

Geometric properties of the Bertotti–Kasner space-time

Bertotti–
Kasner
space-time

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Abstract

Purpose – The purpose of this paper is to study the Bertotti–Kasner space-time and its geometric properties.

Design/methodology/approach – This paper is based on the features of λ -tensor and the technique of six-dimensional formalism introduced by Pirani and followed by W. Borgiel, Z. Ahsan *et al.* and H.M. Manjunatha *et al.* This technique helps to describe both the geometric properties and the nature of the gravitational field of the space-times in the Segre characteristic.

Findings – The Gaussian curvature quantities specify the curvature of Bertotti–Kasner space-time. They are expressed in terms of invariants of the curvature tensor. The Petrov canonical form and the Weyl invariants have also been obtained.

Originality/value – The findings are revealed to be both physically and geometrically interesting for the description of the gravitational field of the cylindrical universe of Bertotti–Kasner type as far as the literature is concerned. Given the technique of six-dimensional formalism, the authors have defined the Weyl conformal λ_W -tensor and discussed the canonical form of the Weyl tensor and the Petrov scalars. To the best of the literature survey, this idea is found to be modern. The results deliver new insight into the geometry of the nonstatic cylindrical vacuum solution of Einstein's field equations.

Keywords Bertotti–Kasner space-time, Weyl invariants, Gaussian curvature, Gravitational field

Paper type Research paper

1. Introduction

Einstein's relativity theory is one of the successful theories of space-time and gravity. The space-time geometry describes one of the fundamental interactions in nature, namely gravity. Einstein's theory of relativity successfully reveals that space becomes curved in the presence of the gravitational field. The matter distribution determines the geometry of space-time. Albert Einstein introduced the field equations in 1915. Einstein's field equation (EFE) is a remarkable contribution in determining the motion of matter in a gravitational field as well as in determining the gravitational field from the distribution of matter. Among well-known exact solutions of EFE, the Schwarzschild solution is the most important. It preserves spherical symmetry. In 2011, Włodzimierz Borgiel [1] investigated the Schwarzschild space-time and its gravitational field. In [2], Musavvir Ali and Zafar Ahsan have studied the Kerr–Newman solution, which is the generalization of other well-known exact solutions of Einstein–Maxwell equations. The metric of Kerr–Newman space-time goes over into the Kerr

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metric, Reissner–Nordström metric, and Schwarzschild metric if the electric charge, the angular momentum per unit mass and both of them, respectively, are equal to zero. It reduces to Minkowski metric if the physical parameters such as mass, the angular momentum per unit mass and the electric charge vanish. They have studied the Schwarzschild soliton and its geometric properties in [3].

The Schwarzschild-de Sitter (SdS) solution is the generalization of the Schwarzschild solution. It is the spherically symmetric vacuum solution of EFE with a non-vanishing cosmological constant. SdS solution is not the only possible generalization of the Schwarzschild solution. Another possible generalization is the Bertotti–Kasner solution [4].

The Bertotti–Kasner solution is the non-static cylindrical vacuum solution of EFE. The Bertotti–Kasner space-time metric in Schwarzschild coordinates (t, r, θ, ϕ) with relativistic units ($G = c = 1$) is as follows: (see [4, 5])

$$ds^2 = dt^2 - e^{2\sqrt{\Lambda}t} dr^2 - \frac{1}{\Lambda} d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $\Lambda > 0$ denotes the cosmological constant.

According to Bertotti [6], Kasner [7] introduced the existence of this solution in 1925, but the explanation was not clear in the problem of the signature. In [6], it is found that the Bertotti–Kasner solution exists in the absence of the electromagnetic field. In the 1960s, many geometers and physicists have studied the spherically symmetric vacuum solutions. The Bertotti–Kasner solution characterizes the geometry of our universe as cylindrical. It is distinct from the solution due to the field around a spherical distribution of mass. So, Bonnor [8] neglected the Bertotti–Kasner solution. However, it has been drawn the attention of many geometers and physicists in recent days. One can see the discussion of geodesics on Bertotti–Kasner space, and hyper-spherical Bertotti–Kasner space in [9] with the famous Kruskal–Szekeres procedure. The Bertotti–Kasner solution is constructed in [10] from multiplets of scalar fields with a self-interacting potential in $3 + 1$ – dimensions. A discussion on Killing’s equations, Killing vectors, and time-like Killing vectors on Bertotti–Kasner space-time is found in [11].

