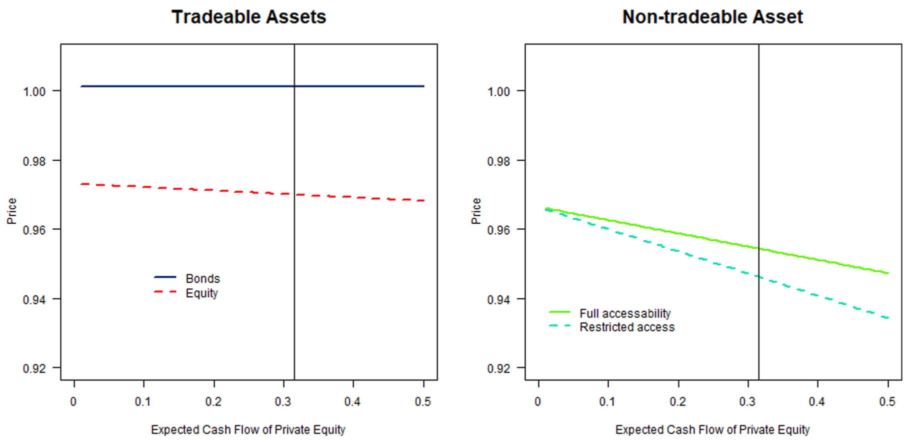


Figure 7. Sensitivity of asset prices to the size of private equity

6.4.2 Non-tradable assets. To assess the impact of covariances across market segments on the shadow price of non-tradable assets, we reformulate Equation (21) to

$$p_N^S = \mu_N - \frac{1}{\tau^E}\Sigma_{NN}e_N - \frac{1}{\tau}\Sigma'_{TN}\left(e_T - \frac{\tau^F}{\tau^E}\mathbb{B}e_N\right). \tag{23}$$

The risk premium for non-tradable assets in Equation (23) consists of two parts. First, the covariances among non-tradable assets matter. Here, limited risk sharing is reflected by dividing by the risk tolerance of the endowed investor only. Second, there is a term that contains covariances between tradable and non-tradable assets. To assess their impact, note that the sum of these covariances is reduced by the scaled projection of non-tradable assets on tradable assets, with the ratio of risk tolerances of F to E as the scaling factor. This remaining part of the covariance risk is spread over all investors, E and F, indicated by the divisor τ. Hence, we see from Equation (23) that the effect of increasing correlations between tradable and non-tradable assets on the shadow prices of the latter is ambiguous. Consider the polar cases of investor F having zero or infinite risk tolerance. Taking Equation (23) to the limits gives

$$\lim_{\tau^F\to 0} p_N^S = \mu_N - \frac{1}{\tau^E}\left(\Sigma_{NN}e_N + \Sigma'_{TN}e_N\right) \tag{24}$$

$$\lim_{\tau^F\to \infty} p_N^S = \mu_N - \frac{1}{\tau^E}\left(\Sigma_{NN}e_N - \Sigma'_{TN}\mathbb{B}e_N\right) \tag{25}$$

For an infinitely risk averse financial investor F, higher covariances translate into higher risk premiums as E's basket – consisting of all non-tradable and tradable assets – is less well diversified. Yet if F is infinitely risk tolerant there is maximum risk sharing. E only bears the residual risk that cannot be hedged via changing her positions in the tradable asset. Hence, more extreme correlations, both positive and negative, facilitate hedging and lead to lower risk premiums.

