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On primality of Cartesian product of graphs

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Abstract

Purpose – The present work focuses on the primality and the Cartesian product of graphs.
Design/methodology/approach – Given a graph G, a subset M of V (G) is a module of G if, for a, b ∈ M and x ∈ V (G) \ M, xa ∈ E(G) if and only if xb ∈ E(G). A graph G with at least three vertices is prime if the empty set, the single-vertex sets and V (G) are the only modules of G.
Findings – Motivated by works obtained on "the Cartesian product of graphs" and "the primality," this paper creates a link between the two notions.
Originality/value – In fact, we study the primality of the Cartesian product of two connected graphs minus k vertices, where k ∈ {0, 1, 2}.

Keywords Graphs, Prime, Module, Indecomposability graph, Cartesian product of graphs
Paper type Research paper

1. Introduction

In our paper, $G = (V, E)$ always denotes a finite undirected graph where $V = V(G)$ is a non-empty and finite set, called the *vertex-set* of G and $E = E(G)$ is a set of pairs of distinct vertices called the *edge-set* of G. An edge $\{u, v\}$ of G is denoted by uv. Two distinct vertices u and v of G are *adjacent* whenever $uv \in E$; otherwise u and v are said to be *non-adjacent*. Given a finite and non-empty set V, $(V, \varnothing)$ is the empty graph on V, whereas $(V, [V]^2)$ is the complete graph where $[V]^2$ is the set of pairs of V. The *complement* of each graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ such that, for $x \neq y \in V$, $xy \in \overline{E}$ if and only if $xy \notin E$. Any graph with just one vertex is referred to as *trivial*.

Let $G = (V, E)$ be a graph and x be a vertex of G. A *neighbor* of x is vertex y of G such that $xy \in E$. The family of neighbors of x is called the *neighborhood* of x denoted by $N_G(v)$. The vertex x is said to be *pendant* if it has a unique neighbor.

The *degree* of x, denoted by $d_G(x)$, is to the number of its neighbors. For example, a vertex with degree zero is called an *isolated vertex*. The minimum vertex degree, known as the *minimum degree* of G is the smallest vertex degree of G denoted by $\delta(G)$.

The notation $u — v$ signifies that $uv \in E$ while $u \ldots v$ means that $uv \notin E$. For any two disjoint subsets I and J of V, $I — J$ (resp. $I \ldots J$) signifies for each $(x, y) \in I \times J$, $x — y$ (resp. $x \ldots y$). In particular whenever $I = \{x\}$, we denote $x — J$ (resp. $x \ldots J$). Furthermore, $x \sim J$ means $x — J$ or $x \ldots J$. The negation is denoted by $x \nsim J$.

Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. Given a non-empty vertex subset X of V, the *subgraph* of G *induced* by X is the subgraph $G[X]$

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$= (X, E \cap [X]^2)$. If X is a proper subset of V, $G[V \setminus X]$ is also denoted by $G - X$ and by $G - x$ whenever $X = \{x\}$.

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. A bijection f from V onto V' is an *isomorphism* from G onto H provided that, for $x \neq y \in V$, $xy \in E$ if and only if $f(x)f(y) \in E'$.

Given a graph $G = (V, E)$, a subset M of V is a *module* [1] (or a *clan* [2,3] or an *interval* [4,5]) of G provided that, for all $a, b \in M$ and $x \in V \setminus M$, $xa \in E$ if and only if $xb \in E$. Thus, M is a module of G if for all $x \in V \setminus M$, $x \sim M$. For example, the empty set, V and $\{x\}$ where $x \in V$ are modules of G called *trivial modules*. A two-element module of G is known as a *duo* [6]. The graphs that have no duo, are called *duo-free* graphs. A graph is *indecomposable* [5] if all its modules are trivial. An indecomposable graph with at least three vertices is a *prime graph* [7]. All graphs with two vertices at most are indecomposable. However, all the 3-vertex graphs are not prime. Notice that the graphs G and $\overline{G}$ share the same modules. Thus, G is prime if and only if $\overline{G}$ is prime. Given $n \geq 2$, the n-vertex graph denoted by P_n is defined on $\{1, \ldots, n\}$ as follows: for $i, j \in \{1, \ldots, n\}$, ij is an edge of P_n if $|i - j| = 1$. Each graph that is isomorphic to P_n is called a *path*. It is clear that a path with at least 4 vertices is a prime graph. A path with extremities x and y is denoted by (x, y)-path.

Let $G = (V, E)$ be a prime graph. A vertex x of G is *critical* if $G - x$ is not prime. The graph G is *critical* if all its vertices are critical. The *indecomposability graph* of the graph G, denoted by $\Im(G)$, is the graph defined on the set $V(G)$ as follows: for $u \neq v \in V(G)$, uv is an edge of $\Im(G)$ if $G - \{u, v\}$ is prime.