The interesting feature of Bertotti–Kasner space-time metric is its mathematical simplicity and is purely geometric. It leads to the impression that our universe expands more in one particular direction. Some recent experimental evidence shows that our universe may have a particular direction, and in that direction the expansion velocity of the universe is maximum. In a galactic coordinate system, the experimental data [12] of Union2 type Ia supernova has given the evidence for the preferred direction, $(l, b) = (309^\circ, 18^\circ)$ of the universe. The experimental report of the Planck Collaboration [13] has confirmed the deviations from the isotropy with a significance level $\sim 3\sigma$ and hence given the evidence for the preferred direction.

The article is organized as follows. In Section 2, we discuss the canonical form, and the curvature invariants based on the technique of six-dimensional formalism. Hence, we analyze the curvature of Bertotti–Kasner space-time. The description of gravitational field is given by the features of λ -tensor. A glimpse of Weyl conformal λ_W -tensor is also given in Section 2. The paper ends with Section 3, where we have mentioned some important conclusions.

2. Curvature of the Bertotti–Kasner space-time

We have considered that the matrix $(g_{\mu\nu})$ has the signature $(+, -, -, -)$. In the Schwarzschild coordinates $x = (t, r, \theta, \phi)$, the matrix $(g_{\mu\nu})$ is as follows:

$$g_{\mu\nu}(x) = \begin{pmatrix} 1 & & & 0 \\ & -e^{2\sqrt{\Lambda}t} & & \\ & & \frac{1}{\Lambda} & \\ 0 & & & -\frac{\sin^2\theta}{\Lambda} \end{pmatrix}, \quad (2)$$

where $\mu, \nu = 0, 1, 2, 3$. The determinant of a matrix $(g_{\mu\nu})$ is equal to $-\frac{e^{2\sqrt{\Lambda}t}\sin^2\theta}{\Lambda^2}$, which is less than zero. Hence it is a real space-time [14]. The Riemannian metric tensor $g_{\mu\nu}$ determines the nature of gravitational field potential, and for the Bertotti–Kasner space-time metric, it is given by

$$\Phi \approx \frac{1}{2}(g_{00} - 1) = 0.$$

We deduce that the gravitational field potential of Bertotti–Kasner space-time metric is approximately equal to zero. Christoffel symbols are the functions constructed by certain combinations of partial differential coefficients of the metric tensor $g_{\mu\nu}$. Let $\Gamma_{\beta\gamma}^\alpha$ denote the Christoffel symbols of second kind defined by

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\delta}(\partial_\gamma g_{\beta\delta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\gamma\beta}). \quad (3)$$

The independent non-vanishing components of $\Gamma_{\beta\gamma}^\alpha$ are as follows:

$$\begin{aligned} \Gamma_{01}^1 = \Gamma_{10}^1 = \sqrt{\Lambda}, \quad \Gamma_{11}^0 = \sqrt{\Lambda}e^{2\sqrt{\Lambda}t}, \\ \Gamma_{23}^3 = \Gamma_{32}^3 = \frac{\cos\theta}{\sin\theta}, \quad \Gamma_{33}^2 = -\cos\theta \sin\theta. \end{aligned}$$

Riemann and Christoffel introduced the tensor $R_{\beta\gamma\delta}^\alpha$ of type (1, 3). It is formed by metric tensor $g_{\mu\nu}$ and its partial derivatives up to second order. The curvature tensor (or Riemann tensor) of type (0, 4) can be expressed as

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\partial_{\beta\gamma}^2 g_{\alpha\delta} + \partial_{\alpha\delta}^2 g_{\beta\gamma} - \partial_{\beta\delta}^2 g_{\alpha\gamma} - \partial_{\alpha\gamma}^2 g_{\beta\delta}) + g_{\mu\nu}(\Gamma_{\beta\gamma}^\mu \Gamma_{\alpha\delta}^\nu - \Gamma_{\beta\delta}^\mu \Gamma_{\alpha\gamma}^\nu). \quad (4)$$

The independent nonzero components of $R_{\alpha\beta\gamma\delta}$ are

$$\begin{aligned} R_{1010}(x) &= \Lambda e^{2\sqrt{\Lambda}t}, \\ R_{3232}(x) &= -\frac{\sin^2\theta}{\Lambda}. \end{aligned}$$

The Ricci tensor $R_{\alpha\beta}$ is a covariant tensor of order 2 and is given by

$$R_{\alpha\beta} = R_{\alpha\beta\gamma}^\gamma.$$

We found that $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$. Let R denote the scalar curvature. It is a tensor of order zero given by

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

The scalar curvature of Bertotti–Kasner space-time is 4Λ . Therefore, it has a constant scalar curvature. Since components of Ricci tensor are proportional to metric tensor components,

scalar curvature is directly proportional to the cosmological constant, and hence Bertotti–Kasner space-time is an Einstein space.

