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Paley and Hardy's inequalities for the Fourier-Dunkl expansions

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Received 19 December 2021
Revised 26 May 2022
4 June 2022
8 June 2022
Accepted 9 June 2022

Abstract

Purpose – Paley's and Hardy's inequality are proved on a Hardy-type space for the Fourier–Dunkl expansions based on a complete orthonormal system of Dunkl kernels generalizing the classical exponential system defining the classical Fourier series.

Design/methodology/approach – Although the difficulties related to the Dunkl settings, the techniques used by K. Sato were still efficient in this case to establish the inequalities which have expected similarities with the classical case, and Hardy and Paley theorems for the Fourier–Bessel expansions due to the fact that the Bessel transform is the even part of the Dunkl transform.

Findings – Paley's inequality and Hardy's inequality are proved on a Hardy-type space for the Fourier–Dunkl expansions.

Research limitations/implications – This work is a participation in extending the harmonic analysis associated with the Dunkl operators and it shows the utility of BMO spaces to establish some analytical results.

Originality/value – Dunkl theory is a generalization of Fourier analysis and special function theory related to root systems. Establishing Paley and Hardy's inequalities in these settings is a participation in extending the Dunkl harmonic analysis as it has many applications in mathematical physics and in the framework of vector valued extensions of multipliers.

Keywords Fourier–Dunkl expansions, Hardy spaces, BMO spaces, Paley's inequality, Hardy's inequality

Paper type Research paper

1. Introduction

Dunkl operators are differential-difference operators on $\mathbb{R}^N$ related to finite reflection groups. They can be regarded as a generalization of partial derivatives and they lead to a generalization of the classical tools of harmonic analysis. For further details on the corresponding basic theory, one can see Refs [1–3].

In rank-one case, we consider the Dunkl operator $\mathcal{D}^\alpha$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$, given by

$$\mathcal{D}^\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}, \ \alpha \geq -1/2.$$

For $\lambda \in \mathbb{C}$, the following system

$$\begin{cases} \mathcal{D}^\alpha f(x) &= i\lambda f(x), \quad x \in \mathbb{R}, \\ f(0) &= 1, \end{cases} \tag{1}$$

JEL Classification — 42B10, 46F12, 33C50, 30H10, 30H35

Arab Journal of Mathematical Sciences
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-12-2021-0312

admits a unique solution, denoted by $E_\alpha(i\lambda\cdot)$ expressed in terms of the normalized spherical Bessel functions j_α and $j_{\alpha+1}$, namely

$$E_\alpha(i\lambda z) = j_\alpha(\lambda z) + \frac{i\lambda z}{2(\alpha+1)} j_{\alpha+1}(\lambda z),$$

where

$$j_\beta(z) = \begin{cases} 2^\beta \Gamma(\beta+1) \dfrac{J_\beta(z)}{z^\beta}, & \text{if} \quad z \neq 0, \\ 1 & \text{if} \quad z = 0. \end{cases}$$

J_β being the Bessel function of the first kind and order β (see Ref. [4]). For $\alpha = -1/2$, it is clear that $\mathcal{D}^{-1/2} = d/dx$ and $E_{-1/2}(iz) = e^{iz}$.

For $\alpha \geq -1/2$, $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$ the estimate

$$|E_\alpha(i\lambda z)| \leq \exp|\lambda\, Im(z)| \tag{2}$$

holds. In particular, we have

$$|E_\alpha(i\lambda x)| \leq 1, \ \lambda, x \in \mathbb{R}. \tag{3}$$

As a generalization of the classical Fourier transform, the Dunkl transform $\mathcal{F}_\alpha$ of order $\alpha \geq -1/2$ is defined by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} \overline{E_\alpha(i\lambda y)} f(y) d\mu_\alpha(y), \lambda \in \mathbb{R},$$

for $f \in L^1(\mathbb{R}, d\mu_\alpha)$ the space of integrable functions with respect to the Haar measure $d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}dx$.

The aim of the present work is to obtain the analog of Paley and Hardy's inequalities for the Fourier–Dunkl expansions. We recall that if $\mathcal{R}H^1$ is the real Hardy space consisting of the boundary functions $f(\theta) = \lim_{r\to 1} \mathcal{R}F(re^{i\theta})$ where $F \in H^1(\mathbb{D})$ the Hardy space on the unit disc $\mathbb{D}$ which consists of the analytic functions $F(z)$ on $\mathbb{D}$ satisfying

$$\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})| d\theta < \infty,$$

and $\|f\|_{\mathcal{R}H^1} = \|F\|_{H^1}$ with real $F(0)$, then the Paley's inequality is given by (see Ref. [5]):

$$\left[\sum_{k=1}^{\infty} |c_{n_k}(f)|^2 + |c_{-n_k}(f)|^2 \right]^{1/2} \leq C \|f\|_{\mathcal{R}H^1}, \tag{4}$$

where $\{n_k\}_{k=1}^{\infty}$ is an Hadamard sequence, that is, a sequence of positive integers such that $n_{k+1}/n_k \geq \rho$ with a constant $\rho > 1$. And Hardy's inequality is

$$\sum_{n=-\infty}^{\infty} \frac{|c_n(f)|}{|n|+1} \leq C \|f\|_{\mathcal{R}H^1}, \tag{5}$$

where $f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n(f) e^{in\theta}$ in $\mathcal{R}H^1$ and C is independent of f.

