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Zero-divisor graphs of twisted partial skew generalized power series rings

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Abstract

Purpose – The aim of this paper is to investigate the relationship between the ring structure of the twisted partial skew generalized power series ring $[[R^{G,\leq};\Theta]]$ and the corresponding structure of its zero-divisor graph $\overline{\Gamma}[[R^{G,\leq};\Theta]]$.

Design/methodology/approach – The authors first introduce the history and motivation of this paper. Secondly, the authors give a brief exposition of twisted partial skew generalized power series ring, in addition to presenting some properties of such structure, for instance, a-rigid ring, a-compatible ring and (G,a)-McCoy ring. Finally, the study's main results are stated and proved.

Findings – The authors establish the relation between the diameter and girth of the zero-divisor graph of twisted partial skew generalized power series ring $[[R^{G,\leq};\Theta]]$ and the zero-divisor graph of the ground ring R. The authors also provide counterexamples to demonstrate that some conditions of the results are not redundant. As well the authors indicate that some conditions of recent results can be omitted.

Originality/value – The results of the twisted partial skew generalized power series ring embrace a wide range of results of classical ring theoretic extensions, including Laurent (skew Laurent) polynomial ring, Laurent (skew Laurent) power series ring and group (skew group) ring and of course their partial skew versions.

Keywords Twisted partial skew generalized power series ring, Zero-divisor graph, Diameter, Girth

Paper type Research paper

1. Introduction

During the last decades, several papers have studied the relation between the algebraic structure of rings and their related graphs. Perhaps one of the first papers connecting the graphs to rings dates back to 1963 when R. Swan [1] gave an elegant proof to a well-known theorem by Amitsur and Levitzki [2] that "the ring of all $n \times n$ matrices $M_n(R)$ over a commutative ring R satisfies the standard polynomial identity $S_{2n}(x) = 0$." Swan's proof is based completely on the use of graph theory. Also this connection was pointed out by C. Chao and M. Schutzenberger (see [3], p. 167]).

Recently, Beck's results on coloring of a commutative ring attract the interest of many mathematicians to explore the structure of rings through their zero-divisor graph [4]. Beck considered a commutative ring R as a simple graph whose vertices are all elements of R, such that two different vertices $x, y \in R$ are adjacent if and only if $xy = 0$. Beck's investigation of colorings was then continued by Anderson and Naseer in [5]. Anderson and Livingston redefined Beck's graph of a commutative ring R by restricting the vertices to be the set $Z^*(R)$ that consists of nonzero zero divisors of R and called such graph a zero-divisor graph of R (see

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[6]). Thereafter, Redmond extended this concept to noncommutative case, and he gave two different ways to define zero-divisor graph of a noncommutative ring R. The first is directed and denoted by $\Gamma(R)$ such that $x \to y$ is an edge between distinct vertices x and y if and only if $xy = 0$. The second graph is undirected and denoted by $\overline{\Gamma}(R)$ such that two different vertices x and y are adjacent if and only if $xy = 0$ or $yx = 0$ (see [7, 8]).

Afterward, many authors studied the relationship between zero-divisor graph of a ring R and zero-divisor graph of some of its extensions, for example, polynomial ring $R[x]$, formal power series ring $R[[x]]$ and skew generalized power series ring $R[S, \sigma]$ (see for example [9–11]).

In this paper, we consider the (undirected) zero-divisor graph $\overline{\Gamma}(R)$ of a ring R. For two distinct vertices x and y in $\overline{\Gamma}(R)$ the distance between x and y, denoted by $d(x, y)$, constitute the length of the shortest path connecting x and y, if such a path exists; otherwise $d(x, y) := \infty$. The diameter of a graph $\overline{\Gamma}(R)$ $\operatorname{diam}\left(\overline{\Gamma}(R)\right) := \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } \overline{\Gamma}(R)\}$ if $\overline{\Gamma}(R)$ has more than one vertex, and it is zero otherwise. A graph $\overline{\Gamma}(R)$ is called complete if all of its vertices are adjacent. The girth $\overline{\Gamma}(R)$, denoted by $\operatorname{gr}\left(\overline{\Gamma}(R)\right)$, is the length of the shortest cycle in $\overline{\Gamma}(R)$, provided $\overline{\Gamma}(R)$ contains a cycle; otherwise $\operatorname{gr}\left(\overline{\Gamma}(R)\right) := \infty$. Redmond in [8] proved that $\overline{\Gamma}(R)$ for any ring R is connected with $\operatorname{diam}\left(\overline{\Gamma}(R)\right) \leq 3$ and if $\overline{\Gamma}(R)$ contains a cycle then $\operatorname{gr}\left(\overline{\Gamma}(R)\right) = 3$ or 4.

