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An investigation on the existence of warped product irrotational screen-real lightlike submanifolds of metallic semi-Riemannian manifolds

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Abstract

Purpose – The purpose of this paper is to study the geometry of screen real lightlike submanifolds of metallic semi-Riemannian manifolds. Also, the authors investigate whether these submanifolds are warped product lightlike submanifolds or not.

Design/methodology/approach – The paper is design as follows: In Section 3, the authors introduce screen-real lightlike submanifold of metallic semi Riemannian manifold. In Section 4, the sufficient conditions for the radical and screen distribution of screen-real lightlike submanifolds, to be integrable and to be have totally geodesic foliation, have been established. Furthermore, the authors investigate whether these submanifolds can be written in the form of warped product lightlike submanifolds or not.

Findings – The geometry of the screen-real lightlike submanifolds has been studied. Also various results have been established. It has been proved that there does not exist any class of irrotational screen-real r-lightlike submanifold such that it can be written in the form of warped product lightlike submanifolds.

Originality/value – All results are novel and contribute to further study on lightlike submanifolds of metallic semi-Riemannian manifolds.

Keywords Metallic semi-Riemannian manifolds, Warped product lightlike submanifolds, Irrotational lightlike submanifolds

Paper type Research paper

1. Introduction

It is well known that the study of semi-Riemannian manifolds and its submanifolds is more complicated as compared to Riemannian manifolds and its submanifolds. It is observed that the induced metric on submanifolds of semi-Riemannian manifolds has two cases, either non-degenerate or degenerate. In case of non-degenerate, there is no complications to do calculus on these submanifolds. On the other hand, if submanifold has a degenerate metric then there is non-trivial intersection of tangent bundle and normal bundle. Due to this, it is not possible to induce many structures uniquely and with same character as structures of ambient space.

MSC Classification — 53C15, 53C40, 53C50

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The study of degenerate submanifolds is known as lightlike geometry. Due to extensive use as a tool to understand theory of relativity, it becomes a topic of interest for mathematicians and physicists.

In 1996, Duggal and Bejancu gave a detail explanation of lightlike geometry in [1]. Later, many research articles have been published on lightlike geometry. In physics, various spacetime models have been studied with the help of lightlike geometry.

Crasmareanu and Hretcanu [2] introduced golden Riemannian manifolds by using golden ratio. Later, Spinadel introduced generalization of golden means known as metallic means [3–5]. For any positive integers p and q, the positive solutions of the equation $y^2 - py - q = 0$, are known as metallic means and

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

is known as (p, q) metallic number. A metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is a semi-Riemannian manifold with metallic structure $\tilde{P}$ such that $\tilde{g}$ is $\tilde{P}$-compatible metric. Different types of submanifolds of metallic and golden Riemannian manifolds have been studied in [1, 3–9]. Apart from this, the geometry of various submanifolds of metallic and golden semi-Riemannian manifolds have been studied in [10–12]. This paper is categorized as follows:

In Section 1, we give brief description of lightlike geometry and metallic semi-Riemannian manifolds. In Section 2, the necessary definitions and theorems required for the current work have been mentioned. In Section 3, we introduce geometry of screen-real lightlike submanifolds of metallic semi-Riemannian manifolds. The necessary and sufficient conditions for integrability and to be totally geodesic foliations of $\mathrm{Rad}(T\check{N})$ and $S(T\check{N})$ have been established. In Section 4, we prove that "there does not exist any class of irrotational screen-real r-lightlike submanifolds that can be written in the form of warped product lightlike submanifold."

2. Preliminaries

A submanifold $(\check{N}^m, \check{g})$ of a semi-Riemannian manifold $(\tilde{N}^{m+n}, \tilde{g})$ with constant index q $(1 \leq q \leq m + n - 1,\ m,\ n \geq 1)$ is known as degenerate (lightlike) submanifold, if the induced metric $\check{g}$ is degenerate [7].