Analogs of these inequalities were established in Refs [6, 7] for the Fourier–Jacobi expansions, and with respect to the Fourier–Bessel expansions in Ref. [8]. Although the difficulties related to the Dunkl settings, the obtained results have strong similarities with (4) and (5), since for $\alpha = -1/2$, we cover the classical case results. As we also cover the inequalities established in Ref. [8] due to the fact that the Bessel transform is the even part of the Dunkl transform.

Now, let us introduce the Fourier–Dunkl expansions and recall the definition of the nonperiodic real Hardy space. It is wellknown that the Bessel function $J_{\alpha+1}(x)$ has an increasing sequence of positive zeros $\{s_n\}_{n\geq 1}$. Then, the real function $Im(E_\alpha(ix)) = \frac{x}{2(\alpha+1)} j_{\alpha+1}(x)$ is odd and it has the infinite sequence of zeros $\{s_n\}_{n\in\mathbb{Z}}$ (with $0 < s_1 < s_2 < \ldots$, $s_{-n} = -s_n$ and $s_0 = 0$).

In Ref. [9], for $\alpha > -1$, the authors normalized the Dunkl kernel E_α to obtain a sequence of functions defining a complete orthonormal system in $L^2(\Delta, |x|^{2\alpha+1}dx)$, where $\Delta = (-1, 1)$. In this work, we define a new sequence of functions $\{e_{\alpha,n}(ix)\}_{n\in\mathbb{Z}}$ presenting a complete orthonormal system of $L^2(\Delta)$, given by

$$e_{\alpha,n}(ix) = d_{\alpha,n}\, |s_n x|^{\alpha+1/2}\, E_\alpha(is_n x),\ n\in\mathbb{Z}\backslash\{0\},\ x\in\Delta, \tag{6}$$

where

$$d_{\alpha,n} = \frac{1}{\sqrt{2}\,|s_n|^{\alpha+1/2}\,|j_\alpha(s_n)|}$$

and

$$e_{\alpha,0}(ix) = \sqrt{\alpha+1}\, x^{\alpha+1/2}.$$

This orthonormal system is a generalization of the classical exponential system defining Fourier series, and we define the Fourier–Dunkl expansion of a function $f(x)$ on Δ, by

$$f(x) \sim \sum_{-\infty}^{\infty} c_n^\alpha(f) e_{\alpha,n}(ix),\ c_n^\alpha(f) = \int_{-1}^{1} f(y) e_{\alpha,n}(iy) dy.$$

We should mention that the theory of Hardy spaces on $\mathbb{R}^d$ was initiated by Stein and Weiss [10]. Then, real variable methods were introduced in Ref. [11] and led to a characterization of Hardy spaces via the so-called "atomic decomposition", obtained by Coifman [12] when $n = 1$, and in higher dimensions by Latter [13]. A real-valued function a on Δ, is a Δ-atom if there exists a subinterval $I \subset \Delta$, satisfying the following conditions:

(1) $supp\,(a) \subset I$,

(2) $\int_I a(y)dy = 0$,

(3) $\|a\|_\infty \leq |I|^{-1}$, where $|I|$ is the length of the interval I.

The function $a(x) = \frac{1}{2}x,\ x\in\Delta$, is a Δ-atom.

The nonperiodic real Hardy space is defined to be the set of functions representable in the form:

$$f = \sum_{n=0}^{\infty} \lambda_n a_n, \tag{7}$$

where $\lambda_n \in \mathbb{C}$, verifying

$$\sum_{n=0}^{\infty} |\lambda_n| < \infty,$$

and every a_n is a Δ-atom. The series in (7) converges in $L^1(\Delta)$ (the set of integrable functions on Δ with respect to the Lebesgue measure) and also a.e.

The Hardy space $\mathcal{H}(\Delta)$ is endowed with the norm $\|.\|_{\mathcal{H}(\Delta)}$, given by

$$\|f\|_{\mathcal{H}(\Delta)} := \inf\left(\sum_{n=0}^{\infty} |\lambda_n|\right),$$

where the infimum is taken over all those sequences $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C}$ such that f is given by (7) for certain Δ-atoms $\{a_n\}$. Then $\mathcal{H}(\Delta)$ is a Banach space and $\|f\|_{L^1(\Delta)} \leq \|f\|_{\mathcal{H}(\Delta)}$.

