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A frictional contact problem with normal damped response conditions and thermal effects for a thermo-electro-viscoelastic material

Abdelmoumene Djabi
Applied Mathematics Laboratory, Ferhat Abbas- Setif 1 University, Setif, Algeria

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Abstract

Purpose – The paper presents a mathematical problem involving quasistatic contact between a thermo-electro-viscoelastic body and a lubricated foundation, where the contact is described using a version of Coulomb's law of friction that includes normal damped response conditions and heat exchange with a conductive foundation. The constitutive law for the material is thermo-electro-viscoelastic. The problem is formulated as a system that includes a parabolic equation of the first kind for the temperature, an evolutionary elliptic quasivariational inequality for the displacement and a variational elliptic equality for the electric stress. The author establishes the existence of a unique weak solution to the problem by utilizing classical results for evolutionary quasivariational elliptic inequalities, parabolic differential equations and fixed point arguments.
Design/methodology/approach – The author establishes a variational formulation for the model and proves the existence of a unique weak solution to the problem using classical results for evolutionary quasivariational elliptic inequalities, parabolic difierential equations and fixed point arguments.
Findings – The author proves the existence of a unique weak solution to the problem using classical results for evolutionary quasivariational elliptic inequalities, parabolic difierential equations and fixed point arguments.
Originality/value – The author studies a mathematical problem between a thermo-electro-viscoelastic body and a lubricated foundation using a version of Coulomb's law of friction including the normal damped response conditions and the heat exchange with a conductive foundation, which is original and requires a good understanding of modeling and mathematical tools.

Keywords Conductive foundation, Fixed point, Frictional contact, Normal damped response, Thermo-piezoelectric, Variational inequality
Paper type Research paper

JEL Classification — 47J30, 70F40, 74F05, 74M10, 74M15

The author would like to thank the reviewers for the important comments. This article presents an extension of sources [2, 11]. The article is based on the work of the abovementioned sources by extending the mathematical model to a multiphysical thermoelectromechanical law with three elements, introducing complex boundary conditions of different physical types and models.

For the mathematical model, the existence of a unique weak solution to the problem is demonstrated using results on quasivariational elliptical inequalities, parabolic differential equations and fixed point arguments.

These contributions represent an important step forward in the field of boundary problems in contact mechanics.

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1. Introduction

Contact phenomena between deformable bodies or between a deformable body and a foundation are ubiquitous phenomena in everyday life. The contact of a wheel with the ground, the contact of the brake shoe with the wheel or the gradual sinking in a wheelchair during a seated posture, are just a few everyday examples, among many others. Some industrial processes such as metal stamping and metal extrusion lead to evolution problems where contact and friction conditions are decisive. These phenomena call upon sophisticated mathematical models, which are represented by systems of partial differential equations with boundary conditions describing complex contact processes (with or without friction). The mathematical theory of contact problems allows rigorous modeling of contact phenomena based on the principles of continuum mechanics as well as on variational analysis and numerical models.

Important developments concerning the mathematical study, numerical mechanics of the problems resulting from the mechanics of the contact were carried out during XX^{th} century. The first contact problem between a deformable body and a foundation was stated by Signorini and first solved by Fichera. Duvaut and Lions were the first to work on the mathematical theory of contact mechanics; They introduced variational formulations of contact problems and provided existence and uniqueness results. Subsequently, several new works have focused on the resolution of these variational problems such as the work of Refs [1–6]. However, mathematical theory of contact problems is a very broad field of study where many issues remain to be investigated.

The importance of the mathematical study of such problems leads to give coupled conditions for the material and the contact conditions.

Recent researches use coupled laws of behavior between mechanical and electric effects or between mechanical and thermal effects. For the case of coupled laws of behavior between mechanical and electric effects, numerous papers use different electro-mechanical conditions such as [2, 5, 7, 8]. For the case of coupled laws of behavior between mechanical and thermal effects, we can found several models in Refs [4, 6, 7, 9–12]. For this, the new researches use coupled conditions between the mechanical, electrical and thermal behavior of the material see [13–15].

The pyroelectric effect is characterized by a coupling between the electrical and thermal effects and does not produce mechanical effects. The pyroelectric effect used for fire alarm, pyroelectric detectors and sensors. Some pyroelectric applications can be found in Refs [9, 16, 17].

The piezoelectric effect is a coupling between the mechanical and electrical properties of the materials and does not produce heat effects. This coupling, leads to the appearance of electric field in the presence of a mechanical stress and conversely. A mechanical stress is generated when electric potential is applied. The first effect is used in sensors and the reverse effect is used in actuators. During the past few years, a lot of attention has been focused on the piezoelectric effects, such as [8, 18, 19].