Due to generate induced metric on $T\check{N}$, for any $u \in \check{N}$, there exist non zero intersection of $T_u\check{N}$ (m-dimensional) and $T_u\check{N}^{\perp}$ (n-dimensional), which is called $\mathrm{Rad}(T\check{N})$. A lightlike submanifold is known as r-lightlike, if there exists a smooth distribution $\mathrm{Rad}(T\check{N})$ of rank $r > 0$, such that every member u of $\check{N}$ goes to an r-dimensional subspace $\mathrm{Rad}(T_u\check{N})$ of $T_u\check{N}$. Let $S(T\check{N})$ (screen distribution) and $S(T\check{N}^{\perp})$ (screen transversal distribution) are non-degenerate complementary sub-bundles of $\mathrm{Rad}(T\check{N})$ in $T\check{N}$ and $T\check{N}^{\perp}$ respectively. Let $ltr(T\check{N})$(lightlike transversal bundle) and $tr(T\check{N})$(transversal bundle) be complementary but not orthogonal vector bundles to $\mathrm{Rad}(T\check{N})$ and $T\check{N}$ in $S(TN^{\perp})^{\perp}$ and $T\tilde{N}|_{\check{N}}$ respectively.

Then, the orthogonal decomposition of $tr(T\check{N})$ and $T\tilde{N}|_{\check{N}}$ are given by (for detail see [7])

$$tr(T\check{N}) = ltr(T\check{N}) \perp S(T\check{N}^{\perp}) \tag{2.1}$$

and

$$T\tilde{N}|_{\check{N}} = T\check{N} \oplus tr(T\check{N}) = [\mathrm{Rad}(T\check{N}) \oplus ltr(T\check{N})] \perp S(T\check{N}) \perp S(T\check{N}^{\perp}) \tag{2.2}$$

respectively.

Theorem 2.1. [7] Let $(\tilde{N}, \tilde{g})$ be a semi-Riemannian manifold, $(\check{N}, \check{g}, S(T\check{N}), S(T\check{N}^{\perp}))$ be its r-lightlike submanifold. Then there exists a vector bundle $ltr(T\check{N})$ and a basis of $\Gamma(ltr(T\check{N})|u)$ containing a smooth section $\{N_i\}$ of $S(T\check{N}^{\perp})^{\perp}|_u$, for a coordinate neighborhood u of $\check{N}$, such that

$$\tilde{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \tilde{g}_{ij}(N_i, N_j) = 0, \tag{2.3}$$

for any $i, j \in \{1, 2, \ldots, r\}$, where $\{\xi_i\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(T\check{N}))$.

For any $U, V \in \Gamma(T\check{N})$ and $W \in \Gamma(tr(T\check{N}))$, the Gauss and Weingarten formulae are

$$\tilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.4}$$

$$\tilde{\nabla}_U W = -A_W U + \nabla_U^{\perp} W, \tag{2.5}$$

where $\{\nabla_U V, A_W U\}$ and $\{h(U, V), \nabla_Y^{\perp} W\}$ belong to $\Gamma(T\check{N})$ and $\Gamma(tr(T\check{N}))$ respectively, and ∇ is a induced connection on $\check{N}$. Further, from (2.4) and (2.5), we deduce that

$$\tilde{\nabla}_U V = \nabla_U V + h^l(U, V) + h^s(U, V), \tag{2.6}$$

$$\tilde{\nabla}_U N = -A_N U + \nabla_U^l(N) + D^s(U, N), N \in \Gamma(ltr(T\check{N})), \tag{2.7}$$

$$\tilde{\nabla}_U W_1 = -A_{W_1} U + \nabla_U^s(W_1) + D^l(U, W_1), W_1 \in \Gamma(S(T\check{N}^{\perp})). \tag{2.8}$$

Eqns (2.4), (2.6) are known as Gauss equations and (2.5), (2.7), (2.8) are known as Weingarten equations respectively, for the lightlike submanifold $\check{N}$ of $\tilde{N}$.

Using metric connection $\tilde{\nabla}$ and (2.4)-(2.8), we get the following equations:

$$\tilde{g}(h^s(U, V), W') + \tilde{g}(U, D^l(V, W')) = \check{g}(A_{W'} U, V), \tag{2.9}$$

$$\tilde{g}(h^l(U, V), \xi) + \tilde{g}(U, h^l(V, \xi)) = -\check{g}(U, \nabla_V \xi), \tag{2.10}$$

for any $\xi \in \Gamma(\mathrm{Rad}(T\check{N}))$ $U, V \in \Gamma(T\check{N})$, and $W' \in \Gamma(S(T\check{N}^{\perp}))$.