Now, we state our theorem:

Theorem 1.1. *Let $\alpha \geq -1/2$ then the Fourier–Dunkl coefficients $c_n^{\alpha}(f)$ of a function $f \in \mathcal{H}(\Delta)$ satisfy*

$$\left[\sum_{k=1}^{\infty} |c_{n_k}^{\alpha}(f)|^2 + |c_{-n_k}^{\alpha}(f)|^2\right]^{1/2} \leq C\|f\|_{\mathcal{H}(\Delta)}, \tag{8}$$

where $\{n_k\}_{k=1}^{\infty}$ is a Hadamard sequence, and

$$\sum_{n=-\infty}^{\infty} \frac{|c_n^{\alpha}(f)|}{|n|+1} \leq C\|f\|_{\mathcal{H}(\Delta)}, \tag{9}$$

where the constant C is independent of f.

This paper is organized as follows. In Section 2 we state some technical lemmas needed for the proof of Theorem 1.1. In section 3 we recall the duality property between BMO and Hardy spaces, which plays an important role to prove a technical proposition for the proof of (8). In the last section, we give the proof of Theorem 1.1 and we finalize with some remarks.

2. Some technical lemmas

We begin this section by collecting three asymptotic formulas which will be needed later:

(1) Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of the successive positive zeros of $J_{\alpha+1}(x)$, the Bessel function of the first kind of order $\alpha+1$. Then we have, (see Ref. [4])

$$s_n = \pi\left(n + \frac{2\alpha+1}{4} + O(n^{-1})\right). \tag{10}$$

(2) An estimation of the constant $d_{\alpha,n}$ as stated in (6), is

$$d_{\alpha,n} = \frac{\sqrt{\pi}}{2^{\alpha+1}\Gamma(\alpha+1)}(1 + O(n^{-1})). \tag{11}$$

(3) Using the asymptotic formula for the Bessel function $J_\alpha(x)$, the Bessel function of the first kind of order $\alpha \in \mathbb{R}$, when $x \to +\infty$, given by

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - (2\alpha+1)\frac{\pi}{4}\right) + O(x^{-3/2}),$$

we deduce that

$$E_\alpha(ix) = \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}\,x^{\alpha+1/2}}\exp\left[i\left(x - \frac{(2\alpha+1)\pi}{4}\right)\right] + O\left(\frac{1}{x^{\alpha+2}}\right),\ x \to +\infty. \tag{12}$$

We begin with two auxiliary results interesting in themselves. We will denote by C a positive constant which is not necessary the same in each occurrence.

Lemma 2.1. *Let $\alpha \geq -1/2$, then there exists a constant C such that*

$$|e_{\alpha,n}(ix_2) - e_{\alpha,n}(ix_1)| \leq C|n|^\delta |x_2 - x_1|^\delta,\ \ -1 \leq x_1 \leq x_2 \leq 1, \tag{13}$$

where $\delta = 1$ for $\alpha = -1/2$ and $\delta = min\{1, \alpha + 1/2\}$ for $\alpha > -1/2$.

Proof. If $\alpha = -1/2$, then $e_{-\frac{1}{2},n}(ix) = \frac{e^{is_n x}}{\sqrt{2}\,|cos(s_n)|}$, and the inequality (13) is obvious in this case.

For $\alpha \geq -1/2$, we consider the function $\psi_\alpha(u) = |u|^{\alpha+1/2}E_\alpha(iu)$. By (10) and (11), to prove (13) it is enough to show that

$$|\psi_\alpha(u_2) - \psi_\alpha(u_1)| \leq C\,|u_2 - u_1|^\delta, \tag{14}$$

for real numbers u_1 and u_2.

If $|u_2 - u_1| > 1$, then using (2) and (12) it is easy to see that $\sup_{u\in\mathbb{R}}|\psi_\alpha(u)| \leq C$. So (14) is obvious in this case.

Now, if $|u_2 - u_1| \leq 1$, we have to distinguish the following three cases:

(1) If $|u_2 - u_1| \leq 1$, $|u_1| \geq 1$ and $|u_2| \geq 1$, using the fact that $E_\alpha(ix)$ is the unique solution of the system (1) we obtain

$$\psi'_\alpha(u) = i\,u^{\alpha+1/2}E_\alpha(iu) + \left(\alpha + \frac{1}{2}\right)u^{\alpha-1/2}E_\alpha(-iu).$$

By (12) we get

$$\sup_{|u|\geq 1}|\psi'_\alpha(u)| \leq C.$$

And since $0 < \delta \leq 1$, (14) is proved.