Given a graph $G = (V, E)$, a non-empty subset C of V is *a connected component of* G if for $x \in C$ and $y \in V \setminus C$, $xy \notin E$ and if, for $x \neq y \in C$, there is a sequence $x = x_0, \ldots, x_n = y$ of elements of C such that, for each integer i where $0 \leq i \leq n - 1$, $x_i x_{i+1} \in E$. Clearly, an isolated vertex of G constitutes a connected component of G. The graph G is *connected* if it has exactly one connected component.

The *Cartesian product* $G \square H$ of graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph such that the vertex-set is $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \square H)$ whenever $ab \in E_1$ and $x = y$ or $a = b$ and $xy \in E_2$. For any $h \in V_2$, the subgraph of $G \square H$ induced by $V_1 \times \{h\}$ is called a G-fiber and is denoted by G^h. The H-fiber could be defined similarly. Figure 1 gives an example of the Cartesian product of two graphs.

Consider the following immediate observation.

1.1 Observation

(1) A Cartesian product of two graphs is connected if and only if both factors are connected.

(2) Every proper module of a Cartesian product of two connected graphs is included in a fiber.

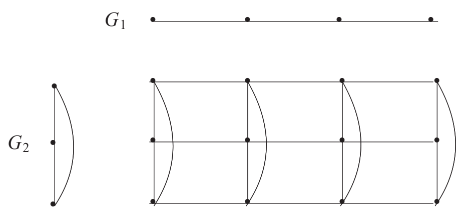

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Figure 1. $G1 \square G2$

The present work focuses on the primality and the Cartesian product of graphs. In the last few years, graph products have emerged again as a flourishing topic in graph theory. It has always been a good method to construct large graphs from small ones. Such products include: the *Categorical product* [8], the *Kronecker product* [9], the *Cardinal product* [10] and the *Cartesian product* [11–13]. The most widely used one that offers interesting problems may be the Cartesian product, which was first introduced by Sabidussi [14]. These types of graph products and other ones have been the subject of several papers [8,9,15–17]. On the other hand, the concept of primality has also been fundamental in the study of finite structures. Many questions on primality revolve around the study of its hereditary aspect in the graphs. Some papers have appeared along these lines [2,4,18–25]. In the case of graphs a first result dates back to D. P. Sumner [26]: *Every prime graph G with at least 4 vertices has a subgraph which is a P_4*. After that, A. Ehrenfeucht and G. Rozenberg [3] affirmed that the prime graphs have the following ascendant hereditary property: *Let X be a subset of a prime graph G such that $G[X]$ is prime. If $|V(G) \setminus X| \geq 2$, then there are $x \neq y \in V(G) \setminus X$ such that $G[X \cup \{x, y\}]$ is prime*. Later, J. H. Schmerl and W. T. Trotter [5] established the following decending. *For each prime graph G with at least 6 vertices, there are $a \neq b \in V(G)$ such that $G - \{a, b\}$ is prime*. To prove this result, the authors have introduced and described the critical graphs.

Motivated by works obtained on ”the Cartesian product of graphs” and ”the primality”, H. Kheddouci proposed to find a link between the two notions. Actually, he asked about the primality of Cartesian product and its subgraphs. This paper provides answers to questions asked by Kheddouci.

First, we establish that the primality of the Cartesian product of two graphs is essentially guaranteed by the connectedness of the two graphs. We obtain the following.

Theorem 1.2 Given two non-trivial connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $|V_1| \geq 3$ or $|V_2| \geq 3$, then the Cartesian product $G_1 \square G_2$ is prime.

Using Observation 1.1, we deduce the following corollary.

Corollary 1.3 Given two non-trivial graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $|V_1| \geq 3$ or $|V_2| \geq 3$, the Cartesian product $G_1 \square G_2$ is connected if and only if it is prime.

Second, we characterize all the vertex pairs of a Cartesian product of two connected graphs such that their suppression results in a prime graph. We obtain the following.

Theorem 1.4 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs such that $|V_1| \geq 3$ and $|V_2| \geq 3$. Consider distinct vertices $a = (x_a, y_a)$ and $b = (x_b, y_b)$ of $G_1 \square G_2$.

The pair $\{a, b\}$ is not an edge of the graph $\Im(G_1 \square G_2)$ if and only if one of the following conditions is satisfied.