The proof of many theorems is based on the following result given by Akbari and Mohammadian in [9].

Theorem 1.1. *Let R be a ring. Then $\overline{\Gamma}(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R)^2 = \{0\}$. Moreover, in the latter case, $Z(R)$ is an ideal of R.*

Axtell *et al.* in [10] proved that if R is a commutative ring with identity and not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then having any one of $\Gamma(R)$, $\Gamma(R[X])$ or $\Gamma(R[[X]])$ complete is enough to imply all three are complete. Using Theorem 1.1, Akbari and Mohammadian in [9] generalized Axtell's result for any arbitrary ring R. For skew generalized power series ring $R[[S, \omega]]$, Moussavi and Paykan in [11] proved the following theorem.

Theorem 1.2. [11, Theorem 3.3] *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring, S an a.n.u.p. monoid and $\omega: S \to End(R)$ a monoid homomorphism. Assume that R is S – compatible. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R[[S, \omega]])$ is complete.*

According to [12], a twisted partial skew generalized power series ring $\left[\left[R^{G,\preceq}; \Theta\right]\right]$ embraces a wide range of classical ring theoretic extensions, including Laurent (skew Laurent) polynomial ring $R\langle x\rangle$ ($R\langle x, \sigma\rangle$), Laurent (skew Laurent) power series ring $R\langle\langle x\rangle\rangle$ ($R\langle\langle x, \sigma\rangle\rangle$) and group (skew group) ring $R[G]$ ($R[G, \sigma]$) and of course their partial skew versions. Our purpose of this paper is to continue study the relationship between zero-divisor graph of a ring R and zero-divisor graph of twisted partial skew generalized power series ring $\left[\left[R^{G,\preceq}; \Theta\right]\right]$.

In the following section, we give a brief exposition of twisted partial skew generalized power series ring $\left[\left[R^{G,\preceq}; \Theta\right]\right]$, in addition to presenting some properties of such structure, for instance, α – rigid ring, α – compatible ring and (G, α) – McCoy ring.

In Section 3, our main results are stated and proved. We establish the relation between the diameter and girth of zero-divisor graph of twisted partial skew generalized power series ring $\left[\left[R^{G,\preceq}; \Theta\right]\right]$ and zero-divisor graph of the ground ring R. We also provide counterexamples to demonstrate that some conditions of the results are not redundant. As well we indicate that some conditions of recent results can be omitted.

2. Twisted partial skew generalized power series ring

The action of groups on sets is one of the crucial tools in study theory of representations and the algebraic structures of groups and rings. Partial action of groups on sets has been raised in functional analysis (see for instance [13, 14]), then it was studied from a purely algebraic point of view. In [15], Dokuchaev and Exel defined partial skew group rings and proved that, under some assumptions, it is an associative ring. As a parallel case of partial skew group ring, Cortes and Ferrero defined partial skew polynomial rings and studied its prime and maximal ideals [16]. Thereafter many contemporaneous researchers were interested in studying the transfer of a lot of properties such as right Goldie, Baer, ACC on right annihilators, right p.p. and right zip properties between partial skew polynomial rings, partial skew Laurent polynomial rings and their ground ring (see for instance [17, 18]). Fahmy *et al.* in [19] studied the transfer of right (left) zip property between the partial skew generalized power series ring $[[R^{G,\le};\alpha]]$ and its ground ring R. The twisted partial skew version was defined and studied in [12, 20].

Let us first recall the definition of an idempotent (unital) twisted partial action, which is inspired by [21, Example 2.1], [22, Section 4] and suits Definition 2.1 of [23].