(1) If $|u_2 - u_1| \leq 1$, $|u_1| \leq 1$ and $|u_2| \leq 1$, the power series representation of the Bessel function leads to the power series of the Dunkl kernel

$$E_\alpha(iu) = \sum_{k=0}^{\infty}\frac{(iu)^k}{\xi_\alpha(k)},$$

where

$$\xi_\alpha(2k) = \frac{2^{2k}k!\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)}\ and\ \xi_\alpha(2k+1) = \frac{2^{2k+1}k!\Gamma(k+\alpha+2)}{\Gamma(\alpha+1)}.$$

So $E_\alpha(iu)$ is an entire function and we have

$$\begin{aligned}
|\psi_\alpha(u_2)-\psi_\alpha(u_1)| &\leq |u_2^{\alpha+1/2}|\,|E_\alpha(iu_2)-E_\alpha(iu_1)| + \||u_2^{\alpha+1/2}| - |u_1^{\alpha+1/2}\|\,|E_\alpha(iu_1)| \\
&\leq |u_2-u_1|\sup_{|u|\leq 1}|E'_\alpha(iu)| + C\,|u_2-u_1|^{\alpha+1/2}\sup_{|u|\leq 1}|E_\alpha(iu)| \\
&\leq C|u_2-u_1|^\delta,
\end{aligned}$$

where C is independent of u_1 and u_2

(1) For the case $|u_2-u_1|<1$, $|u_1|<1$ and $|u_2|>1$, we divide the matter in two parts at the points 1 or -1 and we use the results established in the previous cases.

Lemma 2.2. *Let $-1\leq a<b\leq 1$ and $(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}$. For $\alpha\geq -1/2$, there exists a constant C verifying*

$$\left|\int_a^b e_{\alpha,m}(ix)e_{\alpha,n}(ix)dx\right|\leq C\left\{(b-a)\left|\frac{m}{n}\right|^\delta + \frac{\log^+(|n|(b-a))}{|n|}+\frac{1}{|n|}\right\}, \tag{15}$$

where δ is the same as in Lemma 2.1, and

$$\log^+ x = \begin{cases}\log x & \textit{for} \quad x\geq 1\\ 0 & \textit{for} \quad 0<x<1\end{cases}$$

For $(m,n)=(0,0)$, we have $|\int_a^b(e_{\alpha,0}(ix))^2dx|\leq 1$.

Proof. Let K be the greatest non-negative integer such that $\frac{2\pi K}{|s_n|}\leq b-a$. We have the following three cases:

(1) If $0\leq a<b\leq 1$, let $x_k=a+\frac{2\pi k}{|s_n|}$ for $k\in\{0,1,\ldots,K\}$ and $x_{K+1}=b$. Then we can write

$$\int_a^b e_{\alpha,m}(ix)e_{\alpha,n}(ix)dx = \sum_{k=0}^K A_k^{(1)} + A_k^{(2)},$$

where

$$A_k^{(1)} = \int_{x_k}^{x_{k+1}}\left(e_{\alpha,m}(ix)-e_{\alpha,m}(ix_k)\right)e_{\alpha,n}(ix)\,dx,$$

and

$$A_k^{(2)} = e_{\alpha,m}(ix_k)\int_{x_k}^{x_{k+1}} e_{\alpha,n}(ix)\,dx.$$

From Lemma 2.1 and the inequality (2), we conclude that

$$\begin{aligned} |A_k^{(1)}| &\leq C|m|^\delta \int_{x_k}^{x_{k+1}} |x-x_k|^\delta dx \\ &\leq C|m|^\delta \left(\frac{2\pi}{|s_n|}\right)^\delta (x_{k+1}-x_k) \\ &\leq C\left|\frac{m}{n}\right|^\delta (x_{k+1}-x_k). \end{aligned}$$

The last inequality is a consequence of (10), and we get

$$\sum_{k=0}^{K} |A_k^{(1)}| \leq C\left|\frac{m}{n}\right|^\delta (b-a). \tag{16}$$

For the estimation of the term $A_k^{(2)}$, we remark that for $\alpha \geq -1/2$ and $k \in \{0, K\}$, using (2) we obtain

$$|A_k^{(2)}| \leq C \int_{x_k}^{x_{k+1}} dx \leq C\frac{2\pi}{|s_n|} \leq \frac{C}{|n|}. \tag{17}$$

For $k \in \{1, 2, \ldots, k-1\}$, the asymptotic formulas (10), (11) and (12) permit to see that for $\frac{2\pi}{|s_n|} \leq x_1 \leq x_k \leq x$, we have

$$e_{\alpha,n}(ix) = \frac{2^\alpha \Gamma(\alpha+1)}{\sqrt{\pi}|j_\alpha(s_n)|} \frac{e^{i(s_n x-(\alpha+\frac{1}{2})\frac{\pi}{2})}}{|s_n|^{\alpha+1/2}} + O\left(\frac{1}{|n|x}\right),$$

where O depends only on α. Then for $k \in \{1, 2, \ldots, K-1\}$, we have

$$|A_k^{(2)}| \leq C\left| \int_{x_k}^{x_{k+1}} \left(e^{i(s_n x-(\alpha+\frac{1}{2})\frac{\pi}{2})} + O\left(\frac{1}{|n|x}\right)\right) dx \right|.$$