1. There are distinct vertices x_0, x_1 and x_2 of G_1 and distinct vertices y_0, y_1 and y_2 of G_2 such that $N_{G_1}(x_0) = \{x_1\}$, $x_2 \in N_{G_1}(x_1)$, $N_{G_2}(y_0) = \{y_1\}$, $y_2 \in N_{G_2}(y_1)$ and

$$\{a, b\} = \begin{cases} \{(x_0, y_1), (x_1, y_0)\} & \textit{if } |N_{G_1}(x_1)| \geq 3 \text{ or } |N_{G_2}(y_1)| \geq 3 \\ \text{or} & \\ \{(x_i, y_j), (x_j, y_i)\} \text{ where } i \neq j \in \{0, 1, 2\} & \textit{otherwise}. \end{cases}$$

2. Either $x_a = x_b$, x_a is the neighbor of a pendant vertex of G_1 and $\{y_a, y_b\}$ is a duo of G_2, or $y_a = y_b$, y_a is the neighbor of a pendant vertex of G_2 and $\{x_a, x_b\}$ is a duo of G_1.

For example, for $G = P_3\square P_3$, $E(\overline{\Im}(P_3\square P_3)) = \{\{(1,1),(3,3)\},\{(1,3),(3,1)\},\{(1,2),(3,2)\},\{(1,2),(2,1)\},\{(2,1),(3,2)\},\{(3,2),(2,3)\},\{(2,3),(1,2)\},\{(1,2),(3,2)\},\{(2,1),(2,3)\}\}$ (see Figure 2)

Note that Schmerl and Trotter [5] have confirmed that each prime graph with at least 6 vertices has a non-empty indecomposability graph. Theorem 1.4 describes the edges of the indecomposability graph of the Cartesian product of two connected graphs.

The following corollary is an immediate consequence of Theorem 1.4.

Corollary 1.5 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs such that $|V_1| \geq 3$ and $|V_2| \geq 3$. The graph $\Im(G_1\square G_2)$ is complete if and only if one of the following conditions is satisfied.

(1) *min* $(\delta(G_1), \delta(G_2)) \geq 2$.

(2) *There are* $i \neq j \in \{1, 2\}$ *such that* $\delta(G_i) = 1$, $\delta(G_j) \geq 2$ *and* G_j *is duo-free.*

Finally, we prove that all the vertices of the Cartesian product of two connected graphs with at least 3 vertices are not critical. We obtain the following.

Theorem 1.6 Given two connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $|V_1| \geq 3$ and $|V_2| \geq 3$, then $G_1\square G_2$ has no critical vertex.

The text is organized as follows: Section 2 focuses on the proof of Theorem 1.2 while Section 3 is devoted to the proof of Theorem 1.4. However, Section 4 covers the third main result.

2. Proof of Theorem 1.2

Let two non-trivial connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $|V_1| \geq 3$ or $|V_2| \geq 3$. Denote $G = G_1\square G_2$, $V = V(G_1\square G_2)$ and $E = E(G_1\square G_2)$. Let us prove that $G_1\square G_2$ is prime. On the contrary, suppose that $G_1\square G_2$ is decomposable and consider I a nontrivial module of it. Since by Observation 1.1 $G_1\square G_2$ is connected, I cannot be a connected component of it. Thus, there is $z \in V \setminus I$ such that z— I. It follows from Observation 1.1 that $I \subseteq V(G_2^{x_z}) \cup V(G_1^{y_z})$. Accordingly the two following cases have to be distinguished.

- Case 1. Either $I \cap V(G_2^{x_z}) = \varnothing$ or $I \cap V(G_1^{y_z}) = \varnothing$.

Assume that $I \cap V(G_2^{x_z}) = \varnothing$ (resp. $I \cap V(G_1^{y_z}) = \varnothing$). Then, $I \subseteq G_1^{y_z}$ (resp. $I \subseteq G_2^{x_z}$). Based on Observation 1.1, since G_1 (resp. G_2) is connected, then there is $h \in V_1$ (resp. $h \in V_2$) such that $hx_z \in E_1$ (resp. $hy_z \in E_2$). Using the definition of $G_1\square G_2$ again, there is $\alpha \in V(G_2^h)$ (resp. $\alpha \in V(G_1^h)$) such that $\alpha \nsim I$; impossible.

- Case 2. $I \cap V(G_2^{x_z}) \neq \varnothing$ and $I \cap V(G_1^{y_z}) \neq \varnothing$.

