The current issue and full text archive of this journal is available on Emerald Insight at:
https://www.emerald.com/insight/1319-5166.htm

On classification of $(n+6)$-dimensional nilpotent n-Lie algebras of class 2 with $n \geq 4$

Mehdi Jamshidi and Farshid Saeedi
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran, and
Hamid Darabi
Esfarayen University of Technology, Esfarayen, Iran

Received 24 September 2020
Revised 26 September 2020
Accepted 26 September 2020

Abstract

Purpose – The purpose of this paper is to determine the structure of nilpotent $(n+6)$-dimensional n-Lie algebras of class 2 when $n \geq 4$.

Design/methodology/approach – By dividing a nilpotent $(n+6)$-dimensional n-Lie algebra of class 2 by a central element, the authors arrive to a nilpotent $(n+5)$ dimensional n-Lie algebra of class 2. Given that the authors have the structure of nilpotent $(n+5)$-dimensional n-Lie algebras of class 2, the authors have access to the structure of the desired algebras.

Findings – In this paper, for each $n \geq 4$, the authors have found 24 nilpotent $(n+6)$ dimensional n-Lie algebras of class 2. Of these, 15 are non-split algebras and the nine remaining algebras are written as direct additions of n-Lie algebras of low-dimension and abelian n-Lie algebras.

Originality/value – This classification of n-Lie algebras provides a complete understanding of these algebras that are used in algebraic studies.

Keywords Nilpotent n-lie algebra, Classification, Nilpotent n-Lie algebra of class 2

Paper type Research paper

1. Introduction

In 1985, Filippov [1] introduced the concept of *n-Lie (Filippov) algebras*, as an n-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]].$$

Clearly, such an algebra becomes an ordinary Lie algebra when $n = 2$. Beside presenting many examples of n-Lie algebras, he also extended the notions of simplicity and nilpotency and determined all $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero.

The study of n-Lie algebras is important, since it is related to geometry and physics. Among other results, n-Lie algebras are classified in some cases. For example, Bai *et al.* [2] classified all n-Lie algebras of dimension $n+1$ over a field of characteristic 2. Also, they showed that there is no simple n-Lie algebra of dimension $n+2$. Then, Bai *et al.* [3] classified

**JEL Classification — ** Primary 17B05, 17B30, Secondary 17D99

Arab Journal of Mathematical Sciences
Vol. 27 No. 2, 2021
pp. 139-150
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-09-2020-0075

n-Lie algebras of dimension $n+2$ on the algebraically closed fields with characteristic zero. (see [4–7] for more information on the Filippov algebras).

In 1986, Kasymov [8] studied some properties of nilpotent and solvable n-Lie algebras. An n-Lie algebra A is *nilpotent* if $A^s = 0$ for some nonnegative integer s, where A^i is defined inductively by $A^1 = A$ and $A^{i+1} = [A^i, A, \dots, A]$. The n-Lie algebra A is nilpotent of class c, if $A^{c+1} = 0$ and $A^i = 0$ for each $i \le c$. The ideal $A^2 = [A, \dots, A]$ is called the *derived subalgebra* of A. The *center* of A is defined by

$$Z(A) = \{x \in A : [x, A, \dots, A] = 0\}.$$

Let $Z_0(A) = \langle 0 \rangle$. Then the ith center of A is defined inductively by

$$\frac{Z_i(A)}{Z_{i-1}(A)} = Z\left(\frac{A}{Z_{i-1}(A)}\right)$$

for all $i \ge 1$. Clearly, $Z_1(A) = Z(A)$.

The nilpotent theories of many algebras attract more and more attention. For example, in [9,10], and [11], the authors studied nilpotent Leibniz n-algebras, nilpotent Lie and Leibniz algebras, and nilpotent n-Lie algebras, respectively.

The $(n+3)$-dimensional nilpotent n-Lie algebras and $(n+4)$-dimensional nilpotent n-Lie algebras of class 2 were classified in [12]. Hoseini *et al.* [13] classified $(n+5)$-dimensional nilpotent n-Lie algebras of class 2.

In this paper, we have interest for algebras of class 2 (the minimal class for nonabelian case). The concept of filiform n-Lie algebras (maximal class) has been studied in some papers. For example, see [14].

The rest of our paper is organized as follows: Section 2 includes the results that are used frequently in the last section. In Section 3, we classify $(n+6)$-dimensional n-Lie algebras of class 2 when $n \ge 4$. For the case $n = 2$, this problem is dealt with by Yan *et al.* [15]. Also, the case $n = 3$ stated in [16].

2. Preliminaries

In this section, we introduce some known and necessary results. We denote d-dimensional abelian n-Lie algebra by $F(d)$. An important category of n-Lie algebras of class 2, which plays an essential role in classification of nilpotent n-Lie algebras, are algebras whose derived and center are equal. We call an n-Lie algebra A, a generalized Heisenberg of rank k, if $A^2 = Z(A)$ and $\dim A^2 = k$. The particular case $k = 1$, is called the special Heisenberg n-Lie algebras. The structure of this algebras defined as follows.

Theorem 2.1. [17] Every special Heisenberg n-Lie algebra has dimension $mn+1$ for some natural number m, and it is isomorphic to

$$H(n, m) = \langle x, x_1, \dots, x_{nm} : [x_{n(i-1)+1}, x_{n(i-1)+2}, \dots, x_{ni}] = x, i = 1, \dots, m \rangle.$$

Theorem 2.2. [18] Let A be a d-dimensional nilpotent n-Lie algebra, and let $\dim A^2 = 1$. Then, for some $m \ge 1$, it follows that

$$A \cong H(n, m) \oplus F(d - mn - 1).$$

Theorem 2.3. [18] Let A be a nonabelian nilpotent n-Lie algebra of dimension $d \le n+2$. Then A is isomorphic to $H(n, 1)$, $H(n, 1) \oplus F(1)$ or $A_{n,n+2,1}$, where
$A_{n,n+2,1} = \langle e_1, \dots, e_{n+2} : [e_1, \dots, e_n] = e_{n+1}, [e_2, \dots, e_{n+1}] = e_{n+2} \rangle$.