Definition 2.1. An idempotent twisted partial action of a group G on a ring R is a triple $\Theta = (D;\ \alpha,\tau)$, where $D = \{D_s|\ D_s \text{ is a two}-\text{sided ideal in } R,\ s\in G\}$, $\alpha = \{\alpha_s|\ \alpha_s \text{ is a ring isomorphism from } D_{s^{-1}} \text{ to } D_s,\ s\in G\}$, and τ is a twisted map from $G\times G$ to $U(R)$, the group of units of R, satisfying the following postulates, for all u, v and w in G:

(i) D_u is generated by a central idempotent 1_u;

(ii) $D_{1_G} = R$ and α_{1_G} is the identity map of R;

(iii) $\alpha_v^{-1}(D_v\cap D_{u^{-1}}) = D_{v^{-1}}\cap D_{(uv)^{-1}}$;

(iv) $\tau(1_G,u) = \tau(u,1_G) = 1_R$;

(v) $\alpha_u(\alpha_v(a)) = \tau(u,v)\alpha_{uv}(a)\tau(u,v)^{-1}$ for each $a\in D_{v^{-1}}\cap D_{(uv)^{-1}}$;

(vi) $\alpha_u(a\tau(v,w))\tau(u,vw) = \alpha_u(a)\tau(u,v)\tau(uv,w)$ for each $a\in D_{u^{-1}}\cap D_v\cap D_{vw}$.

An ordered group $(G;\cdot,\preceq)$ is called a strictly ordered group if it is satisfying the condition, if $u,v,w\in G$ and $u\prec v$, then $uw\prec vw$ and $wu\prec wv$. A subset X of $(G;\cdot,\preceq)$ is said to be Artinian if every strictly decreasing sequence of elements of X is finite and that X is narrow if every subset of pairwise order-incomparable elements of X is finite.

The twisted partial skew generalized power series ring was introduced in [12, Definition 1.2] as follows.

Definition 2.2. Let R be a ring, $(G,\preceq)$ a strictly ordered group and Θ an idempotent twisted partial action of G on R. The twisted partial skew generalized power series ring $A = [[R^{G,\le};\Theta]]$ is the ring of all maps f: $G\to R$, where $f(s)$ belongs to the corresponding ideal D_s such that $supp(f) = \{s\in G|\ f(s)\neq 0\}$ is Artinian and narrow subset of G, with pointwise addition, and the product operation is defined by

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} \alpha_u(\alpha_{u^{-1}}(f(u))g(v))\tau(u,v)$$

and $(fg)(s) = 0$ if $X_s(f,g) = \varnothing$ for each $f,g\in A$, where

$$X_s(f,g) = \{(u,v)\in G\times G : uv = s,\ u\in supp(f),\ v\in supp(g)\}.$$

According to Krempa [24], an endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies $a = 0$ for $a\in R$. If there exists a rigid endomorphism σ of R, then R is said to be $\sigma-$rigid. In [25], Hashemi and Moussavi generalized $\sigma-$rigid rings by introducing

$\sigma-$ compatible rings. A ring R is called $\sigma-$ compatible if for each $a, b \in R$, $ab=0$ if and only if $a\sigma(b)=0$. If R is a ring, $(S, \preceq)$ a strictly ordered monoid and ω: $S \to End(R)$ a monoid homomorphism, Marks *et al.* in [26] extended such concepts to $S-$ compatible and $S-$ rigid rings. A ring R is said to be $S-$ compatible (respectively $S-$ rigid) if ω_s is compatible (respectively rigid) for every $s \in S$. The partial version of such concepts can be given as follows:

Definition 2.3. Let R be a ring, $(G, \preceq)$ a strictly ordered group and Θ an idempotent twisted partial action of G on R. The ring R is called partial $\alpha-$ compatible if whenever $s \in G$, $a \in D_s$, $b \in R$; $ab=0$ if and only if $\alpha_{s^{-1}}(a)b=0$.

According to Cortes [17] we adopt the following definition.

Definition 2.4. Let R be a ring, $(G, \preceq)$ a strictly ordered group and Θ an idempotent twisted partial action of G on R. The ring R is called partial $\alpha-$ rigid if $a \in D_s$ for $s \in G$ such that $\alpha_{s^{-1}}(a)a=0$, then $a=0$.