Since the induced connection is not necessarily Levi- Civita connection, for any $U_1, U_2, U_3 \in \Gamma(T\check{N})$ and $U, U' \in \Gamma(tr(T\check{N}))$, we have following formula

$$(\nabla_{U_1}\check{g})(U_2, U_3) = \tilde{g}\left(h^l(U_1, U_2), U_3\right) + \tilde{g}\left(h^l(U_1, U_3), U_2\right). \tag{2.11}$$

Let S denote projection map on $S(T\check{N})$ from $T\check{N}$. Then, for any $U, V \in \Gamma(T\check{N})$ and $\xi \in \Gamma(\mathrm{Rad}(T\check{N}))$, we have the following equations:

$$\nabla_U SV = \nabla_U^* SV + h^*(U, SV). \tag{2.12}$$

$$\nabla_V \xi = A_\xi^* V + \nabla_V^{*t}(\xi), \tag{2.13}$$

where $\{h^*(U, PV), \nabla_V^{*t}(\xi)\}$ and $\{\nabla_U^* SV, A_\xi^* V\}$ belong to $\Gamma(\mathrm{Rad}(T\check{N}))$ and $\Gamma(S(T\check{N}))$ respectively.

For detail understanding of Eqns (2.4)–(2.13), see [7] (pp. 196–198).

Definition 2.1. A metallic semi-Riemannian manifold is a smooth manifold with $(1, 1)$ tensor field $\tilde{P}$ on $\tilde{N}$ such that

$$\tilde{P}^2 = p\tilde{P} + qI, \tag{2.14}$$

and $\tilde{g}$ is $\tilde{P}$-compatible, i.e.

$$\tilde{g}(\tilde{P}U, V) = \tilde{g}(U, \tilde{P}V). \tag{2.15}$$

Using (2.14) in (2.15), we obtain

$$\tilde{g}(\tilde{P}U, \tilde{P}V) = p\tilde{g}(\tilde{P}U, V) + q\tilde{g}(U, V), \tag{2.16}$$

for any $U, V \in \Gamma(T\check{N})$ [2, 8].

If $(\tilde{\nabla}_U \tilde{P})V = 0$, then $\tilde{P}$ is called locally metallic structure. Throughout the paper, we assume that $\tilde{P}$ is a locally metallic structure.

3. Screen-real lightlike submanifolds

Definition 3.1. A lightlike submanifold $(\check{N}, \check{g}, S(T\check{N}))$ of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ is said to be a screen-real lightlike submanifold if it satisfies the following:

$$\tilde{P}(\mathrm{Rad}(T\check{N}) = \mathrm{Rad}(T\check{N}) \ \& \ \tilde{P}(S(T\check{N})) \subseteq S(T\check{N}^{\perp}).$$

Clearly,

$$\tilde{P}(ltr(T\check{N}) = ltr(T\check{N}) \ \& \ \tilde{P}(\mu) = \mu.$$

From above decomposition of distributions, we get

$$T\tilde{N}|_{\check{N}} = [\mathrm{Rad}(T\check{N}) \oplus ltr(T\check{N})] \oplus_{orth.} S(T\check{N}) \oplus_{orth.} \tilde{P}(S(T\check{N})) \oplus_{orth.} \mu. \tag{3.1}$$

For any $U \in \Gamma(T\check{N})$, using (3.1), we obtain

$$U = RU + SU,$$

where R and S are projection maps on $\mathrm{Rad}(T\check{N})$ and $S(T\check{N})$ respectively. Applying $\tilde{P}$ on above equation and using (3.1), we obtain

$$\tilde{P}U = RU + S'U, \tag{3.2}$$

where $\tilde{P}RU = RU$, $\tilde{P}SU = S'U$ and R, S' are projection maps on $Rad(T\check{N})$ and $S(T\check{N}^{\perp})$ respectively.

For any $w \in tr(T\check{N})$, we have

$$\tilde{P}(w) = Bw + C_1 w + C_2 w + C_3 w, \tag{3.3}$$

where B, C_1, C_2 and C_3 are projection maps on $S(T\check{N})$, $ltr(T\check{N})$, $\tilde{P}S(T\check{N})$ and μ respectively.