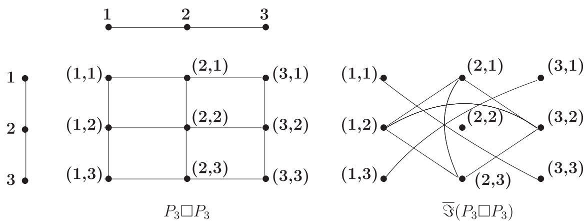

Figure 2. $P_3\square P_3$ and $\overline{\Im}(P_3\square P_3)$

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In this case, there are two distinct elements $u = (x_u, y_u)$ and $v = (x_v, y_v)$ of I such that $x_u = x_z$, $y_v = y_z$ and $x_u \neq x_v$. Using the definition of $G_1 \Box G_2$, the vertex $t = (x_v, y_u)$ verifies $tu \in E$ and $tv \in E$ because z— $\{u, v\}$. In addition, since $I \subseteq V(G_2^{x_z}) \cup V(G_1^{y_z})$, necessarily $t \notin I$. Hence, t—I. Moreover, for each $h \in (V(G_2^{x_z}) \cup V(G_1^{y_z}))\setminus\{u, v\}$, $th \notin E$ and thus $h \notin I$. Therefore, $I = \{u, v\}$. Consequently, if $|V_2| \geq 3$ (resp. $|V_1| \geq 3$), there is $\alpha \in V(G_2^{x_z})$ (resp. $\alpha \in V(G_1^{y_z})$) such that $\alpha u \in E$ and $\alpha v \notin E$ (resp. $\alpha u \notin E$ and $\alpha v \in E$); which is impossible.

3. Proof of Theorem 1.4

We start by the following obvious observations.

3.1 Observation

(1) Let G be a connected graph with at least 3 vertices. Then for any vertices x and y of G, there is a subgraph (not necessarily an induced one) in G containing x and y and isomorphic to P_n where $n \geq 3$.

(2) Let G_1 and G_2 be two connected graphs with at least 3 vertices. Then for any distinct vertices a and b of $G_1 \Box G_2$, there is a subgraph (not necessary an induced one) of $G_1 \Box G_2$ containing a and b and isomorphic to $P_n \Box P_m$ where $n \geq 3$, $m \geq 3$.

(3) Let m and n be two integers such that $m, n \geq 3$, $P_n \Box P_m$ be a Cartesian product and a and b be two distinct vertices of $P_n \Box P_m$. Then

- $(P_n \Box P_m) - \{a\}$ *is connected.*
- $(P_n \Box P_m) - \{a, b\}$ *is connected if and only if* $\{a, b\} \notin \{\{(1,2),(2,1)\}, \{(n-1,1),(n,2)\}, \{(1,m-1),(2,m)\}, \{(n,m-1),(n-1,m)\}$.

Notice that the second assertion of Observation 3.1 is an immediate consequence of the first one.

Lemma 3.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs such that $|V_1| \geq 3$ and $|V_2| \geq 3$ and let a and b be two distinct vertices of $G_1 \Box G_2$. Then $(G_1 \Box G_2) - \{a, b\}$ is not connected if and only if there are x, $x' \in V_1$ and y, $y' \in V_2$ such that $N_{G_1}(x) = \{x'\}$, $N_{G_2}(y) = \{y'\}$ and $\{a, b\} = \{(x', y), (x, y')\}$.

Proof. Consider two connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $|V_1| \geq 3$ and $|V_2| \geq 3$ and two distinct vertices a and b of $G_1 \Box G_2$. Let $G = G_1 \Box G_2$. Assume that $G - \{a, b\}$ is not connected. Then there are two distinct vertices $c = (x_c, y_c)$ and $d = (x_d, y_d)$ of $G - \{a, b\}$ such that there is no (c, d)-path in $G - \{a, b\}$. Based on Observation 3.1, there is a subgraph of G_1 (resp. G_2) containing x_c and x_d (resp. y_c and y_d) which is isomorphic to P_k where $k \geq 3$ (resp. P_l where $l \geq 3$). Hence, consider P'_n (resp P'_m) where $n, m \geq 3$, a longest path of G_1 (resp. G_2) containing x_c and x_d (resp. y_c and y_d) and $H' = P'_n \Box P'_m$.
If $a \notin V(H')$ or $b \notin V(H')$, then Observation 3.1 confirms that there is (c, d)-path in $G - \{a, b\}$; impossible. Presently, assume that both a and b are elements of $V(H')$. Given Observation 3.1, we have to consider only the case where $\{a, b\} \in \{\{(1,2), (2,1)\}, \{(n-1,1), (n,2)\}, \{(1,m-1),(2,m)\}, \{(n,m-1),(n-1,m)\}$. Without loss of generality, let $\{a, b\} = \{(1, 2), (2, 1)\}$. To end the proof, it is enough to prove that $N_{G_1}(1) = \{2\}$ and $N_{G_2}(1) = \{2\}$. On the contrary suppose that, for example, there is a vertex $h = (x_h, y_h)$ such that $x_h \in N_{G_1}(1)\setminus\{2\}$. In case $h \notin V(H')$, we obtain the subgraph defined on $\{1, 2, \ldots, n, x_h\}$ which is a path containing (x_c and x_d) and longer than P'_n, which contradicts the choice of P'_n. In case $h \in V(H')$, $H' - \{a, b\}$ is connected. Thus, there is a (c, d)-path in $G - \{a, b\}$; impossible.
Conversely, assume that there are x, $x' \in V_1$ and y, $y' \in V_2$ such that $N_{G_1}(x) = \{x'\}$,