Recent modeling, analysis and numerical simulations of electro-mechanical, thermo-mechanical and thermo-electro-mechanical contact problems with friction can be found in Refs [2, 4, 5, 7, 10, 11, 14]. General models of energy can be found in Refs [1]. a generalized Coulomb friction version is given in Refs [3, 20]. Indeed, the authors used the normal damped response conditions for a lubricated foundation; see, for instance [21, 22].

Nowadays, there are increasing efforts to investigate coupled-field problems. In this respect, electro-thermo-mechanical coupling is one particular application, which occurs, for example, in Car fan or Computer fan. In this paper we use mixed conditions between electrical, thermal and mechanical conditions. The law of behavior used is given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{G}\varepsilon(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t) - \mathcal{M}\theta(t), \tag{1.1}$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \boldsymbol{B}E(\varphi(t)) + \mathcal{P}\theta(t), \tag{1.2}$$

This law is thermo-electro-viscoelastic Kelvin-Voigt model where $\mathcal{A}$, $\mathcal{G}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively and $E(\varphi) = -\nabla\varphi$, $\mathcal{E} = (e_{ijk})$, $\mathcal{M}$, $\boldsymbol{B}$, $\mathcal{P}$ are respectively electric field, piezoelectric, thermal expansion, electric permittivity, pyroelectric tensors, and $\mathcal{E}^*$ is the transpose of $\mathcal{E}$. Note also that when $\mathcal{E} = 0$ and $\mathbf{D} = 0$, (1.1)–(1.2) becomes the Kelvin-Voigt thermo-viscoelastic constitutive relation used in [10]. Moreover, when $\mathcal{M} = 0$ and $\mathcal{P} = 0$, the relations (1.1)–(1.2) becomes the Kelvin-Voigt electro-viscoelastic.

The evolution of the temperature field obtained from the conservation of energy and defined with the following differential equation

$$\dot{\theta}(t) - div\mathcal{K}(\nabla\theta(t)) = \psi(\mathcal{M}\theta(t), \mathbf{u}(t)) + q_{th}, \tag{1.3}$$

where θ is the temperature, $\mathcal{K}$ denotes the thermal conductivity tensor, $\mathcal{M}$ the thermal expansion tensor, q_{th} is the density of volume heat sources and ψ is a nonlinear function, assumed here depends on thermal expansion tensor and the displacement field.

Processes of contact are present in numerous domestic and industrial applications which may change from body to body depending on the constitutive law of the body studied. In this paper we use mechanical, thermal and electrical contact conditions.

For the mechanical contact conditions, the Coulomb friction is one of the most useful friction laws and known from the literature. This law has two basic ingredients namely the concept of friction threshold and its dependence on the normal stress. We use normal damped response conditions associated with the Coulomb's law of dry friction given by:

$$\begin{cases} \sigma_\nu = -p_\nu(\dot{\mathbf{u}}_\nu(t)), \ ||\boldsymbol{\sigma}_\tau|| \leq p_\tau(\dot{\mathbf{u}}_\nu(t)), \\ \boldsymbol{\sigma}_\tau = -p_\tau(\dot{\mathbf{u}}_\nu(t))\dfrac{\dot{\mathbf{u}}_\tau(t)}{||\dot{\mathbf{u}}_\tau(t)||}, \text{ if } \dot{\mathbf{u}}_\tau(t) \neq 0, \end{cases} \tag{1.4}$$

This condition models frictional contact between the body and lubricated foundation where p_ν and p_τ represent given contact functions, $\dot{\mathbf{u}}_\nu$ and $\dot{\mathbf{u}}_\tau$ denote the normal and tangential velocity field respectively.

On the other hand in the study of this problem, we make the assumption that the foundation is thermo-electrical conductive, the electrical conductivity assumed depends on the linear function H defined as:

$$H(\varphi(t)) = \varphi(t) - \varphi_0 \tag{1.5}$$

Here, we assume that the electrical conductivity H depends only on the electric potential φ and the initial electric potential φ_0

Moreover, for the thermal conductivity we use the following conditions on the contact surface

$$-k_{ij}\theta_{,i}n_j = k_e(\theta(t) - \theta_F) \tag{1.6}$$

where k_e is the heat exchange coefficient between the body and the obstacle, θ_F is the temperature of the foundation.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we introduce the notations, some preliminaries results, list of the assumptions on the data and we give the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result theorem 4.1. The proof of the theorem is based on evolutionary elliptic variational inequalities, ordinary differential equations and fixed point arguments.

2. The model

The physical setting is the following. A thermo-electro-viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. This boundary is divided into three open disjoint parts Γ_1, Γ_2 and Γ_3, on one hand and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. Let $T > 0$ and $[0, T]$ be the time interval of interest. The body is subjected to the action of body forces of density $\mathbf{f}_0$, volume electric charges of density q_0 and a heat source of constant strength q_{th}.The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_b is prescribed on $\Gamma_b \times (0, T)$. Moreover, we suppose that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0, T)$. Moreover, we suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach to the process.