The Kretschmann scalar is found to be

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 8\Lambda^2.$$

For Bertotti–Kasner space-time, Kretschmann scalar is also constant and is directly proportional to the square of the cosmological constant. The tensor $G_{\alpha\beta}$ of type (0, 2) is defined by

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$

is called the Einstein tensor. It plays a supreme role in Einstein’s relativity theory. We found that $G_{\alpha\beta} = -\Lambda g_{\alpha\beta}$. Hence EFE is given by

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0.$$

Given the anti-symmetric property

$$A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}),$$

the Weyl curvature tensor can be evaluated as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\gamma[\alpha}R_{\beta]\delta} - g_{\delta[\alpha}R_{\beta]\gamma} - \frac{1}{3}g_{\gamma[\alpha}g_{\beta]\delta}R. \tag{5}$$

The Weyl tensor is traceless, but it has the symmetric properties as Riemann tensor $R_{\alpha\beta\gamma\delta}$. If we contract on the couple of indices $\alpha\delta$ or $\beta\gamma$, the obtained tensor vanishes. The independent non-vanishing components of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ are as follows:

$$C_{1010}(x) = \frac{2\Lambda e^{2\sqrt{\Lambda}t}}{3}, \quad C_{2020}(x) = -\frac{1}{3}, \quad C_{3030}(x) = -\frac{\sin^2\theta}{3},$$

$$C_{2121}(x) = \frac{e^{2\sqrt{\Lambda}t}}{3}, \quad C_{3131}(x) = \frac{\sin^2\theta e^{2\sqrt{\Lambda}t}}{3}, \quad C_{3232}(x) = -\frac{2\sin^2\theta}{3\Lambda}.$$

We observed that at a point of Bertotti–Kasner space-time, some components of the Weyl tensor are non-vanishing. Hence Bertotti–Kasner space-time is not conformally flat.

Now, we switch onto the six-dimensional formalism to examine the bivector-tensors, the Riemann tensor and the Weyl tensor in a pseudo-Euclidean space \mathbb{R}^6 [15]. Let us consider the following identification to adopt the six-dimensional formalism [16]:

$$\alpha\beta : 23 \ 31 \ 12 \ 10 \ 20 \ 30 \tag{6}$$

$$U : 1 \ 2 \ 3 \ 4 \ 5 \ 6.$$

Given $g_{\alpha\beta}$ are the metric tensor components at a point of Bertotti–Kasner space-time, we define the bivector-tensor as follows:

$$g_{UV} = g_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}.$$

The bivector-tensor g_{UV} is non-singular, and has the signature (+, +, +, −, −, −). It satisfies the symmetric property, that is, $g_{UV} = g_{VU}$. The suffix pairs $\alpha\beta$, $\gamma\delta$ are skew-symmetric. The bivector-tensor g_{UV} has the following non-vanishing components:

$$g_{11}(x) = \frac{\sin^2 \theta}{\Lambda^2}, \quad g_{22}(x) = \frac{e^{2\sqrt{\Lambda}t} \sin^2 \theta}{\Lambda}, \quad g_{33}(x) = \frac{e^{2\sqrt{\Lambda}t}}{\Lambda},$$

$$g_{44}(x) = -e^{2\sqrt{\Lambda}t}, \quad g_{55}(x) = \frac{1}{\Lambda}, \quad g_{66}(x) = -\frac{\sin^2 \theta}{\Lambda}.$$