For any $w_1 \in \Gamma(ltr(T\check{N}))$, $w_2 \in \Gamma(\tilde{P}S(T\check{N}))$ and $w_3 \in \Gamma(\mu)$, (3.3) takes following different forms, respectively

$$\tilde{P}(w_1) = C_1 w_1, \tag{3.4}$$

$$\tilde{P}(w_2) = Bw_2 + C_2 w_2, \tag{3.5}$$

$$\tilde{P}(w_3) = C_3 w_3. \tag{3.6}$$

Example 3.1. Let $(\tilde{N} = \mathbb{R}^6_2, \breve{g}, \tilde{P})$ be a six dimensional semi-Euclidean space, where $\breve{g}$ is a semi-Euclidean metric with signature $(+ - + \ + \ + -)$ Let us define

$$\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6) = (\sigma x_1, \sigma x_2, (p - \sigma)x_3, \sigma x_4, \sigma x_5, (p - \sigma)x_6),$$

where $(x_1, x_2, x_3, x_4, x_5, x_6)$ is the standard coordinate system of $\mathbb{R}^6_2$. Then, it can be easily verified that $\tilde{P}$ is a metallic structure.

Let us define a submanifold $\breve{N}$ of $\tilde{N}$ such that

$$x_1 = \sinh \sigma u_1, \quad x_2 = \cosh \sigma u_1,$$

$$x_3 = \frac{\sigma}{\sqrt{q}} u_4, \quad x_4 = u_4,$$

$$x_5 = u_1, \quad x_6 = 0.$$

Then we can find following tangent vectors of the above submanifold

$$U_1 = \sinh\sigma \frac{\partial}{\partial x_1} + \cosh\sigma \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, \quad U_2 = \frac{\sigma}{\sqrt{q}} \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},$$

such that $T\breve{N} = \mathrm{span}\{U_1, U_2\}$. Clearly, $\breve{N}$ is a lightlike submanifold with

$$\mathrm{Rad}(T\breve{N}) = \mathrm{span}\{U_1 = \xi\},$$

$$S(T\breve{N}) = \mathrm{span}\{U_2\},$$

$$ltr(T\breve{N}) = \mathrm{span}\left\{N = \frac{1}{2}\left(-\sinh\sigma \frac{\partial}{\partial x_1} - \cosh\sigma \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}\right)\right\},$$

$$S(T\breve{N}^{\perp}) = \mathrm{span}\left\{W = -\sqrt{q} \ \frac{\partial}{\partial x_3} + \sigma \frac{\partial}{\partial x_4}\right\},$$

where $\tilde{P}\xi = \sigma\xi$, $\tilde{P}(N) = \sigma N$, $\tilde{P}(U_2) = W$. Since $\tilde{P}$ satisfies $\tilde{P}(\mathrm{Rad}(T\breve{N})) = (\mathrm{Rad}(T\breve{N}))$ and $\tilde{P}(S(T\breve{N})) = S(T\breve{N}^{\perp})$, $\breve{N}$ is a screen real lightlike submanifold.

Theorem 3.2. Let $(\breve{N}, \breve{g}, S(T\breve{N}))$ be a screen-real lightlike submanifold of a metallic semi-Riemannian manifold, then following equations hold:

$$R\nabla_U V + B(h^s(U, V)) = \nabla_U RV - A_{S'V} U,$$
$$C_1 h^l(U, V) = h^l(U, RV) + D^l(U, S'V)$$

and

$$h^s(U, RV) + \nabla^s_U S'V = S'\nabla_U V + C_2 h^s(U, V) + C_3 h^s(U, V).$$

Proof. Using (3.2)-(3.6) in $(\tilde{\nabla}_U \tilde{P})V = 0$, for any $U, V \in \Gamma(T\breve{N})$, we obtain

$$\begin{aligned} &\nabla_U RV + h^l(U, RV) + h^s(U, RV) - A_{S'V} U + \nabla^s_U S'V + D^l(U, S'V) = R\nabla_U V \\ &+ S'\nabla_U V + C_1 h^l(U, V) + B(h^s(U, V)) + C_2 h^s(U, V) + C_3 h^s(U, V). \end{aligned} \quad (3.7)$$

By equating tangential, $ltr(T\breve{N})$ and $S(T\breve{N}^{\perp})$ components in the above equation, we get required results. □

Theorem 3.3. Let $(\breve{N}, \breve{g}, S(T\breve{N}))$ be a screen-real lightlike submanifold of a metallic semi-Riemannian manifold, then