Since $\int_{x_k}^{x_{k+1}} e^{i(s_n x-(\alpha+\frac{1}{2})\frac{\pi}{2})} dx = 0$, for $k \in \{1, 2, \ldots, K-1\}$,

$$|A_k^{(2)}| \leq \frac{C}{|n|} \int_{x_k}^{x_{k+1}} \frac{dx}{x} = \frac{C}{|n|}(\log x_{k+1} - \log x_k).$$

It follows that

$$\begin{aligned} \sum_{k=1}^{K-1} |A_k^{(2)}| &\leq \frac{C}{|n|}(\log x_K - \log x_1) \\ &\leq \frac{C}{|n|}\log K \\ &\leq \frac{C}{|n|}\log^+ \frac{|s_n|}{2\pi}(b-a) \\ &\leq \frac{C}{|n|}(1 + \log^+ |n|(b-a)). \end{aligned} \tag{18}$$

By (16), (17) and (18) we have the inequality (15) in this case.

(1) If $-1 \leq a < b \leq 0$, the same steps as in the first case are applied by taking $x_k = b - \frac{2\pi k}{|s_n|}$ for $k \in \{0, 1, \ldots, K\}$ and $x_{K+1} = a$.

(2) The case where $-1 \leq a < 0 < b \leq 1$, is a consequence from the first and the second cases, since we can write

$$\left| \int_a^b e_{\alpha,m}(ix) e_{\alpha,n}(ix) dx \right| \leq \left| \int_a^0 e_{\alpha,m}(ix) e_{\alpha,n}(ix) dx \right| + \left| \int_0^b e_{\alpha,m}(ix) e_{\alpha,n}(ix) dx \right|.$$

The integrals on the right hand side of the last inequality cover respectively the second and the first cases' conditions. So there exist two positive constants C_1 and C_2, such that

$$\begin{aligned} \left| \int_a^b e_{\alpha,m}(ix) e_{\alpha,n}(ix) dx \right| &\leq C_1 \left\{ (-a) \left| \frac{m}{n} \right|^\delta + \frac{\log^+(|n|(-a))}{|n|} + \frac{1}{|n|} \right\} \\ &\quad + C_2 \left\{ b \left| \frac{m}{n} \right|^\delta + \frac{\log^+(|n|(b))}{|n|} + \frac{1}{|n|} \right\} \\ &\leq C \left\{ (b-a) \left| \frac{m}{n} \right|^\delta + \frac{\log^+(|n|(b-a))}{|n|} + \frac{1}{|n|} \right\}. \end{aligned}$$

3. Duality between BMO and Hardy spaces

The duality between bounded mean oscillation (BMO) and Hardy spaces was studied extensively in Refs [10, 14–16] and others. The nonperiodic $BMO(\Delta)$ space is defined to be the space of functions $f \in L^1(\Delta)$, verifying

$$\|f\|_{BMO} = \mathcal{N}_\Delta(f) + \left| \int_\Delta f(x) dx \right| < \infty,$$

with

$$\mathcal{N}_\Delta(f) = \sup_I \frac{1}{|I|} \int_I |f(x) - f_I| dx,$$

where the supremum is taken over all subintervals I of Δ and

$$f_I = (1/|I|) \int_I f(x) dx.$$

The space $BMO(\Delta)$ endowed with the norm $\|f\|_{BMO}$ is a Banach space and its duality with the Hardy space $(\mathcal{H}(\Delta))^* = BMO(\Delta)$, plays an essential role in the proof of Theorem1.1. In particular, if $g \in L^\infty(\Delta) \subset BMO(\Delta)$ and $f \in \mathcal{H}(\Delta)$, we have the following inequality

$$\left| \int_\Delta f(x) g(x) dx \right| \leq C \|f\|_{\mathcal{H}(\Delta)} \|g\|_{BMO(\Delta)}, \tag{19}$$

where C is an absolute constant.

Remark 3.1. *For every subinterval $I\subset\Delta$ and any constant c, we have*

$$\frac{1}{|I|}\int_I |f(x)-f_I|dx \le \frac{2}{|I|}\int_I |f(x)-c|dx,$$

for a function f on Δ.

The next proposition is the key tool to prove the Paley's inequality.