In the reference configuration, the body is in contact with a foundation, over the contact surface Γ_3. The model of the contact is frictional specified by the normal damped response conditions and it is associated with the Coulomb's law of dry friction for the mechanical contact, an associated temperature boundary condition for the thermal contact and electrical conditions modeling electric potential exchange between the body and the conductive foundation.

The classical formulation of the mechanical problem is as follows.

Problem $\mathcal{P}$. Find the displacement field $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, the electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, the electric displacement field $\mathbf{D} : \Omega \times [0, T] \to \mathbb{R}^d$ and the temperature $\theta : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{G}\varepsilon(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t) - \mathcal{M}\theta(t), \text{in}\, \Omega \times (0, T), \tag{2.1}$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \boldsymbol{B}\nabla(\varphi(t)) + \mathcal{P}\theta(t), \quad \text{in}\, \Omega \times (0, T), \tag{2.2}$$

$$\dot{\theta}(t) - div\mathcal{K}(\nabla\theta(t)) = \psi(\mathcal{M}\theta(t), \mathbf{u}(t)) + q_{th}, \quad \text{in}\, \Omega \times (0, T), \tag{2.3}$$

$$Div\boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in}\, \Omega \times (0, T), \tag{2.4}$$

$$div\mathbf{D} = q_0 \quad \text{in}\, \Omega \times (0, T), \tag{2.5}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on}\, \Gamma_1 \times (0, T), \tag{2.6}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on}\, \Gamma_2 \times (0, T), \tag{2.7}$$

$$\begin{cases} \sigma_\nu = -p_\nu(\dot{\mathbf{u}}_{\boldsymbol{\nu}}(t)), \ ||\boldsymbol{\sigma}_\tau|| \leq p_\tau(\dot{\mathbf{u}}_{\boldsymbol{\nu}}(t)), \\ \boldsymbol{\sigma}_\tau = -p_\tau(\dot{\mathbf{u}}_{\boldsymbol{\nu}}(t))\dfrac{\dot{\mathbf{u}}_\tau(t)}{||\dot{\mathbf{u}}_\tau(t)||}, \text{ if } \dot{\mathbf{u}}_\tau(t) \neq 0, \end{cases} \quad \text{on}\, \Gamma_3 \times (0, T), \tag{2.8}$$

$$-k_{ij}\theta_{,i}n_j = k_e(\theta(t) - \theta_F) \quad \text{on}\, \Gamma_3 \times (0, T), \tag{2.9}$$

$$\mathbf{D}.\boldsymbol{\nu} = H(\varphi(t)) \quad \text{on}\, \Gamma_3 \times (0, T), \tag{2.10}$$

$$\theta = 0 \quad \text{on}\, (\Gamma_1 \cup \Gamma_2) \times (0, T), \tag{2.11}$$

$$\varphi = 0 \quad \text{on}\, \Gamma_a \times (0, T), \tag{2.12}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = q_b \quad \text{on}\, \Gamma_b \times (0, T), \tag{2.13}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \quad \text{in}\, \Omega. \tag{2.14}$$

we now describe problem (2.1)–(2.14) and provide explanation of the equations and the boundary conditions.

Equations (2.1) and (2.2) represent the thermo-electro-viscoelastic constitutive law, the evolution of the temperature field is governed by differential equation given by the relation (2.3) where ψ is the mechanical source of the temperature growth, assumed to be rather general function of the strains. Next equations (2.4) and (2.5) are the steady equations for the stress and electric-displacement field, conditions (2.6) and (2.7) are the displacement and traction boundary conditions. Equation (2.11) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2)$. Next, (2.12) and (2.13) represent the electric boundary conditions for the electrical potential on Γ_a and the electric charges on Γ_b respectively. Equation (2.14) represents the initial displacement field and the initial temperature field where the initial displacement is $\mathbf{u}_0$, and θ_0 is the initial temperature.

We turn to the contact conditions (2.8)–(2.10) describe a mixed contact on the potential contact surface Γ_3. The relation (2.8) describes a normal damped response conditions with the Coulomb's law of dry friction (2.9) represents an associated temperature boundary condition on contact surface. Finally, (2.10) shows models the electric conductivity.