For unification of notation in what follows, the tth d-dimensional n-Lie algebra is denoted by $A_{n,d,t}$.

Theorem 2.4. [12] The $(n+3)$-dimensional nonabelian nilpotent n-Lie algebras for $n>2$ over an arbitrary field are $A_{n,n+3,i}(2\leq i\leq 8)$. Moreover nilpotent classes of $A_{n,n+3,2}$ and $A_{n,n+3,5}$ is two, nilpotent classes of $A_{n,n+3,3}$, $A_{n,n+3,4}$ and $A_{n,n+3,8}$ is three and finally, nilpotent classes of $A_{n,n+3,6}$ and $A_{n,n+3,7}$ is four (maximal class).

Theorem 2.5. [12] The only $(n+4)$-dimensional nilpotent n-Lie algebras of class 2 are $H(n,1)\oplus F(3)$, $A_{n,n+4,1}$, $A_{n,n+4,2}$, $A_{n,n+4,3}$, $H(2,2)\oplus F(1)$, $H(3,2)$, $L_{6,22}(\varepsilon)$, and $L^2_{6,7}(\eta)$.

Theorem 2.6. [13] The $(n+5)$-dimensional nilpotent n-Lie algebras of class 2 for $n>2$ over an arbitrary field are $H(n,1)\oplus F(4)$, $A_{n,n+5,i}(1\leq i\leq 7)$, $H(3,2)\oplus F(1)$ and $H(4,2)$.

Theorem 2.7. [19] Let A be a nilpotent n-Lie algebra of class 2. Then, there exist a generalized Heisenberg n-Lie algebra H and an abelian n-Lie algebra F such that $A=H\oplus F$.

3. Main results

In this section, we classify $(n+6)$-dimensional nilpotent n-Lie algebras of class 2. If n-Lie algebra A is nilpotent of class 2, then A is nonabelian and $A^2\subseteq Z(A)$. The nilpotent n-Lie algebra of class 2 plays an essential role in some geometry problems such as the commutative Riemannian manifold. Additionally, the classification of nilpotent Lie algebras of class 2 is one of the most important issues in Lie algebras.

The following theorems define the structure of generalized Heisenberg n-Lie algebras of rank 2 with dimension at most $2n+3$.

Theorem 3.1. [18] Let A be a nilpotent n-Lie algebra of dimension $d=n+k$ for $3\leq k\leq n+1$ such that $A^2=Z(A)$ and $\dim A^2=2$. Then

$$A\cong\langle e_1,\ldots,e_{n+k}:[e_{k-1},\ldots,e_{n+k-2}]=e_{n+k},[e_1,\ldots,e_n]=e_{n+k-1}\rangle.$$

Remark. In the above theorem for $n=2$ and $k=3$, we obtain

$$A\cong\langle e_1,e_2,e_3,e_4,e_5:[e_1,e_2]=e_4,[e_2,e_3]=e_5\rangle.$$

This algebra appears many times in differential geometry in the study of Pfaffian systems. It was developed by P. Libermann and introduced in [20].

Theorem 3.2. [19] Let A be a generalized Heisenberg n-Lie algebra of rank 2 with dimension $2n+2$. Then

$$A\cong A_{n,2n+2,1}=\langle e_1,\ldots,e_{2n+2}:[e_1,\ldots,e_n]=e_{2n+1},[e_{n+1},\ldots,e_{2n}]=e_{2n+2}\rangle.$$

Theorem 3.3. [19] Let A be a generalized Heisenberg n-Lie algebras of rank 2 with dimension $2n+3$. Then, A is isomorphic to one of the following n-Lie algebras:

$$A_{n,2n+3,1}=\langle e_1,\ldots,e_{2n+3}:[e_1,\ldots,e_n]=e_{2n+3},[e_2,\ldots,e_{n+1}]=[e_{n+2},\ldots,e_{2n+1}]=e_{2n+2}\rangle.$$

$$A_{n,2n+3,2}=\langle e_1,\ldots,e_{2n+3}:[e_1,\ldots,e_n]=[e_{n+1},\ldots,e_{2n}]=e_{2n+3},[e_2,\ldots,e_{n+1}]=[e_{n+2},\ldots,e_{2n+1}]=e_{2n+2}\rangle.$$

For $n=2$, we obtain also a Lie algebra of the previous type.

Now we are going to classify $(n+6)$-dimensional nilpotent n-Lie algebras of class 2.

According to Theorem 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n-Lie algebra of rank 2 and F is abelian. Therefore, first we classify the generalized Heisenberg n-Lie algebra of rank 2.

By the classification of nilpotent n-Lie algebras of class 2, we have the following theorem. All the algebras defined in theorem 3.4 and follow are in Table 1 at the end of the paper.

Theorem 3.4.

(1) The only $(n+4)$-dimensional generalized Heisenberg n-Lie algebra of rank 3 is $A_{n,n+4,3}$.

(2) The only $(n+5)$-dimensional generalized Heisenberg n-Lie algebras of rank 3 are $A_{n,n+5,4}$ and $A_{n,n+5,5}$.

(3) The only $(n+5)$-dimensional generalized Heisenberg n-Lie algebra of rank 4 is $A_{n,n+5,6}$.

The following lemma defines the structure of $(n+6)$-dimensional generalized Heisenberg n-Lie algebras of rank 2.