Proposition 3.1. *Let $\{r_k\}_{k=1}^{\infty}$ be a sequence such that $\sum_{k=1}^{\infty}|r_k|^2<\infty$ and*

$$g_N(x)=\sum_{k=1}^{N} r_k\left(e_{\alpha,n_k}(ix)+e_{\alpha,-n_k}(ix)\right),$$

for a positive integer N. Then

$$\|g_N\|_{BMO(\Delta)} \le C\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2}, \tag{20}$$

with a constant C independent of N and the sequence $\{r_k\}_{k=1}^{\infty}$.

Proof. Knowing that

$$\left|\int_\Delta g_N(x)dx\right| \le 2\,\|g_N\|_{L^2(\Delta)} = 4\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2},$$

to prove (20), it is enough to show that

$$\mathcal{N}_\Delta(g_N) \le C\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2}, \tag{21}$$

where the constant C is independent of I, N and the sequence $\{r_k\}_{k=1}^{\infty}$. According to Remark 3.1, it is sufficient to verify that for every subinterval $I\subset\Delta$, there exists a constant c_I such that

$$\frac{1}{|I|}\int_I |g_N(x)-c_I|\,dx \le C\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2}.$$

Let $I=[x_1,x_2]$ be a subinterval of Δ, then if $|I|>1/n_1$, we have

$$\begin{aligned}\frac{1}{|I|}\int_I |g_N(x)|dx &\le \left(\frac{1}{|I|}\int_I |g_N(x)|^2dx\right)^{1/2}\\ &\le n_1^{1/2}\left(\int_I |g_N(x)|^2dx\right)^{1/2}\\ &\le n_1^{1/2}\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2}.\end{aligned}$$

If there exists a positive integer M, such that $1/n_{M+1} < |I| < 1/n_M$, we show inequality (21) with $c_I = g_M(x_1)$. We write $g_N(x) = g_M(x) + E_{M,N}(x)$, with

$$E_{M,N}(x) = \sum_{k=M+1}^{N} r_k(e_{\alpha,n_k}(ix) + e_{\alpha,-n_k}(ix)).$$

It follows that

$$\frac{1}{|I|}\int_I |g_N(x) - g_M(x_1)|dx \le \frac{1}{|I|}\int_I |g_M(x) - g_M(x_1)|dx + \frac{1}{|I|}\int_I |E_{M,N}(x)|dx. \tag{22}$$

Using Schwarz's inequality and Lemma 2.1, we get

$$\begin{aligned}
|g_M(x) - g_M(x_1)|^2 &\le \sum_{k=1}^{M}|r_k|^2\sum_{k=1}^{M}|e_{\alpha,n_k}(ix) - e_{\alpha,-n_k}(ix_1) + e_{\alpha,-n_k}(ix) - e_{\alpha,-n_k}(ix_1)|^2 \\
&\le 2\sum_{k=1}^{M}|r_k|^2\sum_{k=1}^{M}|e_{\alpha,n_k}(ix) - e_{\alpha,-n_k}(ix_1)|^2 + |e_{\alpha,-n_k}(ix) - e_{\alpha,-n_k}(ix_1)|^2 \\
&\le C\sum_{k=1}^{M}|r_k|^2\sum_{k=1}^{M}n_k^{2\delta}|x - x_1|^{2\delta} \\
&\le C\,|I|^{2\delta}\sum_{k=1}^{M}|r_k|^2\sum_{k=1}^{M}n_k^{2\delta} \\
&\le C\,|I|^{2\delta}\,n_M^{2\delta}\sum_{k=1}^{M}|r_k|^2.
\end{aligned}$$

Since $\{n_k\}$ is a Hadamard sequence, it is possible to choose M such that $|I|\,n_M \le 1$, so that

$$|g_M(x) - g_M(x_1)|^2 \le C\sum_{k=1}^{M}|r_k|^2$$

and

$$\begin{aligned}
\frac{1}{|I|}\int_I |g_M(x) - g_M(x_1)|dx &\le \left(\frac{1}{|I|}\int_I |g_M(x) - g_M(x_1)|^2 dx\right)^{1/2} \\
&\le C\left(\sum_{k=1}^{M}|r_k|^2\right)^{1/2}.
\end{aligned} \tag{23}$$

Now we estimate the second integral on the right-hand side of (22), we have

$$\begin{aligned}\left(\frac{1}{|I|}\int_I |E_{M,N}(x)|dx\right)^2 &\leq \frac{1}{|I|}\int_I |E_{M,N}(x)|^2 dx\\ &\leq \sum_{\ell,k=M+1}^{N} \frac{|r_\ell||r_k|}{|I|}\left|\int_I (e_{\alpha,n_\ell}(ix)-e_{\alpha,-n_\ell}(ix))(e_{\alpha,n_k}(ix)-e_{\alpha,-n_k}(ix))dx\right|\\ &\leq -2cm\sum_{\ell,k=M+1}^{N}\frac{|r_\ell||r_k|}{|I|}\left[U_{n_\ell,n_k}+U_{n_\ell,-n_k}+U_{-n_\ell,n_k}+U_{-n_\ell,-n_k}\right],\end{aligned}$$