Now we relabel the Riemann tensor $R_{\alpha\beta\gamma\delta}$ as R_{UV} given the scheme of six-dimensional formalism. Because of the property $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$, the tensor R_{UV} satisfies the symmetric property, that is, $R_{UV} = R_{VU}$. The tensor R_{UV} has the following nonvanishing components:

$$R_{11}(x) = -\frac{\sin^2 \theta}{\Lambda}, \quad R_{44}(x) = \Lambda e^{2\sqrt{\Lambda}t},$$

$$R_{22}(x) = R_{33}(x) = R_{55}(x) = R_{66}(x) = 0.$$

Given identification (6) of the six-dimensional formalism, we relabel the Weyl tensor $C_{\alpha\beta\gamma\delta}$ as C_{UV} . Because of the property $C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}$, the tensor C_{UV} is symmetric, that is, $C_{UV} = C_{VU}$. The tensor C_{UV} has the following nonvanishing components:

$$C_{11}(x) = -\frac{2\sin^2 \theta}{3\Lambda}, \quad C_{22}(x) = \frac{\sin^2 \theta e^{2\sqrt{\Lambda}t}}{3}, \quad C_{33}(x) = \frac{e^{2\sqrt{\Lambda}t}}{3},$$

$$C_{44}(x) = \frac{2\Lambda e^{2\sqrt{\Lambda}t}}{3}, \quad C_{55}(x) = \frac{1}{3}, \quad C_{66}(x) = -\frac{\sin^2 \theta}{3}.$$

The λ -tensor is defined as $R_{UV} - \lambda g_{UV}$. The curvature invariants are roots of the characteristic equation $|R_{UV}(x) - \lambda g_{UV}(x)| = 0$, and for Bertotti–Kasner space-time, they are obtained as follows:

$$\lambda_1(r) = \lambda_4(r) = -\Lambda, \quad (7)$$

$$\lambda_2(r) = \lambda_3(r) = \lambda_5(r) = \lambda_6(r) = 0. \quad (8)$$

The curvature tensor has the following canonical form:

$$R_{U'V'} = \begin{pmatrix} -\Lambda & & & & & 0 \\ & 0 & & & & \\ & & 0 & & & \\ & & & \Lambda & & \\ & & & & 0 & \\ 0 & & & & & 0 \end{pmatrix}.$$

Also, we have

$$g_{U'V'} = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix}.$$

From the features of λ -tensor $R_{UV} - \lambda g_{UV}$, we analyze the description of the gravitational field in the Segre symbols (see Ref. [17]), and we found that it is of the type $G_1[(1111)(11)]$.

Under the algebraic structure of the Riemann tensor, we conclude that the Bertotti–Kasner space-time (1) belongs to Type I in the Petrov’s classification (see Ref. [18]). It is important to notice that the geometry of Bertotti–Kasner space-time is both flat and curved. It reduces smoothly into Minkowski space-time and hence will become flat as $\Lambda \rightarrow 0$. This is shown in Figure 1.

The Weyl conformal λ_W -tensor $C_{UV} - \lambda_W g_{UV}$ is constructed from the symmetric tensors C_{UV} and g_{UV} . The roots of the characteristic equation $|C_{UV}(x) - \lambda_W g_{UV}(x)| = 0$ are called Weyl invariants, Petrov invariants or Petrov scalars. For Bertotti–Kasner space-time, Weyl invariants are as follows:

$$\lambda_{W_1}(r) = \lambda_{W_4}(r) = -\frac{2\Lambda}{3}, \tag{9}$$

$$\lambda_{W_2}(r) = \lambda_{W_3}(r) = \lambda_{W_5}(r) = \lambda_{W_6}(r) = \frac{\Lambda}{3}. \tag{10}$$

The determinant of the Weyl conformal λ_W -matrix $C_{UV}(x) - \lambda_W g_{UV}(x)$ is zero for any of the above Weyl invariants. The canonical form of the Weyl tensor or Petrov canonical form is given by

$$C_{U'V'} = \begin{pmatrix} -\frac{2\Lambda}{3} & & & & & 0 \\ & \frac{\Lambda}{3} & & & & \\ & & \frac{\Lambda}{3} & & & \\ & & & \frac{2\Lambda}{3} & & \\ & & & & -\frac{\Lambda}{3} & \\ 0 & & & & & -\frac{\Lambda}{3} \end{pmatrix}.$$