Theorem 3.5. Let A be a generalized Heisenberg n-Lie algebra of rank 2 with dimension $n+6$. Then

$$A \cong A_{n,n+6,1} = \langle e_1, \ldots, e_{n+6} : [e_1, \ldots, e_n] = e_{n+5}, [e_5, \ldots, e_{n+4}] = e_{n+6} \rangle.$$

Proof. For $n=4$, we have $n+6 = 2n+2$. Thus by Theorem 3.2, if $n \geq 5$, then $n+3 < n+6 \leq 2n+1$. Applying Theorem 3.1 completes the proof. ∎

Theorem 3.6. The only $(n+6)$-dimensional generalized Heisenberg n-Lie algebras of rank 3 are

$$A_{n,n+6,2},\ A_{n,n+6,3},\ A_{n,n+6,4},\ A_{n,n+6,5},\ \textit{and}\ A_{n,n+6,6}(\varepsilon).$$

Nilpotent n-Lie algebras of class 2	Nonzero multiplications
$A_{n,n+4,1}$	$[e_1, \ldots, e_n] = e_{n+3},\ [e_2, \ldots, e_{n+1}] = e_{n+4}$
$A_{n,n+4,2}$	$[e_1, \ldots, e_n] = e_{n+3},\ [e_3, \ldots, e_{n+2}] = e_{n+4} \quad (n \geq 3)$
$A_{n,n+4,3}$	$[e_1, \ldots, e_n] = e_{n+1},\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+3},\ [e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4}$
$A_{n,n+5,1}$	$[e_1, \ldots, e_n] = e_{n+4},\ [e_2, \ldots, e_{n+1}] = e_{n+5}$
$A_{n,n+5,2}$	$[e_1, \ldots, e_n] = e_{n+4},\ [e_3, \ldots, e_{n+2}] = e_{n+5} \quad (n \geq 3)$
$A_{n,n+5,3}$	$[e_1, \ldots, e_n] = e_{n+5},\ [e_4, \ldots, e_{n+3}] = e_{n+4} \quad (n \geq 3)$
$A_{n,n+5,4}$	$[e_1, \ldots, e_n] = e_{n+3},\ [e_2, \ldots, e_{n+1}] = e_{n+4},\ [e_1, e_3, \ldots, e_{n+1}] = e_{n+5}$
$A_{n,n+5,5}$	$[e_1, \ldots, e_n] = e_{n+3},\ [e_2, \ldots, e_{n+1}] = e_{n+4},\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+5}$
$A_{n,n+5,6}$	$[e_1, \ldots, e_n] = e_{n+3},\ [e_2, \ldots, e_{n+1}] = e_{n+4},\ [e_3, \ldots, e_{n+2}] = e_{n+5}$
$A_{n,n+5,7}$	$[e_1, \ldots, e_n] = e_{n+1},\ [e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+5},\ [e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4},\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+3}$

Table 1.

Proof. Suppose that A is an $(n+6)$-dimensional generalized Heisenberg n-Lie algebra of rank 3 with basis $\{e_1, \ldots, e_{n+6}\}$, which $n \geq 4$. Also, suppose that $A^2 = \langle e_{n+4}, e_{n+5}, e_{n+6}\rangle$. In this case, $A/\langle e_{n+6}\rangle$ is an $(n+5)$-dimensional nilpotent n-Lie algebra of class 2 with derived algebra of dimension 2. By Theorem 2.6, we have three possibilities for $A/\langle e_{n+6}\rangle$: Case 1: Let $A/\langle e_{n+6}\rangle \cong A_{n,n+5,1}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+4} + \alpha e_{n+6}, & \\ [e_2, \ldots, e_{n+1}] = e_{n+5} + \beta e_{n+6}, & \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+3}] = \gamma_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_n, e_{n+1}, e_{n+3}] = \delta_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_n, e_{n+2}, e_{n+3}] = \lambda_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_n, e_{n+1}, e_{n+2}, e_{n+3}] = \phi_{ijk} e_{n+6}, & 1 \leq i < j < k \leq n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$.

Since $\dim(A/\langle e_{n+4}, e_{n+5}\rangle)^2 = 1$, we have $A/\langle e_{n+4}, e_{n+5}\rangle \cong H(n, 1) \oplus F(3)$. According to the structure of n-Lie algebras, we conclude that one of the coefficients

$$\lambda_{ij} (1 \leq i < j \leq n), \quad \phi_{ijk} (1 \leq i < j < k \leq n)$$

is equal to one, and the others are zero. We have four possibilities:

(1) $\lambda_{12} = 1$, $\lambda_{ij} = 0 (1 \leq i < j \leq n, (i, j) \neq (1, 2))$, and $\phi_{ijk} = 0 (1 \leq i < j < k \leq n)$. In this case, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+4}, \quad [e_2, \ldots, e_{n+1}] = e_{n+5}, \quad [e_3, \ldots, e_n, e_{n+2}, e_{n+3}] = e_{n+6},$$

which we denote it by $A_{n,n+6,2}$.

(2) $\lambda_{23} = 1$, $\lambda_{ij} = 0 (1 \leq i < j \leq n, (i, j) \neq (2, 3))$, and $\phi_{ijk} = 0 (1 \leq i < j < k \leq n)$. In this case, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+4}, \quad [e_2, \ldots, e_{n+1}] = e_{n+5}, \quad [e_1, e_4, \ldots, e_n, e_{n+2}, e_{n+3}] = e_{n+6},$$

which we denote it by $A_{n,n+6,3}$.

(3) $\lambda_{ij} = 0 (1 \leq i < j \leq n)$, $\phi_{123} = 1$, and $\phi_{ijk} = 0 (1 \leq i < j < k \leq n, (i, j, k) \neq (1, 2, 3))$. In this case, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+4}, \quad [e_2, \ldots, e_{n+1}] = e_{n+5}, \quad [e_4, \ldots, e_{n+3}] = e_{n+6}.$$

One can easily see that this algebra is isomorphic to $A_{n+6,3}$.