$N_{G_2}(y) = \{y'\}$ and $\{a, b\} = \{(x', y), (x, y')\}$. It is clear that. $V(G) \setminus \{(x', y), (x, y'), (x, y)\}$ is a connected component of $G - \{a, b\}$. Therefore, $G - \{a, b\}$ is not connected. $\square$

3.2 Proof of theorem. 1.4

Let $a = (x_a, y_a)$ and $b = (x_b, y_b)$ be distinct elements of $V(G_1 \Box G_2)$. Denote $G = G_1 \Box G_2$, $V(G_1 \Box G_2) = V$ and $E(G_1 \Box G_2) = E$. Assume that $ab \notin E(\Im(G_1 \Box G_2))$. Hence, $G - \{a, b\}$ is not prime. If $G - \{a, b\}$ is not connected, then using Lemma 3.2, there are $x, x' \in V_1$ and $y, y' \in V_2$ such that $N_{G_1}(x) = \{x'\}, N_{G_2}(y) = \{y'\}$ and $\{a, b\} = \{(x', y), (x, y')\}$. So by considering $x_0 = x; x_1 = x', y_0 = y; y_1 = y'$ we obtain $\{a, b\} = \{(x_1, y_0), (x_0, y_1)\}$ and thus the first condition of Theorem 1.4 is verified. Assume that $G - \{a, b\}$ is connected. Let I be a non-trivial module of $G - \{a, b\}$. Obviously, I is not a connected component of $G - \{a, b\}$. Then there are $u = (x_u, y_u) \in I$ and $z = (x_z, y_z) \in V \setminus (I \cup \{a, b\})$ such that $uz \in E$. Based on the definition of G, $x_u = x_z$ or $y_u = y_z$. Without loss of generality, we may assume that $x_u = x_z$. Since I is a module of $G - \{a, b\}$, $z — I$. Then $I \subseteq V({G_2}^{x_u}) \cup V({G_1}^{y_z})$. We distinguish the two following cases.

- Case 1. Either $I \cap V({G_1}^{y_z}) = \varnothing$ or $I \cap V({G_2}^{x_u}) = \varnothing$.

Without loss of generality, we may assume that $I \cap V({G_1}^{y_z}) = \varnothing$. Hence, necessarily $I \subseteq V({G_2}^{x_u})$. Since G_1 is connected, there is $h \in V_1$ such that $hx_u \in E_1$. If $|I| \geq 3$, $|G_2^h| \geq 3$. Thus, there is $\alpha \in V(G_2^h) \setminus \{a, b\}$ such that $\alpha \nsim I$; impossible. Consequently, $|I| = 2$. As I is a duo of $G - \{a, b\}, N_{G_1}(x_u) = \{h\}$. Let $v = (x_u, y_v) \in I \setminus \{u\}$. It is clear that $\{y_u, y_v\}$ is a duo of G_2 and $\{a, b\} = \{(h, y_u), (h, y_v)\}$, thus verifying the second condition of Theorem 1.4.

- Case 2. $I \cap V({G_1}^{y_z}) \neq \varnothing$ and $I \cap V({G_2}^{x_u}) \neq \varnothing$.

In this case, there is $v = (x_v, y_v) \in I$ such that $y_v = y_z$ and $x_v \neq x_u$. As G_1 and G_2 are connected, using the definition of G, there is $t = (x_t, y_t) \in V$ such that $x_t = x_v, y_t = y_u$, $tu \in E$ and $tv \in E$.

First, prove that $t \notin \{a, b\}$. On the contrary, suppose that $t \in \{a, b\}$. For instance, assume that $t = a$. Since G_1 is connected and $|V_1| \geq 3$, there is $x_\alpha \in V_1 \setminus \{x_u, x_v\}$ such that $x_\alpha x_u \in E_1$ or $x_\alpha x_v \in E_1$. The two situations are studied as follows.

(1) In case $x_\alpha x_u \in E_1$. Since $I \subseteq V({G_2}^{x_u}) \cup V({G_1}^{y_z})$, $(x_\alpha, y_u) \notin I$. Moreover, $(x_\alpha, y_u) \nsim I$ because $(x_v, y_v) \ldots (x_\alpha, y_u) — (x_u, y_u)$. Thus, $(x_\alpha, y_u) = b$. Since G_2 is connected and $|V_2| \geq 3$, there is $y_\beta \in V_2 \setminus \{y_u, y_v\}$ such that $y_\beta y_u \in E_2$ or $y_\beta y_v \in E_2$. First, assume that $y_\beta y_v \in E_2$, then $(x_v, y_\beta) \notin I$ because $I \subseteq V({G_2}^{x_u}) \cup V({G_1}^{y_z})$. Moreover, based on the definition of G, $(x_u, y_u) \ldots (x_v, y_\beta) — (x_v, y_v)$; which contradicts the fact that I is a module of $G - \{a, b\}$. Second, assume that $y_\beta y_u \in E_2$ and $y_\beta \notin N_{G_2}(y_v)$ because otherwise we return to the first step. Observe that $(x_u, y_\beta) \notin I$, as $z \notin I$, $u \in I$ and $(x_u, y_\beta) \ldots (x_z, y_z) — (x_u, y_u)$. Furthermore, $(x_u, y_\beta) \nsim I$ because $(x_v, y_v) \ldots (x_u, y_\beta) — (x_u, y_u)$; which contradicts the fact that I is a module of $G - \{a, b\}$.