A ring R is called Armendariz if whenever the polynomials $f(x)=\Sigma_{i=0}^{m} a_i x^i$ and $g(x)=\Sigma_{j=0}^{n} b_j x^j$ in the polynomial ring $R[x]$, satisfy $f(x)g(x)=0$ implies that $a_i b_j=0$ for all $0 \le i \le m$ and $0 \le j \le n$. In [19], the partial skew version of Armendariz rings was defined as a natural extension of Definition 2 in [17]. In the light of the above definitions, we get the following:

Definition 2.5. Let R be a ring, $(G, \preceq)$ a strictly ordered group and Θ an idempotent twisted partial action of G on R. The ring R is called $(G, \alpha)-$Armendariz if for any $f, g \in A = [[R^{G,\preceq};\Theta]]$ such that $fg=0$, then $\alpha_{u^{-1}}(f(u))g(v)=0$ for each $u \in supp(f)$ and $v \in supp(g)$.

Similar to [27, Definition 3.11] a ring R is called right $(G, \alpha)-$ McCoy if whenever nonzero elements f, g of $[[R^{G,\preceq};\Theta]]$ satisfy $fg=0$, then there exists $0 \neq r \in R$ such that $fr=0$. Left $(G, \alpha)-$ McCoy rings is defined analogously. If R is both left and right $(G, \alpha)-$ McCoy, then we say R is $(G, \alpha)-$ McCoy ring.

Recall that a monoid S (resp. a group G) is called a unique product monoid (u.p., for short) if for any two nonempty finite subsets $X, Y \subseteq S$ (resp. G) there exist $x \in X$ and $y \in Y$ such that $xy \neq x'y'$ for every $(x', y') \in X \times Y \setminus \{(x, y)\}$, the element xy is called a u.p. element of $XY=\{st$: $s \in X, t \in Y\}$. The class of u.p. monoids (resp. groups) includes the right and the left totally ordered monoids (resp. groups), for more details see [28].

Definition 2.6. [26, Definition 4.11] Let $(S, \preceq)$ be an ordered monoid. Then $(S, \preceq)$ is called an Artinian narrow unique product monoid (or simply an a.n.u.p. monoid) if for every two Artinian and narrow subsets X and Y of S, there exists a u.p. element in the product XY.

3. Main results

In the following lemma, $Z_l^*(R)$ ($Z_r^*(R)$) denote to the set of nonzero left (right) zero divisors of a ring R.

Lemma 3.1. *Let R be a ring, $(G, \preceq)$ a strictly ordered a.n.u.p. group and $\Theta=(D;\alpha,\tau)$ an idempotent twisted partial action of the group G on the ring R.*

(i) If $f \in Z_l^*[[R^{G,\preceq};\Theta]]$ and R is partial $\alpha-$ compatible, then $f(s) \in Z_l^*(R)$ for some $s \in supp(f)$.

(ii) If $f \in Z_r^*[[R^{G,\preceq};\Theta]]$, then $f(s) \in Z_r^*(R)$ for some $s \in supp(f)$.

Proof. (i) Let $f \in Z_l^*\left[\left[R^{G,\preceq};\Theta\right]\right]$. Then there exists a nonzero element $g \in \left[\left[R^{G,\preceq};\Theta\right]\right]$ such that $fg = 0$. Since G is a.n.u.p., there exist $s \in supp(f)$ and $t \in supp(g)$ such that st is u.p. of $supp(f) \cdot supp(g)$. Thus $0 = (fg)(st) = \alpha_s(\alpha_{s^{-1}}(f(s))g(t))\tau(s,t)$. Since R is partial $\alpha -$ compatible, it follows that $f(s)g(t) = 0$. Hence $f(s) \in Z_l^*(R)$.

(ii) Let $f \in Z_r^*\left[\left[R^{G,\preceq};\Theta\right]\right]$. Then there exists a nonzero element $g \in \left[\left[R^{G,\preceq};\Theta\right]\right]$ such that $gf = 0$. Since G is a.n.u.p., there exist $t \in supp(g)$ and $s \in supp(f)$ such that ts is u.p. of $supp(g) \cdot supp(f)$. Therefore, $0 = (gf)(ts) = \alpha_s(\alpha_{s^{-1}}(g(t))f(s))\tau(t,s)$. It follows directly that $f(s) \in Z_r^*(R)$. □

The following example shows that the partial $\alpha -$ compatibility condition for the ring R in part (i) of the previous lemma is not superfluous.