(1) $\mathrm{Rad}(T\check{N})$ is integrable if and only if

$$h^s(\tilde{P}\xi_2, \xi_1) = h^s(\tilde{P}\xi_1, \xi_2).$$

(2) $S(T\check{N})$ is integrable if and only if

$$R(-A_{\tilde{P}V}U + A_{\tilde{P}U}V) = p(h^*(U, V) - h^*(V, U)).$$

Proof. (1) For any $\xi_1, \xi_2 \in \Gamma(\mathrm{Rad}(T\check{N}))$ and $U \in \Gamma(S(T\check{N}))$, $\mathrm{Rad}(T\check{N})$ is integrable if and only if,

$$\tilde{g}([\xi_1, \xi_2], U) = 0.$$

Expanding $\tilde{g}([\xi_1, \xi_2], U)$ and using (2.16), we get

$$\tilde{g}([\xi_1, \xi_2], U) = \frac{1}{q}\tilde{g}\big(\tilde{P}(\tilde{\nabla}_{\xi_1}\xi_2 - \tilde{\nabla}_{\xi_2}\xi_1), \tilde{P}U\big) - \frac{p}{q}\tilde{g}\big(\tilde{\nabla}_{\xi_1}\xi_2 - \tilde{\nabla}_{\xi_2}\xi_1, \tilde{P}U\big). \tag{3.8}$$

Using (2.6), (2.8) and (2.12) in (3.8), we get

$$\tilde{g}([\xi_1, \xi_2], U) = \frac{1}{q}\tilde{g}(h^s(\tilde{P}\xi_2, \xi_1) - h^s(\tilde{P}\xi_1, \xi_2), \tilde{P}U). \tag{3.9}$$

From (3.11), we obtain $\tilde{g}([\xi_1, \xi_2]U) = 0$ if and only if

$$h^s(\tilde{P}\xi_2, \xi_1) = h^s(\tilde{P}\xi_1, \xi_2)$$

(2) For any $U, V \in \Gamma(S(T\check{N}))$ and $N \in \Gamma(ltr(T\check{N}))$, $S(T\check{N})$ is integrable if and only if,

$$\tilde{g}([U, V], N) = 0.$$

Expanding $\tilde{g}([U, V], N)$ and using (2.16), we get

$$\tilde{g}([U, V], N) = \frac{1}{q}\tilde{g}(\tilde{P}(\tilde{\nabla}_U V - \tilde{\nabla}_V U), \tilde{P}N) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_U V - \tilde{\nabla}_V U, \tilde{P}N). \tag{3.10}$$

Using (2.6), (2.8) and (2.12) in (3.10), we obtain

$$\tilde{g}([U, V], N) = \frac{1}{q}\tilde{g}(-A_{\tilde{P}V}U + A_{\tilde{P}U}V, \tilde{P}N) - \frac{p}{q}\tilde{g}(h^*(U, V) - h^*(V, U), \tilde{P}N). \tag{3.11}$$

From (3.11), we obtain $\tilde{g}([U, V], N) = 0$ if and only if

$$\tilde{g}(R(-A_{\tilde{P}V}U + A_{\tilde{P}U}V), \tilde{P}N) = p\tilde{g}(h^*(U, V) - h^*(V, U), \tilde{P}N),$$

i.e.

$$R(-A_{\tilde{P}V}U + A_{\tilde{P}U}V) = p(h^*(U, V) - h^*(V, U)).$$ □

Theorem 3.4. Let $(\check{N}, \breve{g}, S(T\check{N}))$ be a screen-real lightlike submanifold of a metallic semi-Riemannian manifold, then

(1) $\mathrm{Rad}(T\check{N})$ defines a totally geodesic foliation if and only if

$$h^s(\xi_1, \tilde{P}\xi_2) = ph^s(\xi_1, \xi_2).$$

(2) $S(T\check{N})$ defines a totally geodesic foliation if and only if

$$R(A_{\tilde{P}V}U) = -ph^*(V, U).$$

Proof. (1) For any $\xi_1, \xi_2 \in \Gamma(\mathrm{Rad}(T\check{N}))$ and $U \in \Gamma(S(T\check{N}))$, $\mathrm{Rad}(T\check{N})$ defines a totally geodesic foliation if and only if $\breve{g}(\nabla_{\xi_1}\xi_2, U) = 0$.