3. Variational formulation

In order to obtain the variational formulation of the Problem $\mathcal{P}$, we use the following notations and preliminaries

3.1 Notations and preliminaries

We present the notation we recall some preliminary material. For more details, we refer the reader to [23–26]. In what follows the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. We recall that the canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively are given by

$$\begin{aligned} \mathbf{u}.\mathbf{v} &= u_i.v_i\,, \ \|\mathbf{v}\| = \sqrt{\mathbf{v}.\mathbf{v}} \ \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma}.\boldsymbol{\tau} &= \sigma_{ij}.\tau_{ij}\,, \|\boldsymbol{\tau}\| = \sqrt{\boldsymbol{\tau}.\boldsymbol{\tau}} \ \text{ for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with outer Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega = \Gamma$. We introduce the spaces

$$\begin{gathered} H = L^2(\Omega)^d = \{\mathbf{v} = (v_i) : v_i \in L^2(\Omega)\}, \\ \mathcal{H} = \{\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) : \boldsymbol{\tau}_{ij} = \boldsymbol{\tau}_{ji} \in L^2(\Omega)\}, \\ H^1(\Omega)^d = \{\mathbf{v} = (v_i) \in H : \varepsilon(\mathbf{v}) \in \mathcal{H}\}, \\ \mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} : Div\boldsymbol{\tau} \in H\}, \end{gathered}$$

Here $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the linearized deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad Div\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij,j}).$$

The spaces H, $\mathcal{H}$, $H^1(\Omega)^d$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by:

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega u_i.v_i dx, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}dx,$$
$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} = (\mathbf{u},\mathbf{v})_H + (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H .$$

and with the associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. We introduce the closed subspaces of $H^1(\Omega)$ and $H^1(\Omega)^d$ defined by

$$\begin{aligned} V &= \left\{\mathbf{v}\in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_1\right\}, \\ W &= \left\{\phi\in H^1(\Omega)^d : \phi = 0 \text{ on } \Gamma_a\right\}, \\ \mathcal{W} &= \left\{ \mathbf{D} = (\mathbf{D}_i) : \mathbf{D}_i\in L^2(\Omega), div\mathbf{D}\in L^2(\Omega)\right\}, \\ \mathcal{Z} &= \left\{\mathbf{w}\in H^1(\Omega) : \mathbf{w} = 0 \text{ a.e on } \Gamma_1\cup\Gamma_2\right\}, \end{aligned}$$

Since $meas\Gamma_a > 0$ and $meas\Gamma_1 > 0$, the Korn's and Friedrichs-Poincaré inequalities hold, thus,

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}} \geq C_0\|\mathbf{v}\|_{H^1(\Omega)^d}, \quad \forall \mathbf{v}\in V, \tag{3.1}$$

$$\|\nabla\phi\|_{\mathcal{W}} \geq C_1\|\phi\|_{H^1(\Omega)}, \quad \forall\phi\in W, \tag{3.2}$$

$$\|\nabla\mathbf{w}\|_H \geq C_2\|\mathbf{w}\|_{H^1(\Omega)}, \quad \forall\mathbf{w}\in\mathcal{Z}, \tag{3.3}$$

where here and below C_0, C_1 and C_2 are positive constants that depend on the problem data and are independent of the solutions.

On the spaces V, W and $\mathcal{Z}$, we define the following inner products

$$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}, \ \forall\mathbf{u},\mathbf{v}\in V, \tag{3.4}$$

$$(\varphi,\phi)_W = (\nabla\varphi,\nabla\phi)_{\mathcal{W}}, \ \forall\varphi,\phi\in W, \tag{3.5}$$

$$(\mathbf{w},\mathbf{z})_{\mathcal{Z}} = (\nabla\mathbf{w},\nabla\mathbf{z})_H, \ \forall\mathbf{w},\mathbf{z}\in\mathcal{Z}, \tag{3.6}$$

where

$$\begin{aligned} (\varphi,\phi)_W &= \int_\Omega \nabla\varphi.\nabla\phi dx, \\ (\mathbf{D},\mathbf{E})_{\mathcal{W}} &= \int_\Omega \mathbf{D}.\mathbf{E}dx + \int_\Omega div\mathbf{D}.div\mathbf{E}dx . \end{aligned}$$

It follows from (3.1) and (3.4) that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent norms on V, (3.2) and (3.5) follows that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_W$ are equivalent norms on W and from (3.3) and (3.6) we deduce that $\|.\|_{H^1(\Omega)}$ and $\|.\|_{\mathcal{Z}}$ are equivalent norms on $\mathcal{Z}$. Therefore, the spaces $(V,(\cdot,\cdot)_V)$, $(W,(\cdot,\cdot)_W)$ and $(\mathcal{Z},(\cdot,\cdot)_{\mathcal{Z}})$ are real Hilbert spaces. Moreover, by the Sobolev trace theorem and the equalities (3.4)–(3.6), there exists C_0, C_1 and C_2, three positive constants, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq C_0\|\mathbf{v}\|_V, \forall\mathbf{v}\in V, \tag{3.7}$$

$$\|\phi\|_{L^2(\Gamma_3)} \leq C_1\|\phi\|_W, \forall\phi\in W, \tag{3.8}$$

$$\|\mathbf{z}\|_{L^2(\Gamma_3)} \leq C_2\|\mathbf{z}\|_{\mathcal{Z}}, \forall\mathbf{z}\in\mathcal{Z}. \tag{3.9}$$