Example 3.2. Let R be the infinite direct product of copies of a ring A and $(G, \preceq)$ the group of integers $\mathbb{Z}$ with the trivial order. For each positive integer i, consider the isomorphisms $\alpha_i: D_{-i} \to D_i$, where D_i is the ideal of R consists of all elements of R with zero in the first i components, that is, if $a \in D_i$ then a is of the form

$$(\underbrace{0,0,0,\ldots,0}_{i \text{ components}}, a_{i+1}, a_{i+2}, \ldots)$$

and $D_{-i} = R$ such that

$$\alpha_i(a_1, a_2, a_3, \ldots) = (\underbrace{0,0,0,\ldots,0}_{i \text{ components}}, a_1, a_2, a_3, \ldots.)$$

By adding the identity automorphism α_0 of the ring R, we get a construction of a twisted partial skew generalized power series ring $\left[\left[R^{G,\preceq};\Theta\right]\right]$ with trivial twisting, where $D = \{D_i | \text{ for each } i \in \mathbb{Z}\}$ and $\alpha = \left\{\alpha_i | \ \alpha_{-i} = \alpha_i^{-1} \text{ for each } i \in \mathbb{Z}\right\}$. We see that the ring R is not partial $\alpha -$ compatible, since $(1,0,0,0,\ldots)^2 \neq 0_R$ while $(\alpha_1(1,0,0,0,\ldots))(1,0,0,0,\ldots) = 0_R$. Now, consider the element $f \in \left[\left[R^{G,\preceq};\Theta\right]\right]$ defined by $f(-1) = (1,1,1,\ldots)$ and $f(i) = 0_R$ for all $i \in \mathbb{Z} \backslash \{-1\}$ and the element $g \in \left[\left[R^{G,\preceq};\Theta\right]\right]$ defined by $g(0) = (1,0,0,0,\ldots)$ and $g(i) = 0_R$ for all $i \in \mathbb{Z}^*$. Therefore, $fg = 0$, but $f(s) \notin Z^*(R)$ for all $s \in supp(f)$.

Lemma 3.3. *Let R be a ring, $(G, \preceq)$ a strictly ordered a.n.u.p. group and $\Theta = (D; \alpha, \tau)$ an idempotent twisted partial action of the group G on the ring R. Assume $D_s = R$ whenever $D_{s^{-1}} = R$, for any $s \in G$. If $f \in Z^*\left[\left[R^{G,\preceq};\Theta\right]\right]$, then $f(s) \in Z^*(R)$ for some $s \in supp(f)$.*

Proof. Let $f \in Z^*\left[\left[R^{G,\preceq};\Theta\right]\right]$. From Lemma 3.1 ($ii$), it is sufficient that to study the case $fg = 0$ for some nonzero element $g \in \left[\left[R^{G,\preceq};\Theta\right]\right]$. Since G is a.n.u.p., there exist $s \in supp(f)$ and $t \in supp(g)$ such that st is u.p. of $supp(f) \cdot supp(g)$. Therefore, $0 = (fg)(st) = \alpha_s(\alpha_{s^{-1}}(f(s))g(t))\tau(s,t)$. It follows that $\alpha_{s^{-1}}(f(s))g(t) = 0$, that is, $\alpha_{s^{-1}}(f(s)) \in Z^*(R)$. Using our assumption, we have either $\alpha_{s^{-1}}$ is an automorphism of R, therefore $f(s) \in Z^*(R)$, or we have $1_s \neq 1_R$, hence $f(s)(1_s - 1_R) = 0$ and $f(s) \in Z^*(R)$. □

Lemma 3.4. *Let R be a ring, $(G,\preceq)$ a strictly ordered a.n.u.p. group and $\Theta=(D;\alpha,\tau)$ an idempotent twisted partial action of the group G on the ring R. If $Z(R)$ is an ideal of R, then $[[R^{G,\preceq};\Theta]]/\left[\left[(Z(R))^{G,\preceq};\Theta\right]\right]$ is a domain.*