(4) $\lambda_{ij} = 0 (1 \leq i < j \leq n), \phi_{234} = 1, \phi_{ijk} = 0 (1 \leq i < j < k \leq n), (i,j,k) \neq (2,3,4))$. In this case, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+4}, \quad [e_2, \ldots, e_{n+1}] = e_{n+5}, \quad [e_1, e_5, \ldots, e_{n+3}] = e_{n+6},$$

which we denote it by $A_{n,n+6,4}$.

Case 2. Let $A/\langle e_{n+6}\rangle \cong A_{n,n+5,2}$. Then the brackets in A can be written as

$$\begin{cases} [e_1,\dots,e_n] = e_{n+4} + \alpha e_{n+6}, & \\ [e_2,\dots,e_{n+1}] = e_{n+5} + \beta e_{n+6}, & \\ [e_1,\dots,\widehat{e_i},\dots,e_n,e_{n+1}] = \alpha_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\dots,\widehat{e_i},\dots,e_n,e_{n+2}] = \beta_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\dots,\widehat{e_i},\dots,e_n,e_{n+3}] = \gamma_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_n,e_{n+1},e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \le i < j \le n, \\ & (i,j) \ne (1,2), \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_n,e_{n+1},e_{n+3}] = \delta_{ij} e_{n+6}, & 1 \le i < j \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_n,e_{n+2},e_{n+3}] = \lambda_{ij} e_{n+6}, & 1 \le i < j \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,\widehat{e_k},\dots,e_n,e_{n+1},e_{n+2},e_{n+3}] = \phi_{ijk} e_{n+6}, & 1 \le i < j < k \le n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$.

Since $\dim(A/\langle e_{n+4},\, e_{n+5}\rangle)^2 = 1$, we have $A/\langle e_{n+4}, e_{n+5}\rangle \cong H(n,\, 1) \oplus F(3)$. According to the structure of n-Lie algebras and $Z(A) = \langle e_{n+4},\, e_{n+5},\, e_{n+6}\rangle$, we conclude that one of the coefficients

$$\gamma_i (1 \le i \le n), \qquad \delta_{ij} (1 \le i < j \le n),$$
$$\lambda_{ij} (1 \le i < j \le n), \quad \phi_{ijk} (1 \le i < j < k \le n)$$

is equal to one, and the others are zero. Similar to case 1, up to isomorphism, we have the following algebras:

$$[e_1,\dots,\, e_n] = e_{n+4}, [e_2,\dots,\, e_{n+1}] = e_{n+5}, [e_2,\dots,\, e_n,\, e_{n+3}] = e_{n+6},$$
$$[e_1,\dots,\, e_n] = e_{n+4}, [e_2,\dots,\, e_{n+1}] = e_{n+5}, [e_1,\, e_2,\, e_4,\dots,\, e_n,\, e_{n+3}] = e_{n+6},$$
$$[e_1,\dots,\, e_n] = e_{n+4}, [e_2,\dots,\, e_{n+1}] = e_{n+5}, [e_2,\, e_4,\dots,\, e_{n+1},\, e_{n+3}] = e_{n+6},$$
$$[e_1,\dots,\, e_n] = e_{n+4}, [e_2,\dots,\, e_{n+1}] = e_{n+5}, [e_1,\, e_2,\, e_5,\dots,\, e_{n+1},\, e_{n+3}] = e_{n+6}.$$

One can easily see that the first and second algebras are isomorphic to $A_{n,n+6,2}$ and $A_{n,n+6,3}$, respectively. The third and fourth algebras are denoted by $A_{n,n+6,5}$ and $A_{n,n+6,6}$, respectively, that is,

$$A_{n,n+6,5} = \langle e_1,\dots,\, e_{n+6} : [e_1,\dots,\, e_n] = e_{n+4}, [e_2,\dots,\, e_{n+1}] = e_{n+5}, [e_2,\, e_4,\dots,\, e_{n+1},\, e_{n+3}] = e_{n+6}\rangle,$$
$$A_{n,n+6,6} = \langle e_1,\dots,\, e_{n+6} : [e_1,\dots,\, e_n] = e_{n+4}, [e_2,\dots,\, e_{n+1}] = e_{n+5}, [e_1,\, e_2,\, e_5,\dots,\, e_{n+1},\, e_{n+3}] = e_{n+6}\rangle,$$

Case 3. Let $A/\langle e_{n+6}\rangle \cong A_{n,n+5,3}$. Then the brackets in A can be written as

$$\begin{cases} [e_1,\dots,e_n] = e_{n+4} + \alpha e_{n+6}, & \\ [e_2,\dots,e_{n+1}] = e_{n+5} + \beta e_{n+6}, & \\ [e_1,\dots,\widehat{e_i},\dots,e_n,e_{n+1}] = \alpha_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\dots,\widehat{e_i},\dots,e_n,e_{n+2}] = \beta_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\dots,\widehat{e_i},\dots,e_n,e_{n+3}] = \gamma_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_n,e_{n+1},e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \le i < j \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_n,e_{n+1},e_{n+3}] = \delta_{ij} e_{n+6}, & 1 \le i < j \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_n,e_{n+2},e_{n+3}] = \lambda_{ij} e_{n+6}, & 1 \le i < j \le n, \\ [e_1,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,\widehat{e_k},\dots,e_n,e_{n+1},e_{n+2},e_{n+3}] = \phi_{ijk} e_{n+6}, & 1 \le i < j < k \le n, \\ & (i,j,k) \ne (1,2,3). \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$.

