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Structure theorem for Jordan algebra bundles

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Abstract

Purpose – The aims of this paper is to prove that every semisimple Jordan algebra bundle is locally trivial and establish the decomposition theorem for locally trivial Jordan algebra bundles using the decomposition theorem of Lie algebra bundles.

Design/methodology/approach – Using the decomposition theorem of Lie algebra bundles, this paper proves the decomposition theorem for locally trivial Jordan algebra bundles.

Findings – Findings of this paper establish the decomposition theorem for locally trivial Jordan algebra bundles.

Originality/value – To the best of the author's knowledge, all the results are new and interesting to the field of Mathematics and Theoretical Physics community.

Keywords Jordan algebra bundle, Lie algebra bundle, Vector bundle

Paper type Research paper

1. Introduction

In modern mathematics, an important notion is that of non-associative algebra. In Ref. [1], we gave a relationship between two important classes of non-associative algebras, namely, Lie algebras (introduced in 1870 by the Norwegian mathematician Sophus Lie in his study of the groups of transformations) and Jordan algebras (introduced in 1932–1933 by the German physicist Pasqual Jordan (1902–1980) in his algebraic formulation of quantum mechanics [2–4]). These two algebras are interconnected, as was remarked for instance by Kevin McCrimmon [5, p. 622]:

> We are saying that if you open up a Lie algebra and look inside, 9 times out of 10 there is a Jordan algebra (of pair) which makes it work.

Here, we recall some connections between Jordan algebra bundles and Lie algebra bundles [1, 6]. If ξ is a locally trivial Jordan algebra bundle in which each fibre ξ_x has a unit element then

$$K(\xi) = \bigcup_{x\in X} K(\xi_x) = \bigcup_{x\in X}\left(Der\xi_x \oplus L(\xi_x) \oplus \xi_x \oplus \overline{\xi_x}\right)$$

is a Lie algebra bundle, where $Der\xi_x$ is the vector space of all derivation defined on ξ_x, $L(\xi_x)$ is the vector space of all left translations of ξ_x and $\overline{\xi_x}$ an isomorphic copy of ξ_x.

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Lemma 1.1. [1] *Let ξ be a locally trivial Jordan algebra bundle over X, with each fibre ξ_x having a unit element e_x, $x \in X$. Then*

$$g(\xi) = \bigcup_{x\in X} g(\xi_x) = \bigcup_{x\in X} (Der\xi_x + L(\xi_x))$$

is a Lie algebra bundle.

Theorem 1.2. [1] *If ξ is a locally trivial Jordan algebra bundle in which each fibre has a unit element, then $K(\xi) = g(\xi) \oplus \xi \oplus \bar{\xi}$, is a Lie algebra bundle, where $\bar{\xi}$ is an isomorphic copy of the Jordan algebra bundle ξ. Further, ξ can be imbedded in $K(\xi)$ such that the Jordan multiplication on ξ can be given in terms of the Lie multiplication on $K(\xi)$.*

Given a locally trivial Jordan algebra bundle ξ in which each fibre ξ_x has a unit element, consider the Lie algebra bundle $K(\xi) = g(\xi) \oplus \xi \oplus \bar{\xi}$. Let $h(\xi) = \cup_{x\in X} h(\xi_x)$, where each $h(\xi_x) = L(\xi_x) \oplus [L(\xi_x), L(\xi_x)]$ is an ideal of $g(\xi_x) = Der\xi_x + L(\xi_x)$ [7]. Let $\phi : U \times J \to p^{-1}(U)$ be the local triviality of the Jordan algebra bundle ξ, where J is a Jordan algebra. Set $g(\phi) = End\phi|_{U\times g(J)}$, where the vector bundle morphism $End\phi : U \times EndJ \to \cup_{x\in U} End\xi_x$ is given by $(End\phi)(x,f) = \phi_x f \phi_x^{-1}$. Further, $h(\xi)$ is a vector bundle since $g(\phi)|_{U\times h(J)}$ maps $U \times h(J)$ onto $\cup_{x\in U} h(\xi_x)$. Hence $h(\xi)$ is an ideal subbundle of $g(\xi)$. We denote by $L(\xi)$, the ideal subbundle $L(\xi) = h(\xi) \oplus \xi \oplus \bar{\xi}$ of $K(\xi)$. □

In this paper, we prove that any semisimple Jordan algebra bundle is locally trivial and we supply an example to show that the converse need not be true. Further, we prove that a semisimple Jordan algebra bundle can be written as the direct sum of simple ideal bundles.