Using (2.16), we get

$$\breve{g}(\nabla_{\xi_1}\xi_2, U) = \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, U) = \frac{1}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}U) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}U). \tag{3.12}$$

Using (2.6), (2.8) and (2.12) in (3.12), we obtain

$$\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, U) = \frac{1}{q}\tilde{g}(h^s(\xi_1, \tilde{P}\xi_2), \tilde{P}U) - \frac{p}{q}\tilde{g}(h^s(\xi_1, \xi_2), \tilde{P}U). \tag{3.13}$$

From (3.13), we get $\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, U) = 0$ if and only if

$$h^s(\xi_1, \tilde{P}\xi_2) = ph^s(\xi_1, \xi_2).$$

(2) For any $U, V \in S(T\check{N})$ and $N \in ltr(T\check{N})$, $S(T\check{N})$ defines a totally geodesic foliation if and only if, $\breve{g}(\nabla_U V, N) = 0$.

Using (2.16), we get

$$\breve{g}(\nabla_U V, N) = \tilde{g}(\tilde{\nabla}_U V, N) = \frac{1}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_U V, \tilde{P}N) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_U V, \tilde{P}N). \tag{3.14}$$

Using (2.6), (2.8) and (2.12) in (3.14), we obtain

$$\tilde{g}(\tilde{\nabla}_U V, N) = \frac{1}{q}\tilde{g}(-A_{\tilde{P}V}U, \tilde{P}N) - \frac{p}{q}\tilde{g}(-A^*_V U + h^*(V, U), \tilde{P}N). \tag{3.15}$$

From (3.15), we get $\tilde{g}(\nabla_U V, N) = 0$ if and only if

$$R(A_{\tilde{P}V}U) = -ph^*(V, U).$$

□

4. Warped product lightlike submanifolds

Definition 4.1. [7] A product manifold $\check{N} = N_1 \times_\lambda N_2$, where (N_1, g_1) is an r-dimensional totally lightlike submanifold and (N_2, g_2) is an m-dimensional semi-Riemannian submanifold of a semi-Riemannian manifold $\tilde{N}$, is known as a warped product lightlike submanifold with induced degenerate metric $\breve{g}$ defined as

$$\breve{g}(U, V) = \breve{g}_1(\pi_{1*}U, \pi_{1*}V) + (\lambda \, o \, \pi_1)^2 \breve{g}_2(\pi_{2*}U, \pi_{2*}V), \tag{4.1}$$

where $U, V \in \Gamma(T\check{N})$, λ(non constant) $\in C^\infty(N_1, \mathbb{R})$, π_1 and π_2 are projection maps from $N_1 \times N_2$ to N_1 and N_2 respectively and $*$ denotes tangent map.

Theorem 4.1. *Let* $\check{N} = N_1 \times_\lambda N_2$ *be a warped product lightlike submanifold. Then, for any* $\xi \in \Gamma(\mathrm{Rad}(T\check{N}))$ *and* $U \in \Gamma(S(\mathrm{TM}))$, *we have* $\nabla_\xi U \in \Gamma(S(T\check{N}))$.

Proof. For any $\xi \in \Gamma(\mathrm{Rad}(T\check{N}))$ and $U, V \in \Gamma(S(\mathrm{TM}))$, the Koszul formula is

$$2\tilde{g}(\tilde{\nabla}_\xi U, V) = \xi\tilde{g}(U, V) + U\tilde{g}(\xi, V) - V\tilde{g}(\xi, U) + \tilde{g}([\xi, U], V) + \tilde{g}([V, \xi], U) - \tilde{g}([U, V], \xi). \tag{4.2}$$

In the present situation, this reduces to

$$2\tilde{g}(\tilde{\nabla}_{\xi}U,\ V) = \xi\tilde{g}(U,\ V) - \tilde{g}([U,\ V],\ \xi).$$

If $\nabla_{\xi}U \in \Gamma(\mathrm{Rad}(T\check{N}))$, then above equation reduces to

$$\xi\tilde{g}(U,\ V) = \xi(\lambda o \pi_1)^2 \breve{g}_2(U,\ V) = 0.$$

Since $\breve{g}_2$ is constant on $\check{N}_1$, we get

$$\frac{\xi(\lambda)}{\lambda}\breve{g}_2(U,V) = 0.$$

Since λ is non-constant and $\breve{g}_2$ is a positive definite metric, this contradicts our assumption.
Hence, we must have $\nabla_{\xi}U \in \Gamma(S(T\check{N}))$. □

Definition 4.2. [7] An r-lightlike submanifold is said to be a irrotational lightlike submanifold if and only if

$$h^l(\xi,\ U) = 0,\ h^s(\xi,\ U) = 0, \tag{4.3}$$

for any $U \in S(T\check{N})$ and $\xi \in \mathrm{Rad}(T\check{N})$.