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace map $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ, and we denote by v_ν and $\mathbf{v}_\tau$ the *normal* and *tangential* components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}. \tag{3.10}$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ we define its *normal* and *tangential* components by

$$\boldsymbol{\sigma}_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{\sigma}_\nu\boldsymbol{\nu}, \tag{3.11}$$

and for all $\boldsymbol{\sigma} \in \mathcal{H}_1$, $\boldsymbol{\theta} \in H^1(\Omega)^d$ and $\mathbf{D} \in \mathcal{W}$ the following three Green's formulas holds:

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\mathbf{v}da \quad \forall\mathbf{v} \in H^1(\Omega)^d, \tag{3.12}$$

$$(\boldsymbol{\theta}, \nabla\mathbf{w})_H + (div\boldsymbol{\theta}, \mathbf{w})_{L^2(\Omega)} = \int_\Gamma \theta\nu.\mathbf{w}da \quad \forall\mathbf{w} \in H^1(\Omega), \tag{3.13}$$

$$(\mathbf{D}, \nabla\phi)_H + (div\mathbf{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D}\nu.\phi da \quad \forall\phi \in H^1(\Omega), \tag{3.14}$$

where

$$div\theta = (\theta_{i,i}), \ div\mathbf{D} = (\mathbf{D}_{i,i}),$$

We recall the following definition of an Gelfand triple.

Definition 3.1. Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V. We write

$$V \subset H \subset V'.$$

and we say that the inclusions above define a Gelfand triple. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$, and $\|\cdot\|_{V'}$, the norms on the spaces V, H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V. Note that if $f \in H$ then

$$(f, \mathbf{v})_{V'\times V} = (f, \mathbf{v})_H, \forall\mathbf{v} \in H. \tag{3.15}$$

and we recall the following Theorem

Theorem 3.2. *Let $V \subset H \subset V'$ be a Gelfand triple. Assume that A: $V \to V'$ is a hemicontinuous and monotone operator that satisfies*

$$(A\mathbf{v}, \mathbf{v})_{V'\times V} \geq \omega\|\mathbf{v}\|_V^2 + \zeta, \forall\mathbf{v} \in V \tag{3.16}$$

$$\|A\mathbf{v}\|_{V'}^2 \leq C(\|\mathbf{v}\|_V + 1), \forall\mathbf{v} \in V \tag{3.17}$$

For some constants $\omega > 0$, $C > 0$ and $\zeta \in \mathbb{R}$ Then, given $\mathbf{u}_0 \in H$ and $f \in L^2(0, T; V')$, there exists a unique function $\mathbf{u} \in L^2(0, T; V) \cap \mathcal{C}(0, T; H)$ satisfies

$$\begin{aligned}
&\mathbf{u} \in L^2(0, T; V) \cap \mathcal{C}(0, T; H), \dot{\mathbf{u}} \in L^2(0, T; V'),\\
&\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = f(t) \text{ a.e } t \in (0, T),\\
&\mathbf{u}(0) = \mathbf{u}_0
\end{aligned}$$

The proof of this abstract result may be found in [3, p. 141], and will be used in the study of thermal problem presented in Section 5.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$, where $1 \le p \le \infty$ and $k > 1$. For $T > 0$ we denote by $\mathcal{C}(0, T; X)$ and $\mathcal{C}^1(0, T; X)$ the space of continuous and continuously differentiable functions from $[0, T]$ to X, respectively, with the norms

$$\|\mathbf{f}\|_{\mathcal{C}(0,T;X)} = \max_{t\in[0,T]} \|\mathbf{f}(t)\|_X,$$

$$\|\mathbf{f}\|_{\mathcal{C}^1(0,T;X)} = \max_{t\in[0,T]} \|\mathbf{f}(t)\|_X + \max_{t\in[0,T]} \|\dot{\mathbf{f}}(t)\|_X,$$

Moreover, we use the dot above to indicate the derivative with respect to the time variable and if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

3.2 Assumptions on the data

We now list the assumptions on the problem's data.

The *viscosity operator* $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)\| \le L_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\| \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. \mathbf{x} \in \Omega. \\ \text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2))\cdot(\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\|^2 \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varepsilon \in \mathbb{S}^d. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, 0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.18}$$

The *elasticity operator* $\mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \varepsilon_1) - \mathcal{G}(\mathbf{x}, \varepsilon_2)\|. \le L_{\mathcal{G}}\|\varepsilon_1 - \varepsilon_2\| \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varepsilon \in \mathbb{S}^d. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, 0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.19}$$

The *piezoelectric operator* $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a)}\mathcal{E}(x, \tau) = (e_{ijk}\tau_{jk}), \forall \tau = (\tau_{jk}) \in \mathbb{S}^d, a.e.\, x \text{ in } \Omega. \\ \text{(b)}e_{ijk} = e_{ikj} \in L^\infty(\Omega), 1 \le i, j, k \le d. \end{cases} \tag{3.20}$$