Proof. First observe that any nonzero element in $[[R^{G,\preceq};\Theta]]/\left[\left[(Z(R))^{G,\preceq};\Theta\right]\right]$ can be represented as an element $\overline{f}$ where $f(s)\notin Z(R)$ for all $s\in supp(f)$. Now, let $\overline{f},\overline{g}$ be non zero elements in $[[R^{G,\preceq};\Theta]]/\left[\left[(Z(R))^{G,\preceq};\Theta\right]\right]$ such that $\overline{f}\overline{g}=\overline{fg}=\overline{0}$. Then $fg\in\left[\left[(Z(R))^{G,\preceq};\Theta\right]\right]$. Since G is a.n.u.p., there exist $s\in supp(f)$ and $t\in supp(g)$ such that st is u.p. of $supp(f)\cdot supp(g)$. Therefore, $(fg)(st)=\alpha_s(\alpha_{s^{-1}}(f(s))g(t))\tau(s,t)\in Z(R)$. Since $Z(R)$ is an ideal, it follows that R has no nontrivial idempotents and α_s is an isomorphism of R for all $s\in G$. Hence $\alpha_{s^{-1}}(f(s))g(t)\in Z(R)$, so either $f(s)$ or $g(t)$ is nonzero zero divisors, a contradiction. □

Theorem 3.5. *Let $R\ncong\mathbb{Z}_2\times\mathbb{Z}_2$ be a ring, $(G,\preceq)$ a strictly ordered a.n.u.p. group and $\Theta=(D;\alpha,\tau)$ an idempotent twisted partial action of the group G on the ring R. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$ is complete.*

Proof. Adopting the proof of Theorem 3.3 in [11]. Suppose that $\overline{\Gamma}(R)$ is complete. Since $R\ncong\mathbb{Z}_2\times\mathbb{Z}_2$, it follows by [9, Theorem 5] that $Z(R)$ is an ideal of R. So, $R/Z(R)$ is a domain and α is a global action of G on R.

Suppose that $f\in Z[[R^{G,\preceq};\Theta]]\backslash\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$, then there exists a nonzero element $g\in[[R^{G,\preceq};\Theta]]$ such that $fg=0$ (or $gf=0$). Since $f\notin\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$, $\overline{f}$ is a nonzero element in the domain $[[R^{G,\preceq};\Theta]]/\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$ and $\overline{f}\overline{g}=\overline{0}$. We conclude that $g\in\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$. On other hand, consider the set $H=\{s\in supp(f)|\,f(s)\in Z(R)\}$. By Lemma 3.3, H is nonempty; so we can write f as a sum of two maps h and k, where $h(s)=\begin{cases}f(s), & s\in H\\ 0, & s\notin H\end{cases}$ and $k(s)=\begin{cases}f(s), & s\notin H\\ 0, & s\in H\end{cases}$. Since $supp(h)=H\subseteq supp(f)$ and $supp(k)=H^c\cap supp(f)\subseteq supp\ (f)$, it follows that $h,k\in[[R^{G,\preceq};\Theta]]$. Therefore, we have $0=fg=(h+k)g=hg+kg$. Since $h,g\in\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$, $Z(R)^2=0$, by [9, Theorem 5], and α is a global action, it follows that $hg=0$. Hence $kg=0$, which contradicts Lemma 3.3, where $k(s)\notin Z(R)$ for each $s\in supp(k)$. Therefore $Z[[R^{G,\preceq};\Theta]]\subseteq\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$.

Now, let $f,g\in Z[[R^{G,\preceq};\Theta]]\subseteq\left[\left[Z(R)^{G,\preceq};\Theta\right]\right]$. Then $f(s)g(t)=0$ for each $s,t\in G$. Since α is a global action, $\alpha_s(\alpha_{s^{-1}}(f(s))g(t))\tau(s,t)=0$ for each $s,t\in G$. So, $fg=0$ and $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$ is complete.

The converse is clear, since $\overline{\Gamma}(R)$ is induced subgraph of $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$. □

The following example explains why the case of $R\cong\mathbb{Z}_2\times\mathbb{Z}_2$ is excluded in Theorem 3.5.