Since $\dim(A/\langle e_{n+4}, e_{n+5}\rangle)^2 = 1$, we have $A/\langle e_{n+4}, e_{n+5}\rangle \cong H(n, 1) \oplus F(3)$. According to the structure of n-Lie algebra, we conclude that one of the coefficients

$$\alpha_i, \beta_i, \gamma_i (1 \le i \le n), \qquad \chi_{ij}, \delta_{ij}, \lambda_{ij} (1 \le i < j \le n),$$
$$\phi_{ijk} (1 \le i < j < k \le n, (i,j,k) \neq (1,2,3))$$

is equal to one, and the others are zero. Similar to case 1, up to isomorphism, we have the following algebras:

$$[e_1, \ldots, e_n] = e_{n+5}, [e_4, \ldots, e_{n+3}] = e_{n+4}, [e_2, e_3, \ldots, e_{n+1}] = e_{n+6},$$
$$[e_1, \ldots, e_n] = e_{n+5}, [e_4, \ldots, e_{n+3}] = e_{n+4}, [e_1, e_2, e_3, e_5, \ldots, e_{n+1}] = e_{n+6},$$
$$[e_1, \ldots, e_n] = e_{n+5}, [e_4, \ldots, e_{n+3}] = e_{n+4}, [e_2, e_3, e_5, \ldots, e_{n+2}] = e_{n+6}.$$

One can easily see that these algebras are isomorphic to $A_{n,n+6,3}$, $A_{n,n+6,4}$ and $A_{n,n+6,6}$, respectively. Therefore, there is no new algebra in this case. ∎

Theorem 3.7. The only $(n+6)$-dimensional generalized Heisenberg n-Lie algebras of rank 4 are

$$A_{n,n+6,7}, A_{n,n+6,8}, A_{n,n+6,9}, A_{n,n+6,10}, A_{n,n+6,11}, A_{n,n+6,12} \textit{ and } A_{n,n+6,13}.$$

Proof. Suppose that A is an $(n+6)$-dimensional generalized Heisenberg n-Lie algebra of rank 4 with basis $\{e_1, \ldots, e_{n+6}\}$, which $n \ge 4$. Also, suppose that $A^2 = \langle e_{n+3}, e_{n+4}, e_{n+5}, e_{n+6}\rangle$. In this case, $A/\langle e_{n+6}\rangle$ is an $(n+5)$-dimensional nilpotent n-Lie algebra of class 2 with derived algebra of dimension 3. By Theorem LABEL:?, we have three possibilities for $A/\langle e_{n+6}\rangle$:

Case 4. Let $A/\langle e_{n+6}\rangle \cong A_{n,n+5,4}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3} + \alpha e_{n+6}, & \\ [e_2, \ldots, e_{n+1}] = e_{n+4} + \beta e_{n+6}, & \\ [e_1, e_3, \ldots, e_{n+1}] = e_{n+5} + \gamma e_{n+6}, & \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 3 \le i \le n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \le i \le n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \le i < j \le n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$.

Since $\dim(A/\langle e_{n+3}, e_{n+4}, e_{n+5}\rangle)^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}, e_{n+5}\rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n-Lie algebras, we conclude that one of the coefficients

$$\alpha_i \, (3 \le i \le n), \quad \beta_i \, (1 \le i \le n), \quad \chi_{ij} \, (1 \le i < j \le n)$$

is equal to one, and the others are zero. According to $Z(A) = \{e_{n+3}, e_{n+4}, e_{n+5}, e_{n+6}\}$, the coefficients $\alpha_i (3 \le i \le n)$ cannot be equal to one. We have three possibilities:

(1) $\beta_1 = 1$, $\beta_i = 0 \, (2 \le i \le n)$, $\alpha_i = 0 \, (3 \le i \le n)$, $\chi_{ij} = 0 \, (1 \le i < j \le n)$.

In this case, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+3}, \quad [e_2, \ldots, e_{n+1}] = e_{n+4},$$
$$[e_1, e_3, \ldots, e_{n+1}] = e_{n+5}, \quad [e_2, \ldots, e_n, e_{n+2}] = e_{n+6},$$

which we denote it by $A_{n,n+6,7}$.

(2) $\beta_3 = 1$, $\beta_i = 0 \, (1 \le i \le n, n \neq 3)$, $\alpha_i = 0 \, (3 \le i \le n)$, $\chi_{ij} = 0 \, (1 \le i < j \le n)$.

In this case, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+3}, \quad [e_2, \ldots, e_{n+1}]e_{n+4},$$
$$[e_1, e_3, \ldots, e_{n+1}] = e_{n+5}, \quad [e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+6},$$

which we denote it by $A_{n,n+6,8}$.

(3) Only one of $\chi_{ij}s(1 \leq i < j \leq n)$ is equal to one and the others are zero. Up to isomorphism, we have the following algebras:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, & [e_2, \ldots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \ldots, e_{n+1}] = e_{n+5}, & [e_3, \ldots, e_{n+2}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, & [e_2, \ldots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \ldots, e_{n+1}] = e_{n+5}, & [e_2, e_4, \ldots, e_{n+2}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, & [e_2, \ldots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \ldots, e_{n+1}] = e_{n+5}, & [e_1, e_2, e_5, \ldots, e_{n+2}] = e_{n+6}. \end{cases}$$

One can easily see that the first and second algebras are isomorphic to $A_{n,n+6,7}$ and $A_{n,n+6,8}$, respectively. The third algebras is denoted by $A_{n,n+6,9}$.

Case 5. Let $A/\langle e_{n+6}\rangle \cong A_{n,n+5,5}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3} + \alpha e_{n+6}, & \\ [e_2, \ldots, e_{n+1}] = e_{n+4} + \beta e_{n+6}, & \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+5} + \gamma e_{n+6}, & \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$.