The *thermal expansion operator* $\mathcal{M} : \Omega \times \mathbb{R} \to \mathbb{R}$

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{M}} > 0 \text{ such that} \\ \quad \|\mathcal{M}(\mathbf{x},\theta_1) - \mathcal{M}(\mathbf{x},\theta_2)\| \le L_{\mathcal{M}}\|\theta_1 - \theta_2\| \text{ for all } \theta_1,\theta_2 \in \mathbb{R}, a.e. \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto \mathcal{M}(\mathbf{x},\theta) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \theta \in \mathbb{R}. \\ \text{(c) The mapping } \mathbf{x} \to \mathcal{M}(x,0) \in \mathcal{H} \\ \text{(d) } m_{ij} = m_{ji} \in L^\infty(\Omega). \end{cases} \tag{3.21}$$

The *nonlinear constitutive function* $\psi : \Omega \times \mathbb{R} \times V \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_\psi > 0 \text{ such that} \\ \quad \|\psi(\mathbf{x},\mathcal{M}\theta_1,\mathbf{u}_1) - \psi(\mathbf{x},\mathcal{M}\theta_2,\mathbf{u}_2)\| \le L_\psi(\|\mathcal{M}\theta_1 - \mathcal{M}\theta_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\|) \\ \quad \text{for all } \theta_1,\theta_2 \in \mathbb{R}, \text{ for all } \mathbf{u}_1,\mathbf{u}_2 \in V, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \to \psi(x,\mathcal{M}\theta,\mathbf{u}) \text{ is Lebesgue measurable on } \Omega \text{ for any } \theta \in \mathbb{R}, \\ \quad \text{for any } \mathbf{u} \in V. \\ \text{(c) The mapping } \mathbf{x} \to \psi(x,0,0) \in L^2(\Omega). \end{cases} \tag{3.22}$$

The *electric permittivity operator* $\boldsymbol{B} = (\boldsymbol{B}_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } \boldsymbol{B}(x,E) = (\boldsymbol{B}_{ij}(x)E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, a.e. x \in \Omega. \\ \text{(b) } \boldsymbol{B}_{ij} = \boldsymbol{B}_{ji} \in L^\infty(\Omega), 1 \le i,j \le d. \\ \text{(c) There exists a constant } m_B > 0 \text{ such that } \boldsymbol{B}E.E \ge m_B|E|^2 \\ \quad \text{for all } E = (E_i) \in \mathbb{R}^d, \ a.e. \text{ in } \Omega. \end{cases} \tag{3.23}$$

The *pyroelectric operator* $\mathcal{P} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{P}} > 0 \text{ such that} \\ \quad \|\mathcal{P}(\mathbf{x},\theta_1) - \mathcal{P}(\mathbf{x},\theta_2)\| \le L_{\mathcal{P}}\|\theta_1 - \theta_2\| \quad \text{for all } \theta_1,\theta_2 \in \mathbb{R}^d, a.e. \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto \mathcal{P}(\mathbf{x},0) \text{ belongs to } \mathcal{W}. \end{cases} \tag{3.24}$$

The *thermal conductivity operator* $\mathcal{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{K}} > 0 \text{ such that} \\ \quad \|\mathcal{K}(\mathbf{x},r_1) - \mathcal{K}(\mathbf{x},r_2)\| \le L_{\mathcal{K}}\|r_1 - r_2\| \text{ for all } r_1,r_2 \in \mathbb{R}^d, a.e. \mathbf{x} \in \Omega. \\ \text{(b) } k_{ij} = k_{ji} \in L^\infty(\Omega)\, , 1 \le i,j \le d. \\ \text{(c) The mapping } \mathbf{x} \mapsto S(\mathbf{x},0,0) \text{ belongs to } L^2(\Omega). \end{cases} \tag{3.25}$$

The *contact functions* p_s: $\Gamma_3 \times \mathbb{R} \to \mathbb{R}_+, (s = \nu, \tau)$ satisfy

$$\begin{cases} \text{(a) There exists } L_s > 0 \text{ such that} \\ \quad \|p_s(\mathbf{x},\varphi_1) - p_s(\mathbf{x},\varphi_2)\| \le L_s\|r_1 - r_2\| \text{ for all } r_1,r_2 \in \mathbb{R}, a.e. \mathbf{x} \in \Gamma_3. \\ \text{(b) The mapping } \mathbf{x} \mapsto p_s(\mathbf{x},\varphi) \text{ is Lebesgue measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}. \\ \text{(c) The mapping } \mathbf{x} \mapsto p_s(\mathbf{x},0) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{3.26}$$