1.1 Notations and terminology

All Jordan algebra bundles $\xi = (\xi, p, X, \theta)$ are over the arbitrary topological space X unless otherwise mentioned.

2. Preliminaries

Definition 2.1. A Jordan algebra bundle is a vector bundle $\xi = (\xi, p, X)$ together with a vector bundle morphism $\theta : \xi \otimes \xi \to \xi$ inducing a Jordan algebra structure on each fibre ξ_x, $x \in X$.

Definition 2.2. By a trivial Jordan algebra bundle, we mean a trivial vector bundle $(X \times J, p, X)$, where J is a Jordan algebra.

Definition 2.3. A morphism $f : \xi \to \zeta$ of Jordan bundles ξ and ζ is a morphism of the underlying vector bundles such that for every $x \in X$, $f_x : \xi_x \to \zeta_x$ is a Jordan algebra homomorphism. If f is bijective and f^{-1} is continuous, then f is called an isomorphism.

Definition 2.4. By a subalgebra (ideal) bundle of a Jordan algebra bundle $\xi = (\xi, p, X)$, we mean a vector sub-bundle $\xi' = (\xi', p, X)$ of ξ such that each fibre $(\xi')_x$ is a subalgebra (ideal) of ξ_x, $\forall\, x \in X$.

Definition 2.5. By a semisimple Jordan algebra bundle, we mean a Jordan algebra bundle in which each fibre is a semisimple Jordan algebra.

Definition 2.6. If ξ is a Jordan algebra bundle with a nontrivial multiplication $\theta : \xi \otimes \xi \to \xi$ inducing the Jordan algebra bundle structure and if ξ has no ideal bundles except itself and the zero bundle, then we call ξ a simple Jordan algebra bundle.

Definition 2.7. A Jordan algebra bundle ξ is said to be the direct sum of the ideal bundles $\xi_1, \xi_2, \ldots, \xi_n$ provided, $\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$

3. Semisimple Jordan algebra bundles

Definition 3.1. A Jordan algebra bundle $\xi = (\xi, p, X)$ is locally trivial if, for every open set $U \subseteq X$, there exists a Jordan algebra J, together with a (vector bundle) trivialization $\varphi : \xi|_U \to U \times J$, which is fibrewise a homeomorphism of Jordan algebras.

Not every Jordan algebra bundle is locally trivial, as the following example shows.

Example 3.2. Let J be a nonzero real Jordan algebra and $\mathbb{R}$ the field of real numbers. Then $\xi = (\mathbb{R} \times J, p, \mathbb{R})$ is a trivial vector bundle. Let $\theta : \xi \otimes \xi \to \xi$ be defined by, $\theta(t, (u, v)) = t(uv)$ for all $t \in \mathbb{R}$, $u, v \in J$. Then, θ is continuous and induces a Jordan algebra structure on each fibre $\xi_x = x \times J$. Hence, ξ is a Jordan algebra bundle. But ξ is not locally trivial because ξ_t for $t \neq 0$ is a nonzero algebra whereas ξ_0 is the zero algebra.

Remark 3.3. From Atiyah [8, p. 4], we reproduce the following things without any changes. Suppose that V and W are vector spaces and that $E = X \times V$ and $F = X \times W$ are the corresponding product bundles. Then $\phi : E \to F$ determines a map $\Phi : X \to \mathrm{Hom}(V, W)$ by formula $\Phi(x)(v) = \phi(v)$. Moreover, if we give $\mathrm{Hom}(V, W)$ its usual topology, then Φ is continuous; conversely, any such continuous map $\Phi : X \to \mathrm{Hom}(V, W)$ determines a homomorphism $\phi : E \to F$.