Since $\dim(A/\langle e_{n+3}, e_{n+4}, e_{n+5}\rangle)^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}, e_{n+5}\rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n-Lie algebras, we conclude that one of the coefficients

$$\alpha_i \; (3 \leq i \leq n), \quad \beta_i \; (1 \leq i \leq n), \quad \chi_{ij} \; (1 \leq i < j \leq n)$$

is equal to one, and the others are zero. We have two possibilities:

(1) Only one of $\alpha_i(3 \leq i \leq n)$ and $\beta_i(1 \leq i \leq n)$ is equal to one and the others are zero. Without loss of generality, we assume $\alpha_2 = 1$. Thus, the brackets in A can be written as

$$[e_1, \ldots, e_n] = e_{n+3}, \quad [e_2, \ldots, e_{n+1}] = e_{n+4},$$
$$[e_2, \ldots, e_n, e_{n+2}] = e_{n+5}, \quad [e_1, e_3, \ldots, e_{n+1}] = e_{n+6}.$$

One can easily see that this algebra is isomorphic to $A_{n,n+6,7}$.

(2) Only one of $\chi_{ij}s(1 \leq i < j \leq n)$ is equal to one and the others are zero. Up to isomorphism, we have the following algebras:

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, & [e_2, \ldots, e_{n+1}] = e_{n+4}, \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+5}, & [e_3, \ldots, e_{n+2}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1, \ldots, e_n] = e_{n+3}, & [e_2, \ldots, e_{n+1}] = e_{n+4}, \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+5}, & [e_1, e_4, \ldots, e_{n+2}] = e_{n+6}. \end{cases}$$

One can easily see that the first algebra is isomorphic to $A_{n,n+6,7}$. The second algebra is denoted by $A_{n,n+6,10}$.

Case 6. Let $A/\langle e_{n+6}\rangle \cong A_{n,n+5,6}$. Then the brackets in A can be written as

$$\begin{cases} [e_1,\ldots,e_n] = e_{n+3} + \alpha e_{n+6}, \\ [e_2,\ldots,e_{n+1}] = e_{n+4} + \beta e_{n+6}, \\ [e_3,\ldots,e_{n+2}] = e_{n+5} + \gamma e_{n+6}, \\ [e_1,\ldots,\widehat{e}_i,\ldots,e_n,e_{n+1}] = \alpha_i e_{n+6}, & 2 \le i \le n, \\ [e_1,\ldots,\widehat{e}_i,\ldots,e_n,e_{n+2}] = \beta_i e_{n+6}, & 1 \le i \le n, \\ [e_1,\ldots,\widehat{e}_i,\ldots,\widehat{e}_j,\ldots,e_n,e_{n+1},e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \le i < j \le n, (i,j) \ne (1,2). \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$.

Since $\dim(A/\langle e_{n+3}, e_{n+4}, e_{n+5}\rangle)^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}, e_{n+5}\rangle \cong H(n,1) \oplus F(2)$. According to the structure of n-Lie algebras, we conclude that one of the coefficients

$$\alpha_i\ (2 \le i \le n), \quad \beta_i\ (1 \le i \le n), \quad \chi_{ij}\ (1 \le i < j \le n, (i,j) \ne (1,2))$$

is equal to one, and the others are zero. Up to isomorphism, we have the following algebras:

$$\begin{cases} [e_1,\ldots,e_n] = e_{n+3}, & [e_2,\ldots,e_{n+1}] = e_{n+4}, \\ [e_3,\ldots,e_{n+2}] = e_{n+5}, & [e_1,e_3,\ldots,e_{n+1}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1,\ldots,e_n] = e_{n+3}, & [e_2,\ldots,e_{n+1}] = e_{n+4}, \\ [e_3,\ldots,e_{n+2}] = e_{n+5}, & [e_1,e_2,e_4,\ldots,e_{n+1}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1,\ldots,e_n] = e_{n+3}, & [e_2,\ldots,e_{n+1}] = e_{n+4}, \\ [e_3,\ldots,e_{n+2}] = e_{n+5}, & [e_1,e_3,\ldots,e_n,e_{n+2}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1,\ldots,e_n] = e_{n+3}, & [e_2,\ldots,e_{n+1}] = e_{n+4}, \\ [e_3,\ldots,e_{n+2}] = e_{n+5}, & [e_1,e_2,e_4,\ldots,e_n,e_{n+2}] = e_{n+6}, \end{cases}$$
$$\begin{cases} [e_1,\ldots,e_n] = e_{n+3}, & [e_2,\ldots,e_{n+1}] = e_{n+4}, \\ [e_3,\ldots,e_{n+2}] = e_{n+5}, & [e_1,e_2,e_5,\ldots,e_{n+2}] = e_{n+6}. \end{cases}$$

One can easily see that the first and second algebras are isomorphic to $A_{n,n+6,7}$ and $A_{n,n+6,8}$, respectively. The third, fourth and fifth algebras are denoted by $A_{n,n+6,11}$, $A_{n,n+6,12}$ and $A_{n,n+6,13}$, respectively. ∎

Theorem 3.8. The only $(n+6)$-dimensional nilpotent n-Lie algebras of class 2 where $n \ge 4$ are

$$\begin{cases} H(n,1) \oplus F(5) (n \ge 4),\ H(4,2) \oplus F(1),\ H(5,2), \\ A_{n,n+5,1} \oplus F(1),\ A_{n,n+4,1} \oplus F(2),\ A_{n,n+3,1} \oplus F(3),\ A_{n,n+5,4} \oplus F(1) \\ A_{n,n+5,5} \oplus F(1),\ A_{n,n+4,3} \oplus F(2),\ A_{n,n+5,6} \oplus F(1),\ and\, A_{n,n+6,i} (1 \le i \le 14). \end{cases}$$