The *electrical conductivity function* $H : \Omega \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) There exists } m_{H_e} > 0 \text{ such that} \\ \quad (H(\mathbf{x},\varphi_1) - H(\mathbf{x},\varphi_2))\cdot(\varphi_1 - \varphi_2) \geq m_H \|\varphi_1 - \varphi_2\|^2 \\ \quad \text{for all } \varphi_1, \varphi_2 \in \mathbb{R}, a.e. \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto H(\mathbf{x},\varphi) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varphi \in \mathbb{R}. \\ \text{(c) The mapping } \mathbf{x} \mapsto H(\mathbf{x},0) \text{ belongs to } L^2(\Omega). \end{cases} \tag{3.27}$$

The density of volume forces, traction, volume electric charges, surface electric charges and the temperature evolution increase satisfy

$$\mathbf{f}_0 \in \mathcal{C}(0,T;H), \quad \mathbf{f}_2 \in \mathcal{C}\left(0,T;L^2(\Gamma_2)^d\right), \tag{3.28}$$

$$q_0 \in \mathcal{C}(0,T;L^2(\Omega)), \quad q_2 \in \mathcal{C}(0,T;L^2(\Gamma_b)), \tag{3.29}$$

$$\exists L_\theta > 0, \|\theta_1 - \theta_2\|_{\mathcal{Z}} \leq L_\theta \|\varphi_1 - \varphi_2\|_W \text{ for all } \theta_i \in \mathcal{Z}, \varphi_i \in W, \ i = 1,2. \tag{3.30}$$

The initial displacement, the potential of the foundation, the initial temperature and the temperature of the foundation fields satisfy

$$\mathbf{u}_0 \in V, \varphi_0 \in L^2(\Gamma_3), \ \theta_0 \in \mathcal{Z}, \ \theta_F \in L^2(\Gamma_3), \tag{3.31}$$

and the initial temperature field satisfies

$$q_{th} \in L^2(0,T;\mathcal{Z}'). \tag{3.32}$$

Using the above notation and Green's formulas given by (3.12)–(3.14), we obtain the variational formulation of the mechanical problem (2.1)–(2.14) for all functions $\mathbf{v} \in V$, $\mathbf{w} \in \mathcal{Z}$, $\phi \in W$ and a.e $t \in (0,T)$ given as follows,

3.3 Problem $\mathcal{P}_V$

Find the displacement field $\mathbf{u}$: $[0,T] \to V$, the stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$, the electric potential φ: $[0,T] \to W$, the electric displacement field $\mathbf{D}$: $[0,T] \to H$ and the temperature θ: $[0,T] \to V$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{G}\varepsilon(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t) - \mathcal{M}\theta(t), \tag{3.33}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t),\mathbf{v}) - j(\dot{\mathbf{u}}(t),\dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{3.34}$$

$$(\dot{\theta}(t),\mathbf{w})_{\mathcal{Z}'\times\mathcal{Z}} + (\mathcal{K}(\nabla\theta(t)),\nabla\mathbf{w}) = (G_1(\theta,\mathbf{w}) + \psi(\mathcal{M}\theta(t),\mathbf{u}(t)) + q_{th},\mathbf{w})_{\mathcal{Z}'\times\mathcal{Z}}, \tag{3.35}$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \boldsymbol{B}\nabla\varphi(t) + \mathcal{P}\theta(t), \tag{3.36}$$

$$(\mathbf{D}(t),\nabla\phi)_H + (q_e(t),\phi)_W = G_2(\varphi,\phi), \tag{3.37}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \quad \text{in } \Omega. \tag{3.38}$$

where $j : V \times V \to \mathbb{R}$, $\mathbf{f} : [0,T] \to V$, $q_e : [0,T] \to W$, $G_1 : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ and $G_2 : W \times W \to \mathbb{R}$ are respectively, defined by

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu)v_\nu da + \int_{\Gamma_3} p_\tau(u_\nu)\|v_\tau\| da, \tag{3.39}$$

$$(\mathbf{f}(t),\mathbf{v})_V = \int_\Omega \mathbf{f}_0(t)\cdot\mathbf{v}dx + \int_{\Gamma_2} \mathbf{f}_2(t)\cdot\mathbf{v}da, \tag{3.40}$$

$$(q_e(t),\phi)_W = \int_\Omega q_0(t)\phi dx - \int_{\Gamma_b} q_b(t)\phi da, \tag{3.41}$$

$$G_1(\theta,\mathbf{w}) = -\int_{\Gamma_3} k_e(\theta.\boldsymbol{\nu} - \theta_F.\boldsymbol{\nu})\mathbf{w}da, \tag{3.42}$$

$$G_2(\varphi,\phi) = \int_{\Gamma_3} H(\varphi(t))\phi da, \tag{3.43}$$

for all $\mathbf{u},\mathbf{v}\in V, \theta, \mathbf{w}\in\mathcal{Z}$ and $\phi\in W$ and $t\in[0,T]$. We note that the definitions of $\mathbf{f}$ and q_e are based on the Riesz representation theorem. Moreover, conditions (3.28) and (3.29) imply that