Theorem 3.4. *Every semisimple Jordan algebra bundle is locally trivial.*

Proof. Let ξ be a semisimple Jordan algebra bundle. The local triviality of ξ as a vector bundle is given by the vector bundle isomorphism $\alpha : U \times V \to p^{-1}(U)$. Then $\theta : \xi \otimes \xi \to \xi$ induces the morphism $\widehat{\theta} : U \times (V \otimes V) \to U \times V$ given by,

$$\widehat{\theta}(x, (v, w)) = \alpha_x^{-1}\theta(\alpha_x(v), \alpha_x(w)). \tag{3.1}$$

Then, from above Remark (3.3), $x \mapsto \widehat{\theta}_x$ defines a continuous mapping from U to $M \subseteq \mathrm{Hom}(V \times V, V)$, the space of all Jordan multiplications defined on V endowed with the subspace topology. Since the Jordan algebra $J_0 = (V, \widehat{\theta}_{x_0})$, $x_0 \in U$ is semisimple, J_0 is rigid by [9, Corollary 1.3]. That is the orbit

$$G(x_0) = \{ g.\theta_{x_0} |\ g \in Aut(V) \},$$

with respect to the Lie group $G = Aut(V)$ is open in M.

Let $U' = \{x \in U | \widehat{\theta}_x \in G(x_0)\}$. Then, for each $x \in U'$, there exists a g_x in G such that

$$\widehat{\theta}_x = g_x.\widehat{\theta}_{x_0} = g_x \widehat{\theta}_{x_0}(g_x^{-1} \oplus g_x^{-1}). \tag{3.2}$$

Further, since G and $G(x_0)$ satisfy the hypothesis of Aren's theorem [10] G/G_0 is homeomorphic to $G(x_0)$, where G_0 is the stability subgroup corresponding to $\widehat{\theta}_{x_0}$. Also $G \to G/G_0$ is a principal bundle [11, p. 33] together with a local cross section given by $gG_0 \to g$ [12, p. 126]. Hence, the map $x \mapsto g_x$ becomes the composition of the continuous maps

$$x \mapsto \widehat{\theta}_x \mapsto g_x G_0 \mapsto g_x.$$

Therefore, we can define the vector bundle isomorphism $\phi : U \times J_0 \to p^{-1}(U)$ by $\phi(x, v) = \alpha_x(g_x(v))$. The map ϕ preserves the Jordan multiplication

$$
\begin{aligned}
\phi\left(x, \widehat{\theta}_{x_0}(a,b)\right) &= \alpha_x g_x \widehat{\theta}_{x_0}(a,b) \\
&= \alpha_x \widehat{\theta}_x(g_x(a), g_x(b)) \quad \text{by } (3.2) \\
&= \alpha_x \alpha_x^{-1} \theta_x(\alpha_x g_x(a), \alpha_x g_x(b)) \quad \text{by } (3.1) \\
&= \theta_x(\phi(x,a), \phi(x,b)).
\end{aligned}
$$

Hence, ϕ gives the required local triviality of the Jordan algebra bundle ξ. □

4. Decomposition theorem for Jordan algebra bundle

Lemma 4.1. *Let $(X \times V, q, X)$ be a trivial vector bundle and $(X \times J, p, X)$ a trivial semisimple Jordan bundle. Suppose $\phi : X \times V \to X \times J$ is a vector bundle monomorphism such that for each $x \in X$, $\phi(x, V)$ is an ideal in J. Then, there exists a finite open partition $\cup_i X_i = X$ such that $\phi_x(V) = \phi_y(V)$ for $x, y \in X_i$. In particular, if X is connected, for all $x, y \in X$, $\phi_x(V) = \phi_y(V)$.*

Proof. The map $\phi : X \times V \to X \times J$ being a vector bundle morphism, $x \mapsto \phi_x$ is a continuous map from X to Hom(V, J), the vector space of all linear transformations from V to J. If $\tilde{I}$ denotes the collection of all distinct ideals of J whose dimension is equal to that of V, then by the semisimplicity of J, $\tilde{I}$ is a finite set [13, Corollary.4.6, p. 98]. Let $\tilde{I} = \{I_1, I_2, \ldots, I_n\}$ and let $X_i = \{x \in X |\ \phi_x(V) = I_i\}$ for $i = 1, 2, \ldots, n$. Let $x \in X$. Since $\phi_x(V)$ is an ideal in J, and $\dim \phi_x(V) = \dim V$, we have $\phi_x(V) = I_i$ for some i. Thus, $x \in X_i$, and consequently $X = \cup_i X_i$.