Example 3.6. Let $R=\mathbb{Z}_2[x]/\langle x^2+x\rangle$ and $(G,\preceq)$ the group of integers $\mathbb{Z}$ with the trivial order. Let $D_0=R$, $D_1=\langle\overline{1+x}\rangle$, $D_{-1}=\langle\overline{x}\rangle$ and $D_i=\langle\overline{0}\rangle$ for each $i\in\mathbb{Z}\backslash\{0,\pm1\}$. Consider the identity automorphism α_0 of R and the isomorphism α_1: $D_{-1}\to D_1$ defined by $\alpha_1(\overline{x})=\overline{1+x}$. Then

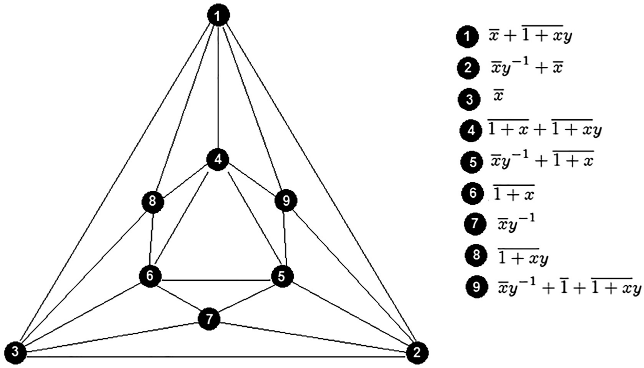
$[[R^{G,\preceq};\Theta]] \cong \{\sum_i a_i y^i \mid a_i \in D_i \text{ for each } i \in \mathbb{Z}\}$ with pointwise addition, and the product operation is defined by

$$(a_i y^i)(a_j y^j) = \alpha_i(\alpha_{-i}(a_i)a_j)y^{i+j}.$$

We conclude that

$$Z[[R^{G,\preceq};\Theta]] = \{a_0 + a_i y^i \mid a_0 \in Z(R), a_i \in D_i, i \in \mathbb{Z}\} \cup \{\overline{x}y^{-1} + \overline{1} + (\overline{1+x})y\}$$

and $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$ is given by the following planar graph.



Before continuing, it worth mention here that accurate tracking of the proof of Proposition 3.21 in [11] shows that the condition on S to be a.n.u.p. is superfluous. Therefore, we can give the following:

Proposition 1. *Let R be a ring that is not a domain, S a nontrivial monoid and $\omega: S \to End(R)$ a monoid homomorphism. Assume that R is $S-$compatible and $R[[S,\omega]]$ the skew generalized power series ring. Then $gr(\overline{\Gamma}(R[[S,\omega]]))$ is either 3 or 4. In particular, if R is not reduced, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$.*

Proof. We have two cases:

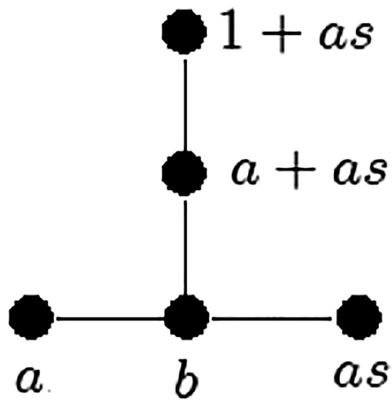
Case 1: If $|Z(R)| > 2$. Let $ab = 0$ for distinct elements $a, b \in Z^*(R)$. Using the $S-$compatibility of R, we find that $(c_a — c_b — c_a e_s — c_b e_s — c_a)$ is a $4-$cycle in $\overline{\Gamma}(R[[S,\omega]])$ for any $s \in S\backslash\{1\}$.

Case 2: If $|Z(R)| = 2$. Let $a^2 = 0$ for the nonzero element $a \in Z(R)$. Using the $S-$compatibility of R, we find that $(c_a — c_a e_s — (c_a + c_a e_s) — c_a)$ is a $3-$cycle in $\overline{\Gamma}(R[[S,\omega]])$ for any $s \in S\backslash\{1\}$. □

Unfortunately, the following example shows that the twisted partial skew version of Proposition 1 is not true.