(2) In case $x_\alpha x_v \in E_1$, we may also assume that $x_\alpha x_u \notin E_1$ because otherwise we return to the first situation. Obviously, $(x_\alpha, y_z) \ldots (x_z, y_z)$ as $x_\alpha x_u \notin E_1$. Hence, $(x_\alpha, y_z) \notin I$. Moreover, $(x_\alpha, y_z) \nsim I$ because $(x_u, y_u) \ldots (x_\alpha, y_z) — (x_v, y_v)$. Thus $b = (x_\alpha, y_z)$. Since G_2 is a connected graph with at least 3 vertices, there is $y_\beta \in V_2 \setminus \{y_u, y_v\}$ such that $y_\beta y_u \in E_2$ or $y_\beta y_v \in E_2$. First, assume that $y_\beta y_v \in E_2$. Then $(x_v, y_\beta) \notin I$. Besides, using the definition of G, $(x_u, y_u) \ldots (x_v, y_\beta) — (x_v, y_v)$; which contradicts the fact that I is a module of $G - \{a, b\}$. Second assume that $y_\beta y_u \in E_2$ and $y_\beta y_v \notin E_2$ because otherwise we return to the first step. Evidently, $(x_u, y_\beta) \notin I$, since $z \notin I$, $v \in I$ and $(x_u, y_\beta) \ldots (x_z, y_z) — (x_u, y_u)$. Furthermore, $(x_u, y_\beta) \nsim I$, because $(x_v, y_v) \ldots (x_u, y_\beta) — (x_u, y_u)$; which contradicts the fact that I is a module of $G - \{a, b\}$.

In what remains, assume that $t \notin \{a, b\}$. Since $I \subseteq V(G_2{}^{x_u}) \cup V(G_1{}^{y_z})$, we have $t \notin I$. Moreover, since $u \in I$ and $tu \in E$, t — I. Using the definition of G, for each $h \in (V(G_2{}^{x_u}) \cup V(G_1{}^{y_z}))\setminus\{u, v\}$, $th \notin E$ and then $h \notin I$. Therefore, $I = \{u, v\}$.

First, assume that $d_{G_1}(x_u) = 1$ and $d_{G_2}(y_v) = 1$. Then necessarily $N_{G_1}(x_u) = \{x_v\}$ and $N_{G_2}(y_v) = \{y_u\}$. Since $|V_1| \geq 3$, $|V_2| \geq 3$ and $\{u, v\}$ is a module of $G - \{a, b\}$, we obtain $N_{G_1^{y_v}}(v)\setminus\{z\} = \{a\}$ or $\{b\}$ and $N_{G_2^{x_u}}(u)\setminus\{z\} = \{a\}$ or $\{b\}$. Thus, by considering, for example, $x_0 = x_u, x_1 = x_v, x_2 = x_a, y_0 = y_v, y_1 = y_u$ and $y_2 = y_b$, we have $\{a, b\} = \{(x_2, y_0), (x_0, y_2)\}$, thus verifying the first condition of Theorem 1.4.