where

$$U_{p,q} = \left|\int_I e_{\alpha,p}(ix)e_{\alpha,q}(ix)dx\right|.$$

Under the assumption $n_\ell \leq n_k$ and by Lemma 2.2, we obtain

$$\frac{1}{|I|}\left|\int_I (e_{\alpha,n_\ell}(ix)-e_{\alpha,-n_\ell}(ix))(e_{\alpha,n_k}(ix)-e_{\alpha,-n_k}(ix))dx\right| \leq C\left\{\left|\frac{n_\ell}{n_k}\right|^\delta + \frac{\log^+(n_k|I|)}{|I|n_k} + \frac{1}{|I|n_k}\right\}.$$

Since $\{n_k\}$ is a Hadamard sequence, we have

$$\left(\frac{n_\ell}{n_k}\right)^\delta \leq \left(\frac{1}{\rho^\delta}\right)^{k-\ell}.$$

If we fix a positive number μ, with $0<\mu<1$, then there exists a constant C_μ verifying:

$$\begin{aligned}\frac{\log^+(n_k|I|)}{|I|n_k} &\leq C_\mu\left(\frac{1}{|I|n_k}\right)^\mu\\ &\leq C_\mu\left(\frac{1}{\rho^\mu}\right)^{k-\ell}.\end{aligned}$$

For the last inequality we used the fact that $|I|n_l > 1$ for $l \geq M+1$. Also, we have $1/(|I|n_k) \leq (1/\rho)^{k-\ell}$. So we deduce that there exist two constants C and σ, with $0<\sigma<1$, such that

$$\frac{1}{|I|}\left|\int_I (e_{\alpha,n_\ell}(ix)-e_{\alpha,-n_\ell}(ix))(e_{\alpha,n_k}(ix)-e_{\alpha,-n_k}(ix))dx\right| \leq C\sigma^{|k-\ell|},$$

for $\ell, k \geq M+1$. As a consequence, there exists a constant C for which

$$\frac{1}{|I|}\int_I |E_{M,N}(x)|dx \leq C\left(\sum_{\ell,k=1}^{\infty}\sigma^{|k-\ell|}|r_\ell||r_k|\right)^{1/2}.$$

We estimate the sum in the right-hand side of the last inequality as follows

$$\sum_{\ell,k=1}^{\infty}\sigma^{|k-\ell|}|r_\ell||r_k| = \sum_{k=1}^{\infty}|r_k|^2 + 2\sigma\sum_{k=1}^{\infty}|r_{k+1}||r_k| + \ldots + 2\sigma^p\sum_{k=1}^{\infty}|r_{k+p}||r_k| + \ldots$$

Using Schwarz inequality, we deduce that

$$\sum_{\ell,k=1}^{\infty}\sigma^{|k-\ell|}|r_\ell||r_k| \le (1+2\sigma+\ldots+2\sigma^p+\ldots)\sum_{k=1}^{\infty}|r_k|^2$$

$$\le C\sum_{k=1}^{\infty}|r_k|^2. \tag{24}$$

Combining (23) and (24), we prove (20).

4. Proof of the theorem

Now, we come to the proof of Paley's inequality (8). Let $\{r_k\}_{k=1}^{\infty}$ be a sequence such that $\sum_{k=1}^{\infty}|r_k|^2 < \infty$ and $g_N(x) = \sum_{k=1}^{N} r_k(e_{\alpha,n_k}(ix) + e_{\alpha,-n_k}(ix))$, for $N = 1, 2, \ldots$ By (19), we obtain

$$\left|\int_{\Delta} f(x)g_N(x)dx\right| \le C\|f\|_{\mathcal{H}(\Delta)}\|g_N\|_{BMO(\Delta)},$$

for $f \in \mathcal{H}(\Delta)$. Since

$$\int_{\Delta} f(x)g_N(x)dx = \sum_{k=1}^{N}(c_{n_k}^{(\alpha)}(f) + c_{-n_k}^{(\alpha)}(f))r_k.$$

Using Proposition 3.1, we get

$$\left|\sum_{k=1}^{N}(c_{n_k}^{(\alpha)}(f) + c_{-n_k}^{(\alpha)}(f))\right| \le C\left(\sum_{k=1}^{\infty}|r_k|^2\right)\|f\|_{\mathcal{H}(\Delta)},$$

which leads to the inequality

$$\left\{\sum_{k=1}^{N}|c_{n_k}^{(\alpha)}(f)|^2 + |c_{-n_k}^{(\alpha)}(f)|^2\right\}^{1/2} \le C\|f\|_{\mathcal{H}(\Delta)},$$

Taking the limit as $N \to \infty$, we obtain (8).