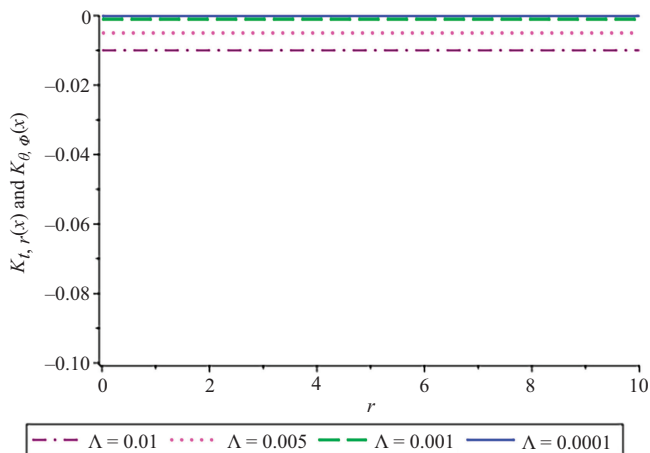


Figure 1.
The quantities $K_{t,r}(x)$ and $K_{\theta,\phi}(x)$ of Gaussian curvature for different values of Λ

Now, let us consider that $\theta = 0$ or $\theta = \pi$. This implies that $d\theta = 0$. Then the Bertotti–Kasner space-time metric (1) reduces to the following form:

$$ds'^2 = dt^2 - e^{2\sqrt{\Lambda}t} dr^2. \quad (11)$$

The matrix $(g'_{\mu\nu})$ of metric tensor components in coordinates $x' = (t, r)$ is given by

$$g'_{\mu\nu}(x') = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\sqrt{\Lambda}t} \end{pmatrix}, \quad (12)$$

where $\mu, \nu = 0, 1$. The matrix $(g'_{\mu\nu})$ has the determinant $-e^{2\sqrt{\Lambda}t}$. The hypersurface H'_0 or H'_π degenerates to the two-dimensional surface. Let $\Gamma'^\alpha_{\beta\gamma}$ denote the Christoffel symbols of second kind. The non-vanishing components of $\Gamma'^\alpha_{\beta\gamma}$ are

$$\Gamma'^0_{01} = \Gamma'^0_{10} = \sqrt{\Lambda}, \quad \Gamma'^0_{11} = \sqrt{\Lambda}e^{2\sqrt{\Lambda}t}.$$

For the metric (11), the Riemann tensor has only one nonzero component, and is given by

$$R'_{1010}(x') = \Lambda e^{2\sqrt{\Lambda}t}.$$

Moreover, the Gaussian curvature at each point $x' = (t, r)$ of the hypersurface H'_0 or H'_π is as follows:

$$K'(x') = \frac{R'_{1010}(x')}{\begin{vmatrix} 1 & 0 \\ 0 & -e^{2\sqrt{\Lambda}t} \end{vmatrix}} = -\Lambda. \quad (13)$$

We conclude that the Gaussian curvature of the hypersurface H'_0 or H'_π is constant. Further, every point is isotropic in the two-dimensional surface.

Next, for the case $\theta \in (0, \pi)$ and $\phi = 0$, the Bertotti–Kasner space-time metric (1) takes the form

$$ds''^2 = dt^2 - e^{2\sqrt{\Lambda}t} dr^2 - \frac{1}{\Lambda} d\theta^2. \quad (14)$$

The matrix $(g''_{\mu\nu})$ of metric tensor components in coordinates $x'' = (t, r, \theta)$ has the form

$$g''_{\mu\nu}(x'') = \begin{pmatrix} 1 & & 0 \\ & -e^{2\sqrt{\Lambda}t} & \\ 0 & & -\frac{1}{\Lambda} \end{pmatrix}, \quad (15)$$

where $\mu, \nu = 0, 1, 2$. The matrix $(g''_{\mu\nu})$ has the determinant $\frac{e^{2\sqrt{\Lambda}t}}{\Lambda}$. Let $\Gamma''^\alpha_{\beta\gamma}$ denote the Christoffel symbols of the second kind. The non-vanishing components of $\Gamma''^\alpha_{\beta\gamma}$ for the metric (14) are as follows:

$$\Gamma''^1_{01} = \Gamma''^1_{10} = \sqrt{\Lambda}, \quad \Gamma''^0_{11} = \sqrt{\Lambda}e^{2\sqrt{\Lambda}t}.$$

Similar to the case of a two-dimensional surface, we have only one non-vanishing component of the Riemann tensor for the metric (14) and is given by

$$R''_{1010}(x'') = \Lambda e^{2\sqrt{\Lambda}t}.$$

Hence the curvature of three-dimensional space at each point x'' is determined by one Gaussian curvature quantity $K''_{\theta}(x'')$. Further, we have

$$K''_{\theta}(x'') = \frac{R''_{1010}(x'')}{\begin{vmatrix} 1 & 0 \\ 0 & -e^{2\sqrt{\Lambda}t} \end{vmatrix}} = -\Lambda. \tag{16}$$