Second, assume that $d_{G_1}(x_u) \geq 2$ or $d_{G_2}(y_v) \geq 2$. Without loss of generality, assume that $d_{G_1}(x_u) \geq 2$. Then there is $h \in V(G_1{}^{y_u})$ such that $h \in N_G(u)\setminus\{t\}$ and $h \nsim \{u, v\}$. Therefore, $h \in \{a, b\}$. For instance, assume that $h = a$. Since $|V_2| \geq 3$ and G_2 is connected, there is $w \in V_2$ such that $w \in N_{G_2}(y_u)$ or $w \in N_{G_2}(y_v)$. To begin with, assume that $w \in N_{G_2}(y_u)$. Since $(x_u, w) \nsim \{u, v\}$, $(x_u, w) = b$. Necessarily, $d_{G_1}(x_v) = 1$ and $d_{G_1}(x_u) = 2$ because otherwise there is $k \in V\setminus\{a, b\}$ such that $k \nsim \{u, v\}$. Similarly, $d_{G_2}(y_v) = 1$ and $d_{G_2}(y_t) = 2$. Hence, by considering $x_0 = x_v$, $x_1 = x_u, x_2 = x_a, y_0 = y_v, y_1 = y_u$ and $y_2 = y_b$, we obtain $\{a, b\} = \{(x_2, y_1), (x_1, y_2)\}$, thus verifying the first condition of Theorem 1.4. Now, we may assume that $w \in N_{G_2}(y_v)$ and $w \notin N_{G_2}(y_u)$ because otherwise we return to the first situation. Since $(x_v, w) \nsim \{u, v\}$, $(x_v, w) = b$. Necessarily, $d_{G_1}(x_v) = 1$ and $d_{G_1}(x_u) = 2$ because otherwise there is $k \in V\setminus\{a, b\}$ such that $k \nsim \{u, v\}$. Similarly, $d_{G_2}(y_u) = 1$ and $d_{G_2}(y_v) = 2$. It results, by considering $x_0 = x_v, x_1 = x_u$, $x_2 = x_a, y_0 = y_u, y_1 = y_v$ and $y_2 = y_b$ that $\{a, b\} = \{(x_2, y_0), (x_0, y_2)\}$, thus verifying the first condition of Theorem 1.4.

Conversely, assume that one of the conditions of Theorem 1.4 is satisfied and let us prove that $ab \notin E(\Im(G))$.

First, assume that the first condition is satisfied. Then there are distinct vertices x_0, x_1 and x_2 of G_1 and distinct vertices y_0, y_1 and y_2 of G_2 such that $N_{G_1}(x_0) = \{x_1\}$, $x_2 \in N_{G_1}(x_1)$, $N_{G_2}(y_0) = \{y_1\}$, $y_2 \in N_{G_2}(y_1)$ and

$$\{a, b\} = \begin{cases} \{(x_0, y_1), (x_1, y_0)\} & \textit{if } |N_{G_1}(x_1)| \geq 3 \text{ or } |N_{G_2}(y_1)| \geq 3 \\ \textit{or} & \\ \{(x_i, y_j), (x_j, y_i)\} \text{ where } i \neq j \in \{0, 1, 2\} & \textit{otherwise}. \end{cases}$$

Therefore, $V\setminus\{(x_0, y_1), (x_1, y_0), (x_0, y_0)\}$ or $\{(x_0, y_1), (x_1, y_0)\}$ or $\{(x_1, y_1), (x_0, y_0)\}$ is a non-trivial module of $G - \{a, b\}$. Thus, $ab \notin E(\Im(G))$.

Second, assume that the second condition is satisfied. If $x_a = x_b$, x_a is the neighbor of a pendant vertex of G_1 and $\{y_a, y_b\}$ is a duo of G_2 (resp. If $y_a = y_b$, y_a is the neighbor of a pendant vertex of G_2 and $\{x_a, x_b\}$ is a duo of G_1), in this case, consider x_1 (resp. y_1) to be a pendant vertex of G_1 (resp. G_2) such that $N_{G_1}(x_1) = \{x_a\}$ (resp. $N_{G_2}(y_1) = \{y_a\}$). Clearly, $\{(x_1, y_a), (x_1, y_b)\}$ (resp. $\{(x_a, y_1), (x_b, y_1)\}$) is a non-trivial module of $G - \{a, b\}$. Thus, $ab \notin E(\Im(G))$.□

4. Proof of Theorem 1.6

The proof of Theorem 1.6 is an immediate consequence of Theorem 1.2, the following result due to Y. Boudabbous and P. Ille and also the lemma below.

Lemma 4.1. [21] Let G be a prime graph with at least 5 vertices. For each critical vertex x of G, $|N_{\Im(G)}(x)| \leq 2$.

Lemma 4.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs such that $|V_1| \geq 3$ and $|V_2| \geq 3$. Then for each vertex $a \in V(G_1 \square G_2)$, $|N_{\Im(G_1 \square G_2)}(a)| \geq 3$.

Proof. Let $a=(x_a,y_a)\in V(G_1\square G_2)$ and note that $G=G_1\square G_2$, $V(G_1\square G_2)=V$ and $E(G_1\square G_2)=E$. The result is obvious when $|N_{\Im(G)}(a)|=|V|-1$. Then assume that $|N_{\Im(G)}(a)|<|V|-1$. Thus, there is $b=(x_b,y_b)\in V\setminus\{a\}$ such that $ab\notin E(\Im(G))$. Then one of the conditions of Theorem 1.4 is satisfied.