To prove Hardy's inequality associated with the Fourier–Dunkl expansion, we consider the function $f \in \mathcal{H}(\Delta)$, there exists a unique sequence $\{a_k\}_{k=0}^{\infty}$ of Δ-atoms and a sequence $\{\lambda_k\}_{k=0}^{\infty}$, such that $f(x) = \sum_{k=0}^{\infty}\lambda_k a_k(x)$ *a.e.*, with

$$\sum_{k=0}^{\infty}|\lambda_k| \le C\|f\|_{\mathcal{H}(\Delta)}. \tag{25}$$

By (2), we see that

$$c_n^{(\alpha)}(f) = \sum_{k=0}^{\infty}\lambda_k c_n^{(\alpha)}(a_k),$$

and

$$\sum_{n=-\infty}^{\infty} \frac{|c_n^{(\alpha)}(f)|}{|n|+1} \le C \sum_{k=0}^{\infty} |\lambda_k| \sum_{n=-\infty}^{\infty} \frac{|c_n^{(\alpha)}(a_k)|}{|n|+1}.$$

Using (25), to show Hardy's inequality for the Fourier–Dunkl expansion, it is enough to show that

$$\sum_{n=-\infty}^{\infty} \frac{|c_n^{(\alpha)}(a)|}{|n|+1} \le C, \tag{26}$$

for any Δ-atom a and C independent of a. For the special case where $a = 1$ on Δ, the Schwarz's inequality and the Parseval's identity yield

$$\sum_{n=-\infty}^{\infty} \frac{|c_n^{(\alpha)}(a)|}{|n|+1} \le \left(\sum_{n=-\infty}^{\infty} \frac{1}{(|n|+1)^2} \right)^{1/2} \left(\int_{-1}^{1} dx \right)^{1/2} \le C.$$

If a is a Δ-atom with $I = [b, b+h]$ as a support interval, then we have

$$c_n^{\alpha}(a) = \int_b^{b+h} a(x) e_{\alpha,n}(ix) dx.$$

Since $\int_I a(x) dx = 0$, we can write

$$|c_n^{\alpha}(a)| = \left| \int_b^{b+h} a(x)(e_{\alpha,n}(ix) - e_{\alpha,n}(ib)) dx \right|.$$

Lemma 2.1 leads to

$$\begin{aligned} |c_n^{\alpha}(a)| &\le C \int_b^{b+h} |a(x)| \, |n|^{\delta} (x-b)^{\delta} \, dx \\ &\le C |n|^{\delta} \, \|a\|_2 \, h^{\delta+1/2}, \end{aligned}$$

where $\|a\|_2^2 = \int_\Delta |a(x)|^2 dx$.

On the other hand, the Δ-atom a satisfies the inequality $h \le \|a\|_2^{-2}$, so

$$|c_n^{\alpha}(a)| \le C |n|^{\delta} \, \|a\|_2^{-2\delta}. \tag{27}$$

If we denote $\gamma = \|a\|_2^2$, we write

$$\sum_{n=-\infty}^{\infty} \frac{|c_n^{\alpha}(a)|}{|n|+1} = \sum_{|n| \le \gamma} \frac{|c_n^{\alpha}(a)|}{|n|+1} + \sum_{|n| > \gamma} \frac{|c_n^{\alpha}(a)|}{|n|+1}. \tag{28}$$

Using (27), we get

$$\begin{aligned} \sum_{|n| \le \gamma} \frac{|c_n^{\alpha}(a)|}{|n|+1} &\le C \, \|a\|_2^{-2\delta} \sum_{|n| \le \gamma} \frac{|n|^{\delta}}{|n|+1} \\ &\le C \, \|a\|_2^{-2\delta} \gamma^{\delta} \le C. \end{aligned} \tag{29}$$

For the second sum in the right-hand side of (28), using Parseval's identity and Schwarz's inequality, we have

$$\sum_{|n|>\gamma} \frac{|c_n^\alpha(a)|}{|n|+1} \leq \|a\|_2 \left(\sum_{|n|>\gamma} \frac{1}{(|n|+1)^2} \right)^{1/2} \leq C \|a\|_2 \gamma^{-1/2} \leq C. \tag{30}$$

Combining (29) and (30) yields (26). This completes the proof of Hardy's inequality for the Fourier–Dunkl expansion.

At the end of this work, we should mention that Hardy space $\mathcal{H}(\Delta)$ can not be replaced by $L^1(\Delta)$ in (8) and (9). This condition is wellknown in the classical case and also in Ref. [8], where the author proved the existence of functions $f, g \in L^1(0,1)$ such that the series $\sum_{n=1}^{\infty} |c_n(f)|^2$ and the series $\sum_{n=1}^{\infty} |c_n(g)|/n$ diverge, where $c_n(f), (n = 1, 2, \ldots)$, represent the Bessel–Fourier coefficients of the function f.

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