Since the Riemann tensor components $R''_{2020}(x'')$ and $R''_{1212}(x'')$ are equal to zero, the Gaussian curvature quantities $K''_r(x'')$ and $K''_t(x'')$ vanish at each point. Further, we have observed that the quantity $K''_{\theta}(x'')$ (Eqn (16)) is identical with $\lambda_1(r)$, $\lambda_4(r)$ and the quantities $K''_r(x'')$ and $K''_t(x'')$ are identical with $\lambda_2(r)$, $\lambda_3(r)$, $\lambda_5(r)$, $\lambda_6(r)$.

We pointed out that the curvature of Bertotti–Kasner space-time is determined by two quantities $K_{t,r}(x)$ and $K_{\theta,\phi}(x)$ of Gaussian curvature at each point x . The four quantities $K_{t,\theta}(x)$, $K_{t,\phi}(x)$, $K_{r,\theta}(x)$, and $K_{r,\phi}(x)$ of Gaussian curvature vanish at each point. The curvature index [19] of $K_{t,r}(x)$ and $K_{\theta,\phi}(x)$ is -1 , and that of $K_{t,\theta}(x)$, $K_{t,\phi}(x)$, $K_{r,\theta}(x)$ and $K_{r,\phi}(x)$ is 0. The six quantities of Gaussian curvature are shown in Eqn (17):

$$\left. \begin{aligned} K_{\theta,\phi}(x) &= K''_{\theta}(x'') = K'(x') = \lambda_4(r) = -\Lambda, \\ K_{t,\phi}(x) &= K''_t(x'') = \lambda_2(r) = 0, \\ K_{r,\phi}(x) &= K''_r(x'') = \lambda_3(r) = 0, \\ K_{t,r}(x) &= \lambda_1(r) = -\Lambda, \\ K_{t,\theta}(x) &= \lambda_5(r) = 0, \\ K_{r,\theta}(x) &= \lambda_6(r) = 0. \end{aligned} \right\} \tag{17}$$

From Eqn (17) we have observed that all the six Gaussian curvature quantities are expressed in terms of curvature invariants. The radial coordinate r is plotted versus $K_{t,r}(x)$ and $K_{\theta,\phi}(x)$ in Figure 1. Here, we have considered different values of the cosmological constant like $\Lambda = 0.01$, $\Lambda = 0.005$, $\Lambda = 0.001$, and $\Lambda = 0.0001$. We deduce from Figure 1 that the quantities $K_{t,r}(x)$ and $K_{\theta,\phi}(x)$ of Gaussian curvature tend to zero as Λ decreases. So that as $\Lambda \rightarrow 0$, all six quantities of Gaussian curvature in Eqn (17) approach to zero, and hence Bertotti–Kasner space-time smoothly becomes flat. In other words, it reduces into Minkowski space-time.

3. Conclusions

We have studied the geometric properties of the Bertotti–Kasner space-time. It is found that in Bertotti–Kasner space-time, every point is isotropic. In two orientations $K = -\Lambda$ and the other four orientations $K = 0$. Further, we have analyzed the following:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \\ R'_{\alpha\beta\gamma\delta} &= K(g'_{\alpha\gamma}g'_{\beta\delta} - g'_{\alpha\delta}g'_{\beta\gamma}), \\ R''_{\alpha\beta\gamma\delta} &= K(g''_{\alpha\gamma}g''_{\beta\delta} - g''_{\alpha\delta}g''_{\beta\gamma}), \end{aligned}$$

where K denotes the Gaussian curvature equal to $-\Lambda$, hence we conclude that Bertotti–Kasner space-time is completely isotropic.

The canonical forms of tensors R_{UV} and C_{UV} are achieved in the pseudo-Euclidean space \mathbb{R}^6 concerning the orthonormal basis $p^1, p^2, p^3, p^4, p^5, p^6$ by six-dimensional formalism. Both curvature invariants and canonical form of the Riemann tensor R_{UV} describe the space-

time curvature, and hence they lead to the analysis of nature of the gravitational field. In four orthonormal directions p^i ($i = 1, \dots, 4$), the curvature invariants are equal to zero. However, in the remaining two orthonormal directions, they are nonzero and equal to $-\Lambda$ each. Therefore, we may deduce that the Bertotti–Kasner space-time has a constant gravitational field.

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