It is enough to prove that each X_i is open in X. Let $\tilde{J}_i$ be the vector subspace $\tilde{J}_i = \{T \in \text{Hom}(V, J) |\ T(V) \subseteq I_i\}$. Then $\tilde{J}_i$ is a closed subset of Hom(V, J) being a linear subspace of the vector space Hom(V, J). Since $x \mapsto \phi_x$ is continuous and $\tilde{J}_i$ is closed in Hom(V, J), $\tilde{X}_i = \{x \in X |\ \phi_x \in \tilde{J}_i\}$ is closed in X.

To prove: $\tilde{X}_i = X_i$

$x \in X_i \Rightarrow \phi_x(V) = I_i \Rightarrow \phi_x \in \tilde{J}_i \Rightarrow x \in \tilde{X}_i \Rightarrow X_i \subseteq \tilde{X}_i$. Now if $x \in \tilde{X}_i$, $\phi_x \in \tilde{J}_i$. Then, $\phi_x(V) \subseteq I_i$. But since $\dim \phi_x(V) = \dim I_i$, $\phi_x(V) = I_i$. So $x \in X_i$. Thus, $X_i = \tilde{X}_i$.

Consider $\tilde{X}_i \cap \tilde{X}_j$ and let $i \neq j$

$$x \in \tilde{X}_i \cap \tilde{X}_j \Rightarrow \phi_x \in \tilde{J}_i \text{ and } \phi_x \in \tilde{J}_j \Rightarrow \phi_x(V) \subseteq I_i \cap J_j = 0$$

Therefore, $\tilde{X}_i \cap \tilde{X}_j = \varnothing$

Thus, $\tilde{X}_i{}'s$ are all disjoint collection of closed sets. Hence, X_j is the complement of $\cup_{i \neq j} X_i$. So X_j is open in X. We here note that if X is connected, then for all $x, y \in X$, $\phi_x(V) = \phi_y(V)$. □

Lemma 4.2. *Let ξ' be an ideal bundle of a semisimple Jordan algebra bundle ξ. Then, $h_1(\xi') = \cup_{x \in X} h_1(\xi'_x)$ is an ideal bundle of $h(\xi)$ where*

$$h_1(\xi'_x) = \{T \in h(\xi_x) |\ T(\xi_x) \subseteq \xi'_x\}$$

Proof. Each $h_1(\xi'_x)$ is an ideal in $h(\xi_x)$. It is enough to prove that $h_1(\xi')$ is a vector bundle. Since the bundle ξ is semisimple and ξ' is semisimple being its ideal bundle, ξ and ξ' are locally trivial by Theorem (3.4). So we have Jordan algebras J and J' and Jordan bundle isomorphisms $\Phi : U \times J \to \cup_{x \in X} \xi_x$ and $\Psi : U \times J' \to \cup_{x \in X} \xi'_x$. Then, $\Phi^{-1}\Psi : U \times J' \to U \times J$ is a vector bundle monomorphism satisfying the hypotheses of Lemma (4.1). Hence, there exists a finite open cover

$\{U_i\}$ for U such that $(\Phi^{-1}\Psi)(x, J') = (\Phi^{-1}\Psi)(x', J')$ for all $x, x' \in U_i$. Consequently, the neighbourhood U can be shrunk so that there exists an ideal I of J and such that Φ maps $U \times I$ onto $\cup_{x\in U}\xi'_x$. Let $h_1(I) = \{T \in h(J) | \ T(J) \subseteq I\}$. Given $T \in h_1(I)$, consider $g(\Phi)_x(T) = \Phi_x T \Phi_x^{-1}$.

$$\begin{aligned} g(\Phi)_x(T)(\xi_x) &= \left(\Phi_x T \Phi_x^{-1}\right)(\xi_x) \\ &= (\Phi_x T)(J) \subseteq \Phi_x(I) = \xi'_x. \end{aligned}$$