6.4.3 Sensitivities to correlation. Figure 8 illustrates how optimal holdings of E and F change when the correlation of private equity and public equity varies, holding standard deviations and the other correlations constant. Given E's share in total risk tolerance, it is intuitive that E will hold about 50% of the supply of equity when equity and private equity are uncorrelated. The negative correlation of bonds with both equity and private

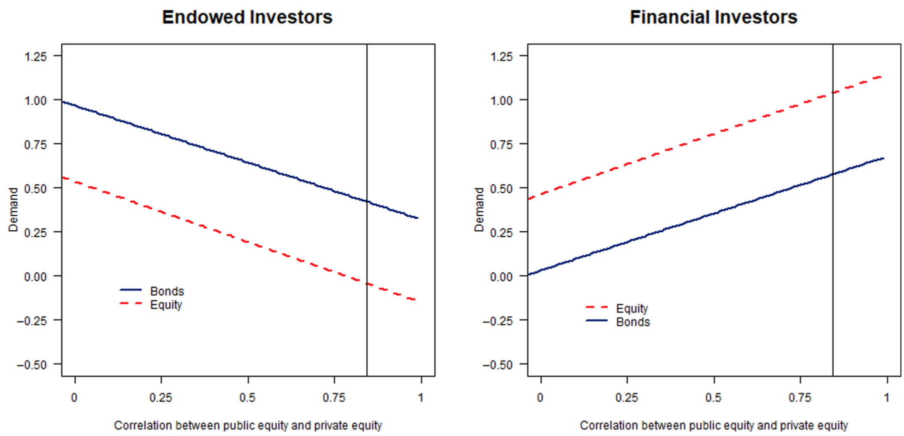


Figure 8.
Sensitivity of demand to correlation of public equity with private equity

equity makes this asset class an attractive investment for E. With higher public-private equity correlations, E's demand for equity gets dramatically lower and even turns negative. While the demand for bonds gets reduced, it stays clearly positive even for extremely high public-private equity correlations. Investor F takes positions such that the market clears, adding to her holdings in tradable assets with increasing correlation.

To isolate the effect of a single correlation, we have set all correlations with the exception of equity - private equity to zero in Figure 9. Here, the demand for bonds equals the neutral 50% for both investors, irrespective of the public-private equity correlation, while investor E (F) reduces (increases) her demand for stocks when this correlation gets higher.

Reduced risk sharing is reflected in the decreasing price of stocks with an increasing public-private equity correlation, shown in the left panel of Figure 10 [12]. To the right, note that the difference between the price of private equity under full accessibility and its valuation when it is non-tradable is largest at slightly positive public-private equity correlations. A correlation of 1 goes with a zero valuation difference. In the range of very high (positive) correlation, the gain from improved risk sharing via better hedging is more important than the loss of diversification, leading to a shadow price of private equity that increases in correlation.

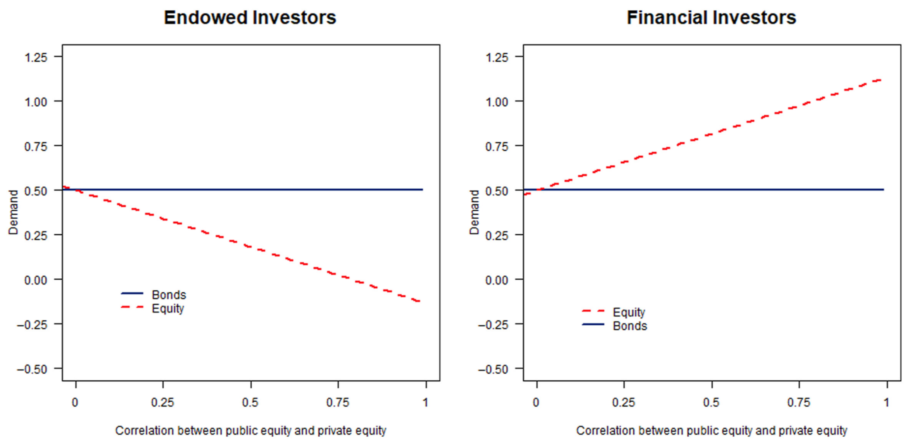


Figure 9.
Sensitivity of demand to correlation of public equity with private equity when all other correlations are zero