Example 3.7. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0 = (0,0), a = (1,0), b = (0,1), 1 = (1,1)\}$, $G = \{1, s \mid s^2 = 1\}$, $D_1 = R$, $D_s = \langle a \rangle$, and α_u be the identity automorphism of D_u, $u \in G$. Then the twisted partial skew generalized power series ring

$[[R^{G,\preceq};\Theta]]$ with trivial twisting map is isomorphic to $\{0,1,a,b,as,1+as,a+as,b+as\}$, which is a subring of the group ring $R[G]$. Consequently, $\mathrm{gr}(\overline{\Gamma}[[R^{G,\preceq};\Theta]])=\infty$ and $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$ is the following tree



Proposition 2. *Let R be a ring, G a nontrivial group, and $[[R^{G,\preceq};\Theta]]$ a twisted partial skew generalized power series ring. If R is partial $\alpha-$compatible and $a\in D_u$, $b\in D_v$ are nonzero elements such that $ab=0$ where $u,v\in G\backslash\{1\}$, then $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$ contains a cycle. In particular, if a is nilpotent, then $gr(\overline{\Gamma}[[R^{G,\preceq};\Theta]])=3$.*

Proof. Let $a\in D_u, b\in D_v$ be nonzero distinct elements such that $ab=0$, where $u,v\in G\backslash\{1\}$. Using the partial $\alpha-$compatibility of R, we find that $(c_a\text{——}c_b\text{——}c_ae_u\text{——}c_be_v\text{——}c_a)$ is a $4-$cycle in $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$. In particular, if $a=b$, then $(c_a\text{——}c_ae_u\text{——}(c_a+c_ae_u)\text{——}c_a)$ is a $3-$cycle in $\overline{\Gamma}[[R^{G,\preceq};\Theta]]$. □

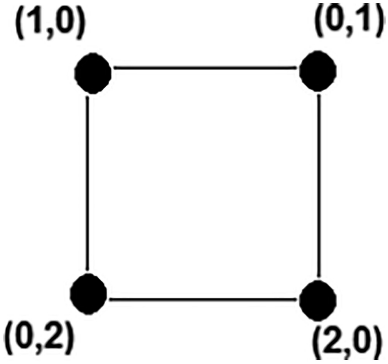
Remark 3.8. By [8, Theorem 3.3], we note that if R is partial $\alpha-$compatible, G a nontrivial group and $a\in D_u$, $b\in D_v$ are nonzero elements such that $ab=0$ where $u,v\in G\backslash\{1\}$ then $\mathrm{gr}(\overline{\Gamma}[[R^{G,\preceq};\Theta]])$ is either 3 or 4.

Theorem 3.9. *Let R be a ring, G a nontrivial a.n.u.p. group, and $[[R^{G,\preceq};\Theta]]$ a twisted partial skew generalized power series ring. If R is partial $\alpha-$rigid and $\overline{\Gamma}(R)$ contains a cycle, then $gr(\overline{\Gamma}(R))=gr(\overline{\Gamma}[[R^{G,\preceq};\Theta]])$.*

Proof. The proof is similar to the proof of Theorem 3.22 in [11]. □

The next example shows that the a.n.u.p. condition in Theorem 3.9 is not superfluous.

Example 3.10. Let $R=\mathbb{Z}_3\times\mathbb{Z}_3$, $G=\{1,s\,|\,s^2=1\}$, $D_1=R$, $D_s=\langle(1,0)\rangle$, and α_u be the identity automorphism of D_u, $u\in G$. Then the twisted partial skew generalized power series ring $[[R^{G,\preceq};\Theta]]$ with trivial twisting map is isomorphic to $\{a+bs|\,a\in R, b\in D_s\}$, which is a subring of the group ring $R[G]$. Since $[(1,0)+(1,0)s],[(1,0)+(2,0)s],[(0,1)+(0,1)s]\in[[R^{G,\preceq};\Theta]]$ consist a three-cycle, $\mathrm{gr}([[R^{G,\preceq};\Theta]])=3$. However $\mathrm{gr}(\overline{\Gamma}(R))=4$, since $\overline{\Gamma}(R)$ is the following four-cycle



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