Proof. Assume that A is an $(n+6)$-dimensional nilpotent n-Lie algebra of class 2, where $n \ge 4$ and $A = \langle e_1, \ldots, e_{n+6}\rangle$. If $\dim A^2 = 1$, then by Theorem 2.2, A is isomorphic to one of the following algebras:

$$H(n,1) \oplus F(5) (n \ge 4), \quad H(4,2) \oplus F(1), \quad H(5,2).$$

Now, assume that $\dim A^2 \ge 2$ and that $\langle e_{n+5}, e_{n+6}\rangle \subset A^2$. Therefore, $A/\langle e_{n+6}\rangle$ is an $(n+5)$-dimensional nilpotent n-Lie algebra of class 2. It follows from Theorem 2.5 that $A/\langle e_{n+6}\rangle$ is one of the following forms:

$$H(n,1) \oplus F(4), \quad H(4,2), \quad A_{n,n+5,i} (1 \le i \le 7).$$

If $A/\langle e_{n+6}\rangle$ is isomorphic to $H(n,1) \oplus F(4)$ or $H(4,2)$, then $\dim A^2 = 2$. According to Lemma 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n-Lie algebra of rank 2 and F is abelian. The center of A has a dimension at most 5; thus the possible cases of A are H_0, $H_1 \oplus F(1)$, $H_2 \oplus F(2)$, $H_3 \oplus F(3)$, where H_0, H_1, H_2, H_3 are generalized Heisenberg n-Lie algebras of rank 2 with dimensions $n+6$, $n+5$, $n+4$, $n+3$, respectively. These algebras read as follows:

$$A_{n,n+6,1}, \quad A_{n,n+5,1} \oplus F(1), \quad A_{n,n+4,1} \oplus F(2), \quad A_{n,n+3,1} \oplus F(3).$$

If $A/\langle e_{n+6}\rangle$ is isomorphic to $A_{n,n+5,1}$, $A_{n,n+5,2}$ or $A_{n,n+5,3}$, then $\dim A^2 = 3$. According to Lemma 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n-Lie algebra of rank 3 and F is abelian. According to ?, these algebras read as follows:

$$A_{n,n+6,2}, \quad A_{n,n+6,3}, \quad A_{n,n+6,4}, \quad A_{n,n+6,5}, \quad A_{n,n+6,6},$$
$$A_{n,n+5,4} \oplus F(1), \quad A_{n,n+5,5} \oplus F(1), \quad A_{n,n+4,3} \oplus F(2).$$

Also, If $A/\langle e_{n+6}\rangle$ is isomorphic to $A_{n,n+5,4}$, $A_{n,n+5,5}$ or $A_{n,n+5,6}$, then $\dim A^2 = 4$. According to Lemma 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n-Lie algebra of rank 4 and F is abelian. According to ?, these algebras read as follows:

$$A_{n,n+6,7}, \quad A_{n,n+6,8}, \quad A_{n,n+6,9}, \quad A_{n,n+6,10},$$
$$A_{n,n+6,11}, \quad A_{n,n+6,12}, \quad A_{n,n+6,13}, \quad A_{n,n+5,6} \oplus F(1).$$

Finally, If $A/\langle e_{n+6}\rangle \cong A_{n,n+5,7}$, then $A^2 = Z(A) = \langle e_{n+1}, e_{n+3}, e_{n+4}, e_{n+5}, e_{n+6}\rangle$. The brackets in A can be written as

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+1} + \alpha e_{n+6}, \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+3} + \beta e_{n+6}, \\ [e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4} + \gamma e_{n+6}, \\ [e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+5} + \phi e_{n+6}, \\ [e_1, \ldots, \widehat{e}_i, \ldots, e_n, e_{n+2}] = \beta_i e_{n+6}, \qquad 4 \le i \le n. \end{cases}$$

With a suitable change of basis, one can assume that $\alpha = \beta = \gamma = \phi = 0$. Thus, the brackets in A are

$$\begin{cases} [e_1, \ldots, e_n] = e_{n+1}, \\ [e_2, \ldots, e_n, e_{n+2}] = e_{n+3}, \\ [e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4}, \\ [e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+5}, \\ [e_1, \ldots, \widehat{e}_i, \ldots, e_n, e_{n+2}] = \beta_i e_{n+6}, \quad 4 \le i \le n. \end{cases}$$

By $\dim Z(A)$, we must have $\beta_i \neq 0$ for some $4 \le i \le n$. Without loss of generality, assume that $\beta_4 \neq 0$. By applying the transformations

$$e_4' = e_4 + \sum_{j=5}^{n} (-1)^j \frac{\beta_i}{\beta_4} e_j, \quad e_i' = e_i \, (1 \le i \le n+5, \, i \neq 4), \quad e_{n+6}' = \beta_4 e_{n+6},$$

we conclude that

$$A = \langle e_1, \ldots, e_{n+6} : [e_1, \ldots, e_n] = e_{n+1}, [e_2, \ldots, e_n, e_{n+2}] = e_{n+3},$$
$$= e_{n+4}, [e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+5}, = e_{n+6}\rangle,$$

which we denote it by $A_{n,n+6,14}$. ∎

In Table 1, we show all $(n+4)$-dimensional and $(n+5)$-dimensional nilpotent n-Lie algebras of class 2.

In Table 2, we show all n-Lie algebras obtained in this paper.