Theorem 4.2. Let $(\check{N},\ g,\ S(T\check{N}))$ be an irrotational screen-real r-lightlike submanifold of a metallic semi-Riemannian manifold, then the induced connection is a metric connection.

Proof. Let ∇ be a connection induced from the ambient connection $\tilde{\nabla}$. Then, for any $U, V \in \Gamma(T\check{N})$, ∇ is said to be a metric connection if and only if $h^l(U, V) = 0$.

From (4.2), $h^l(U, \xi) = 0$. Now, it is enough to show that $h(U, V) = 0$, if $U, V \in S(T\check{N})$.

Using (2.16), we get

$$\tilde{g}(\tilde{\nabla}_U V,\ \xi) = \frac{1}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_U V,\ \tilde{P}\xi) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_U V,\ \tilde{P}\xi). \tag{4.4}$$

Since $\tilde{\nabla}$ is a metric connection, equation (4.4) reduces to

$$\tilde{g}\left(h^l(U,V),\ \xi\right) = -\frac{1}{q}\tilde{g}(\tilde{P}V,\ \tilde{\nabla}_U\tilde{P}\xi) + \frac{p}{q}\tilde{g}(\tilde{P}V,\ \tilde{\nabla}_U\xi). \tag{4.5}$$

Using (2.6) in (4.5), we get

$$\tilde{g}(h^l(U,\ V),\ \xi) = -\frac{1}{q}\tilde{g}(\tilde{P}V,\ h^s(U,\ \tilde{P}\xi)) + \frac{p}{q}\tilde{g}(\tilde{P}V,\ h^s(U,\ \xi)). \tag{4.6}$$

Since $h^s(U,\ \tilde{P}\xi) = 0$ and $h^s(U,\ \xi) = 0$, (4.6) becomes

$$\tilde{g}\left(h^l(U,\ V),\ \xi\right) = 0$$

This implies $h^l(U, V) = 0$. This completes the proof. □

Theorem 4.3. There does not exist any class of irrotational screen-real r-lightlike submanifolds that can be written in the form of warped product lightlike submanifolds.

Proof. If possible, let there exist a class of irrotational screen-real r- lightlike submanifolds such that any $\check{N}$ in this class can be written as warped product lightlike submanifolds i.e. $\check{N} = N_1 \times_{\lambda} N_2$.

Using Theorem (4.1) in (4.1), we obtain

$$\tilde{g}(\tilde{\nabla}_{\xi}U,\ V) = \frac{\xi(\lambda)}{\lambda}\breve{g}_2(U,\ V). \tag{4.7}$$

Since $\check{N}$ is irrotational, using Theorem (4.2), for any $U, V \in \Gamma(T\check{N})$, we get $h^l(U, V) = 0$, From (2.4), we obtain

$$\tilde{g}\Big(\xi, h^l(U, V)\Big) = -\tilde{g}(\tilde{\nabla}_\xi U, V) = 0. \tag{4.8}$$

From (4.7) and (4.8), we get

$$\frac{\xi(\lambda)}{\lambda}\breve{g}_2(U, V) = 0.$$

This implies that either λ is constant or $\breve{g}_2$ is a degenerate metric. In either case, it is a contradiction. This completes the proof. □

5. Conclusion

Our aim in this paper is to investigate whether it is possible to write lightlike submanifolds of metallic semi-Riemannian manifolds in the form of warped product lightlike submanifolds or not. In this context, we introduce the screen real lightlike submanifolds and find that, it is difficult to say that screen real lightlike submanifolds are warped product lightlike submanifolds or not. We find a special class of screen real lightlike submanifolds namely, “irrotational screen real lightlike submanifolds”that can never be written in the form of warped product lightlike submanifolds.

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Further reading

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