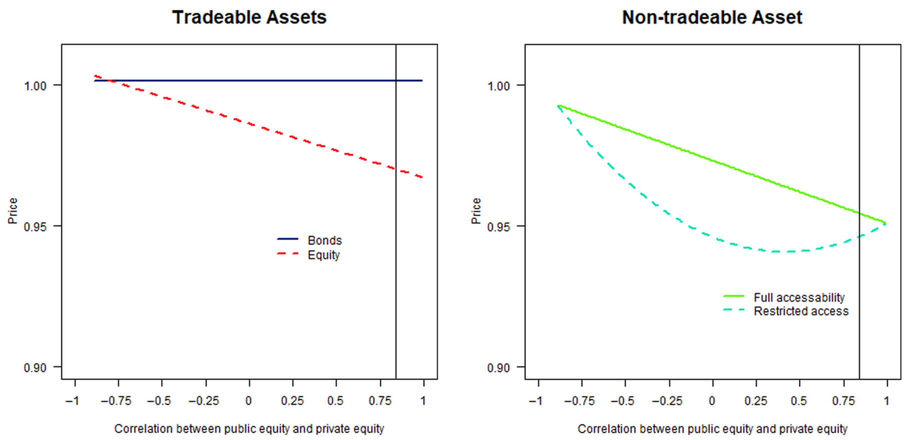


Figure 10.
Sensitivity of prices to correlation of public equity with private equity

7. Conclusion

We analyze portfolio choices and equilibrium prices when some investors (type E) are endowed with illiquid assets whereas others (type F investors) are not. In equilibrium, investors F tilt their portfolios toward tradable assets to which non-tradable assets exhibit positive betas. Using plausible parameter values, F investors' portfolios consist of 60–80% tradable equity, although tradable equity makes up only about 50% of the market portfolio of traded assets. Naive application of "reverse optimization" in the spirit of the Black Litterman model leads to significantly biased estimates of expected risk premiums. Making non-tradable assets tradable does not affect risk premiums of tradable assets, but lowers them for non-tradable assets. Accounting for a plausibly sized non-tradable private equity market leads to a significantly higher equilibrium risk premium on traded equity.

Notes

1. For example, in Gârleanu and Pedersen (2018), only large informed investors choose active portfolio managers, whereas small uninformed investors cannot access this "asset class" of actively managed funds. In our context, active funds are interpreted as funds that invest in opaque, illiquid and hard to trade assets, such as private equity.
2. Examples include Moskowitz and Vissing-Jørgensen (2002), Kartashova (2014) and Heaton and Lucas (2000b). Anderson (2009) discusses the relative merits of the respective approaches.
3. Buyout funds seem to be most representative of the overall private equity market. They also find that venture capital funds have a beta of 2.7 (and highly negative alpha). Various other studies use data on private equity funds. Ang *et al.* (2018) use a similar method on more recent data finding similar numbers. With respect to risk of private equity, Jegadeesh *et al.* (2015) find market betas not significantly different from 1 and positive loadings on the size factor. This is consistent with Franzoni *et al.* (2012) who find market betas to vary from 0.95 (CAPM) to 1.4 (market beta in Fama French model). Phalippou and Gottschalg (2009) find a market beta above 1 for buyout funds from matching portfolio companies with public companies using industry and size.
4. These asset classes constitute the majority of investments, and can be measured with reasonable accuracy. Further asset classes that are sometimes discussed in the analysis of specific investor groups' portfolios are not suitable for our equilibrium approach. For example, the bulk of housing wealth is either already contained in business assets (via its equity or bond components) or is in aggregate offset by housing needs. Other asset classes suffer from double counting, e.g. hedge funds, which are just investment vehicles into other assets. The situation is similar for commodities: many resources are owned by corporations, so we cannot include them in order to avoid double

counting; commodities traded via futures on exchanges have per definition of the instruments a net market capitalization of zero since long and short positions offset each other.