Nilpotent n-Lie algebras of class 2	Nonzero multiplications
$A_{n,n+6,1}$	$[e_1, \ldots, e_n] = e_{n+5}$, $[e_5, \ldots, e_{n+4}] = e_{n+6}$
$A_{n,n+6,2}$	$[e_1, \ldots, e_n] = e_{n+4}$, $[e_2, \ldots, e_{n+1}] = e_{n+5}$, $[e_3, \ldots, e_n, e_{n+2}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,3}$	$[e_1, \ldots, e_n] = e_{n+4}$, $[e_2, \ldots, e_{n+1}] = e_{n+5}$, $[e_1, e_4, \ldots, e_n, e_{n+2}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,4}$	$[e_1, \ldots, e_n] = e_{n+4}$, $[e_2, \ldots, e_{n+1}] = e_{n+5}$, $[e_1, e_5, \ldots, e_{n+3}] = e_{n+6}$
$A_{n,n+6,5}$	$[e_1, \ldots, e_n] = e_{n+4}$, $[e_3, \ldots, e_{n+2}] = e_{n+5}$, $[e_2, e_4, \ldots, e_{n+1}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,6}$	$[e_1, \ldots, e_n] = e_{n+4}$, $[e_3, \ldots, e_{n+2}] = e_{n+5}$, $[e_1, e_2, e_5, \ldots, e_{n+1}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,7}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_1, e_3, \ldots, e_{n+1}] = e_{n+5}$, $[e_3, \ldots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,8}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_1, e_3, \ldots, e_{n+1}] = e_{n+5}$, $[e_2, e_4, \ldots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,9}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_1, e_3, \ldots, e_{n+1}] = e_{n+5}$, $[e_1, e_2, e_5, \ldots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,10}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_2, \ldots, e_n, e_{n+2}] = e_{n+5}$, $[e_1, e_4, \ldots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,11}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_3, \ldots, e_{n+2}] = e_{n+5}$, $[e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+6}$
$A_{n,n+6,12}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_3, \ldots, e_{n+2}] = e_{n+5}$, $[e_1, e_2, e_4, \ldots, e_n, e_{n+2}] = e_{n+6}$
$A_{n,n+6,13}$	$[e_1, \ldots, e_n] = e_{n+3}$, $[e_2, \ldots, e_{n+1}] = e_{n+4}$, $[e_3, \ldots, e_{n+2}] = e_{n+5}$, $[e_1, e_2, e_5, \ldots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,14}$	$[e_1, \ldots, e_n] = e_{n+1}$, $[e_2, \ldots, e_n, e_{n+2}] = e_{n+3}$, $[e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+4}$, $[e_3, \ldots, e_{n+2}] = e_{n+5}$, $[e_1, e_3, \ldots, e_n, e_{n+2}] = e_{n+6}$

Table 2.

References

1. Filippov VT. n-Lie algebras, Sib. Math. J. 1985; 26: 879-91.
2. Bai R, Wang XL, Xiao WY, An HW. Structure of low dimensional n-Lie algebras over a field of characteristic 2, Linear Algebra Appl. 2008; 428(8-9): 1912-20.
3. Bai R, Song G, Zhang Y. On classification of n-Lie algebras, Front. Math. China. 2011; 6(4): 581-606.
4. de Azcárraga J, Izquierdo JM, Picón M. Contractions of filippov algebras. J. Math. Phys. 2011; 52(1): 24, 013516.
5. Darabi H and Saeedi F. On the Schur multiplier of n-Lie algebras, J. Lie Theory. 2017; 27: 271-81.
6. Furuuchi K, Tomino D, Supersymmetric reduced models with a symmetry based on filippov algebra, J. High Energy Phys. 2009; 5(70): 17.
7. Grabowski J, Marmo G. On Filippov algebroids and multiplicative Nambu-Poisson structures. Differential Geom. Appl. 2000; 12: 35-50.
8. Kasymov SM. Theory of n-Lie algebras, Algebra Logic. 1987; 26: 155-66.
9. Camacho LM, Casas JM, Gomez JR, Ladra M and Omirov BA. On nilpotent Leibniz n-algebras, J. Algebra Appl. 11(3) 2012: 1250062(17).
10. Ray CB, Combs A, Gin N, Hedges A, Hird JT, Zack L. Nilpotent Lie and Leibniz algebras, Comm. Algebra. 2014; 42: 2404-10.
11. Williams MP. Nilpotent n-Lie algebras, Comm. Algebra. 2009; 37(6): 1843-49.

12. Eshrati M, Saeedi F, Darabi H. Low dimensional nilpotent n-Lie algebras, arXiv:1810.03782.
13. Hoseini Z, Saeedi F, Darabi H. On classification of $(n+5)$-dimensional nilpotent n-Lie algebras of class two, Bull. Iranian Math. Soc. 2019; 45: 939-49.
14. Goze M, Goze N, Remm E. n-Lie algebras. Afr. J. Math. Phys. 2010; 8: 17-28.
15. Yan Z, Deng S. The classification of two step nilpotent complex Lie algebras of dimension 8, Czechoslovak Math. J. 2013; 63(138): 847-63.
16. Darabi H, Imanparast M. On classification of 9-dimensional nilpotent 3-ary algebras of class two. Bull. Iranian Math. Soc. 2020: 1-9.
17. Eshrati M, Saeedi F, Darabi H. On the multiplier of nilpotent n-Lie algebras, J. Algebra. 2016; 450: 162-72.
18. Darabi H, Saeedi F, Eshrati M. A characterization of finite dimensional nilpotent Filippov algebras, J. Geom. Phys. 2016; 101: 100-7.
19. Hoseini Z, Saeedi F, Darabi H. Characterization of capable nilpotent n-Lie algebras of class two by their Schur multipliers. Rend. Circ. Mat. Palermo. 2019; 68(2): 541–55.
20. Goze M, Haraguchi Y. Sur les r-systemes de contact. CR Acad. Sci. Paris. 1982; 294: 95-97.

Corresponding author
Hamid Darabi can be contacted at: darabi@esfarayen.ac.ir