Hence, $g(\Phi)_x(T) \in h_1(\xi'_x)$. That is, $g(\Phi)$ maps $U \times h_1(I)$ onto $\cup_{x\in U} h_1(\xi'_x)$. That is, $h_1(\xi')$ is an ideal sub-bundle of $h(\xi)$. □

Theorem 4.3. *Every semisimple Jordan algebra bundle can be uniquely written as the direct sum of simple ideal bundles.*

Proof. Let ξ be a semisimple Jordan algebra bundle. Each fibre ξ_x has a unit element being a semisimple Jordan algebra [13, Theorem 4.7, p. 99]. Hence, the corresponding Lie algebra bundle $L(\xi)$ exists [1]. The semisimplicity of ξ_x implies that of $L(\xi_x)$ [7, p. 805] and so $L(\xi)$ is a semisimple Lie algebra bundle. Then, $L(\xi)$ can be written as follows:

$$L(\xi) = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where each L_i is a simple ideal bundle of $L(\xi)$ [14]. Also each L_i is of the form $L_i = \cup_{x\in X}(h_i)_x \oplus (A_i)_x \oplus (\overline{B_i})_x$, [7, Lemma 1, p. 789]. Each $\xi_i = \cup_{x\in X}(A_i)_x$ is an ideal bundle of ξ and each ξ_i is simple since L_i is simple. Hence, $\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$.

Let us prove the uniqueness of the decomposition. Let, ξ be expressed as follows:

$$\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n = \xi = \xi'_1 \oplus \xi'_2 \oplus \cdots \oplus \xi'_m,$$

where each ξ_i and ξ'_j are simple ideal bundles. Consider $\oplus_1^n h_1(\xi_i)_x$, where $h_1(\xi_i)_x = \{T \in h(\xi_x) \big| T(\xi_x) \subseteq (\xi_i)_x\}$. We have $\oplus_1^n h_1(\xi_i)_x \subseteq h(\xi_x)$. Let $L(a) \in h(\xi_x)$, then a can be written as $a = \oplus_1^n a_i, \ a_i \in (\xi_i)_x$. Since $(\xi_i)_x$ is an ideal in ξ_x, $L(a_i)$ maps ξ_x into $(\xi_i)_x$. So

$$L(a) = \oplus_{i=1}^n L(a_i) \in \oplus_1^n h_1(\xi_i)_x,$$

Let $[L(a), L(b)] \in h(\xi_x)$, where $a = \oplus_1^n a_i$ and $b = \oplus_1^n b_i$, $a_i, b_i \in (\xi_i)_x$. Since $[L(a_i), L(b_j)](u) \in (\xi_i)_x \cap (\xi_j)_x = 0$ for $i \neq j$, we obtain that

$$[L(a), L(b)] = \oplus_1^n [L(a_i), L(b_j)] \in \oplus_1^n h_1(\xi_i)_x.$$

Consequently, $h(\xi_x) \subseteq \oplus_1^n h_1(\xi_i)_x$. Therefore, $h(\xi) = h_1(\xi_1) \oplus h_1(\xi_2) \oplus \cdots \oplus h_1(\xi_n)$. Similarly, $h(\xi) = \oplus_1^n h_1(\xi'_j)$. Hence

$$\begin{aligned} L(\xi) &= L_1(\xi_1) \oplus L_1(\xi_2) \oplus \cdots \oplus L_1(\xi_n) \\ &= L_1(\xi'_1) \oplus L_1(\xi'_2) \oplus \cdots \oplus L_1(\xi'_m) \end{aligned}$$

where $L_1(\xi_i) = h_1(\xi_i) \oplus \xi_i \oplus \xi_i$ and $L_1(\xi'_j) = h_1(\xi'_j) \oplus \xi'_j \oplus \xi'_j$ are ideal bundles of $L(\xi)$ by Lemma (4.2). Then by [14, Theorem 2.8], we obtain that $m = n$ and $L_1(\xi_i)$ coincides with one of the $L_1(\xi'_j)$. Then, obviously ξ'_j coincides with ξ_i except for the order. □

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