First, assume that there are distinct vertices x_0, x_1 and x_2 of G_1 and distinct vertices y_0, y_1 and y_2 of G_2 such that $N_{G_1}(x_0)=\{x_1\}$, $x_2\in N_{G_1}(x_1)$, $N_{G_2}(y_0)=\{y_1\}$, $y_2\in N_{G_2}(y_1)$ and

$$\{a,b\}=\begin{cases}\{(x_0,y_1),(x_1,y_0)\} & if\,|N_{G_1}(x_1)|\geq 3 \text{ or } |N_{G_2}(y_1)|\geq 3\\ or & \\ \{(x_i,y_j),(x_j,y_i)\} \text{ where } i\neq j\in\{0,1,2\} & otherwise.\end{cases}$$

To begin with, assume that $\{a,b\}=\{(x_0,y_1),(x_1,y_0)\}$. For instance, we may assume that $a=(x_0,y_1)$ and $b=(x_1,y_0)$. Consider the three vertices $c=(x_0,y_0)$, $d=(x_0,y_2)$ and $e=(x_1,y_1)$ of G. Since $\{y_a,y_0\}$, $\{y_a,y_2\}$ are not duos of G_2 and $\{x_a,x_1\}$ is not a duo of G_1, then using Theorem 1.4, the vertices c, d and e are neighbors of a in $\Im(G)$ and thus $|N_{\Im(G)}(a)|\geq 3$.

Now, assume that $\{a,b\}\in\{\{(x_0,y_2),(x_2,y_0)\},\{(x_1,y_2),(x_2,y_1)\}\}$. If $\{a,b\}=\{(x_0,y_2),(x_2,y_0)\}$, for example, we may assume that $a=(x_0,y_2)$ and $b=(x_2,y_0)$. Consider the two distinct vertices $c=(x_1,y_2)$ and $d=(x_0,y_1)$ of G. Since $\{y_a,y_1\}$ is not a duo of G_2 and $\{x_a,x_1\}$ is not a duo of G_1, then based on Theorem 1.4 the vertices c and d are neighbors of a in $\Im(G)$. Presently, consider the vertex $e=(x_0,y_0)$ of G. Since $N_{G_1}(x_0)=\{x_1\}$, x_0 is not a neighbor of a pendant vertex of G_1. So using Theorem 1.4, e is neighbor of a in $\Im(G)$ and thus $|N_{\Im(G)}(a)|\geq 3$.

At present, assume that $\{a,b\}=\{(x_1,y_2),(x_2,y_1)\}$. For instance, consider $a=(x_1,y_2)$ and $b=(x_2,y_1)$. Let $c=(x_0,y_2)$, $d=(x_2,y_2)$ and $e=(x_1,y_1)$. Since $\{y_a,y_1\}$ is not a duo of G_2 and $\{x_a,x_0\}$ and $\{x_a,x_2\}$ are not duos of G_1, then given Theorem 1.4, the vertices c, d and e are neighbors of a in $\Im(G)$ and thus $|N_{\Im(G)}(a)|\geq 3$.

Second, either $x_a=x_b$, x_a is the neighbor of a pendant vertex of G_1 and $\{y_a,y_b\}$ is a duo of G_2, or $y_a=y_b$, y_a is the neighbor of a pendant vertex of G_2 and $\{x_a,x_b\}$ is a duo of G_1. Without loss of generality, we may assume that $x_a=x_b$, x_a is the neighbor of a pendant vertex of G_1 and $\{y_a,y_b\}$ is a duo of G_2. Let $x_0\in V_1$ such that $N_{G_1}(x_0)=\{x_a\}$. As G_1 is a connected graph and $|V_1|\geq 3$, there is $x_1\in N_{G_1}(x_a)\setminus\{x_0\}$. Let $c=(x_0,y_a)$ and $d=(x_1,y_a)$. Observe that $\{x_0,x_a\}$ and $\{x_a,x_1\}$ are not duos of G_1, then $c,d\in N_{\Im(G)}(a)$ follows from Theorem 1.4. Since G_2 is connected, $|V_2|\geq 3$ and $\{y_a,y_b\}$ is a duo of G_2, there is $y_0\in V_2$ such that y_0— $\{y_a,y_b\}$.

If $y_ay_b\in E_2$, consider $e=(x_0,y_0)$. If y_0 is not a neighbor of a pendant vertex of G_2, then based on Theorem 1.4, $e\in N_{\Im(G)}(a)$ and thus $|N_{\Im(G)}(a)|\geq 3$. In case y_0 is a neighbor of a pendant vertex y_α of G_2, $\{y_a,y_\alpha\}$ is not a duo of G_2 because $y_ay_b\in E_2$ and y_α is a pendant vertex. Thus, Theorem 1.4 implies that $(x_a,y_\alpha)\in N_{\Im(G)}(a)$ and then $|N_{\Im(G)}(a)|\geq 3$. If $y_ay_b\notin E_2$, consider $e=(x_a,y_0)$. It is clear that $\{y_a,y_0\}$ is not a duo of G_2, therefore, based on Theorem 1.4, $e\in N_{\Im(G)}(a)$ and thus $|N_{\Im(G)}(a)|\geq 3$. □

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