5. There has been a lot of debate on disappearing public companies over the fact that the number of listed domestic firms on CRSP has halved over the last 20 years. Interestingly, looking at market capitalization as the relevant size measure, this trend is not in the data. If in the future a combined trend of low IPO numbers and increased occurrences of companies going private lead to a smaller public market, the analysis of our paper would be even more relevant.
6. We use the quarterly time series FL884122005.Q for the total bond market and FL364122005.Q for government bonds, and obtain the size of corporate bond market as the difference between the two.
7. We thank an anonymous referee for pointing out that for an exact conversion of the return covariance matrix into a cash flow covariance matrix one should use the initial asset weights. Since the expected asset returns are different across assets, one cannot have a normalization that makes both the initial asset weights and the expected cash flow weights sum up to 1 simultaneously. If one normalizes the initial asset weights to have $\sum_1^n w_i = 1$, then it follows that $e'\mu = \sum_1^n [w_i \mathbb{E}(R_i)]$, where $\mathbb{E}(R_i)$ is the expected return of asset i.
8. In this paper, we provide formal derivations of comparative statics in section 6.
9. See for example Litterman (2003).
10. Re-engineering expected returns from covariances and weights was first introduced in the context of stock portfolios by Sharpe (1974).
11. As Σ is a covariance matrix of full rank, it is positive definite as well as its submatrices Σ_{NN} and Σ_{TT}. Then it is known from matrix algebra that $\Sigma_{NN} - \Sigma'_{TN}\Sigma_{TT}^{-1}\Sigma_{TN}$ is also positive definite.
12. Only parameter values where the resulting covariance matrix is positive definite are shown in Figure 10.

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Appendix

Symbol	Description
n_T	number > 1 of tradable assets
n_N	Number ≥1 of non-tradable assets
n	Total number of assets, equals $n_T + n_N$
CF	$n \times 1$ vector of cash flows of assets
μ_T	$n_T \times 1$ vector of expected cash flows of tradable assets
μ_N	$n_N \times 1$ vector of expected cash flows of non-tradable assets
μ	$n \times 1$ vector of expected cash flows of assets
Σ_{TT}	$n_T \times n_T$ covariance matrix of tradable assets' cash flows
Σ_{NN}	$n_N \times n_N$ covariance matrix of non-tradable assets' cash flows
Σ_{TN}	$n_T \times n_N$ matrix of covariances of tradable with non-tradable assets' cash flows
Σ	$n \times n$ covariance matrix of all assets' cash flows
e_T	$n_T \times 1$ vector of ones
e_N	$n_N \times 1$ vector of ones
e	$n \times 1$ vector of ones
E	Denotes the investor type owning non-tradable assets, denoted *endowed* investor
F	Denotes the investor type with access only to tradable assets, denoted *financial* investor
τ^E	Risk tolerance of investor E
τ^F	Risk tolerance of investor F
τ	Aggregate risk tolerance, sum of τ^E and τ^F
ω_T^E	$n_T \times 1$ vector of investor E's initial endowment of tradable assets
ω_N^E	$n_N \times 1$ vector of investor E's initial endowment of non-tradable assets
ω^E	$n \times 1$ vector of investor E's initial endowment of all assets
ω_T^F	$n_T \times 1$ vector of investor F's initial endowment of tradable assets
ω_N^F	$n_N \times 1$ vector of investor F's initial endowment of non-tradable assets
ω^F	$n \times 1$ vector of investor F's initial endowment of all assets
x_T^K	$n_T \times 1$ vector of investor K's demand of tradable assets, where $K \in \{E, F\}$
x_N^K	$n_N \times 1$ vector of investor K's demand of non-tradable assets
x^K	$n \times 1$ vector of investor K's demand of all assets
p_T	$n_T \times 1$ vector of tradable assets' prices
p_N	$n_N \times 1$ vector of non-tradable assets' (shadow) prices
p	$n \times 1$ vector of all assets' prices
MV_T^E	$n_T \times 1$ vector of market value of tradable assets held by investor E
MV_T^F	$n_T \times 1$ vector of market value of tradable assets held by investor F
$\mathbb{B}$	$n_T \times n_N$ matrix of cash flow betas
σ_i	Standard deviation of cash flows of asset i
$\sigma_{i,j}$	Covariance of cash flows of assets i and j
$\rho_{i,j}$	Correlation of cash flows of assets i and j
R_i	Realized return of asset i
β_i^{PF}	Beta of asset i's returns with respect to portfolio PF

Table A1.
Table of notations

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