$$\mathbf{f}\in\mathcal{C}(0,T;V),\ q_e\in\mathcal{C}(0,T;W). \tag{3.44}$$

4. Existence and uniqueness of a solution

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that (3.18)–(3.32) hold. Then there exists a constant α_0 which depends only on $\Omega, \Gamma_1, \Gamma_3$ and $\mathcal{A}$ such that if*

$$(L_\nu + L_\tau) < \alpha_0, \tag{4.1}$$

where $\alpha_0 = \frac{m_\mathcal{A}}{C_0^2}$ such that $m_\mathcal{A}$ is defined in (3.18) and C_0 defined by (3.7).Then there exists a unique solution $\{\mathbf{u},\boldsymbol{\sigma},\theta,\varphi,\mathbf{D}\}$ to problem $\mathcal{P}_V$. Moreover, the solution satisfies

$$\mathbf{u}\in\mathcal{C}^1(0,T;V), \tag{4.2}$$

$$\boldsymbol{\sigma}\in\mathcal{C}(0,T;\mathcal{H}_1), \tag{4.3}$$

$$\theta\in L^2(0,T;\mathcal{Z})\cap\mathcal{C}(0,T;L^2(\Omega)), \tag{4.4}$$

$$\varphi\in\mathcal{C}(0,T;W), \tag{4.5}$$

$$\mathbf{D}\in\mathcal{C}(0,T;\mathcal{W}), \tag{4.6}$$

The proof of Theorem 4.1 is carried in several steps. It is based on results of evolutionary variational inequalities, ordinary differential equations and fixed point arguments.

To prove the theorem we consider the following three auxiliary problems for given $\eta\in\mathcal{C}(0,T;V), \chi\in L^2(0,T;\mathcal{Z}')$ we consider the following three auxiliary problems:

4.1 Problem $\mathcal{PV}_\eta$

Find a displacement field $\mathbf{u}_\eta$: $[0,T]\to V$ and a stress field $\boldsymbol{\sigma}_\eta : [0,T]\to\mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(t))) + \mathcal{G}\varepsilon(\mathbf{u}_\eta(t)) + \boldsymbol{\eta}(t), \tag{4.7}$$

$$(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(t))), \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\mathcal{G}\varepsilon(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) \tag{4.8}$$

$$-j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \tag{4.9}$$

$$\geq (\mathbf{f}(t), \mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V - (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}}, \tag{4.10}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \text{ in } \boldsymbol{\Omega}. \tag{4.11}$$

for all $\mathbf{u}_\eta$, $\mathbf{v} \in V$ and $t \in (0, T)$,

4.2 Problem $\mathcal{PV}_\chi$

Find the temperature $\theta_\chi : [0, T] \to \mathcal{Z}$ which is solution of the variational problem

$$\left(\dot{\theta}_\chi(t), \mathbf{w}\right)_{\mathcal{Z}' \times \mathcal{Z}} + (\mathcal{K}(\nabla\theta(t)), \nabla\mathbf{w})_{\mathcal{Z}' \times \mathcal{Z}} = (\chi(t) + q_{th}(t), \mathbf{w})_{\mathcal{Z}' \times \mathcal{Z}}, \tag{4.12}$$

$$\theta_\chi(0) = \theta_0, \quad \text{in } \boldsymbol{\Omega}. \tag{4.13}$$

for all $\theta_\chi, \mathbf{w} \in \mathcal{Z}$, a.e. $t \in (0, T)$,

4.3 Problem $\mathcal{PV}_\varphi$

Find an electrical potential φ: $[0, T] \to W$, $\mathbf{D} : [0, T] \to \mathcal{W}$ such that

$$\mathbf{D}_\eta(t) = \mathcal{E}\boldsymbol{\varepsilon}\left(\mathbf{u}_\eta(t)\right) - \boldsymbol{B}\nabla\varphi(t) + \mathcal{P}\theta, \tag{4.14}$$

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H - \left(\mathcal{E}\boldsymbol{\varepsilon}\left(\mathbf{u}_\eta(t)\right), \nabla\phi\right)_H - (\mathcal{P}\theta, \nabla\phi)_H + G_2(\varphi, \phi) = (q_e(t), \phi)_W. \tag{4.15}$$

for all φ, $\phi \in W$, $t \in (0, T)$.

We begin with an auxiliary result on the properties of the functionals $j : V \times V \to \mathbb{R}$ and $G_1 : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ defined by (3.39) and (3.42), respectively.

Lemma 4.2. *Under the hypotheses (3.18)–(3.32), the functionals j and G_1 satisfy*

$$j(\mathbf{u}, .) \text{ is convex and lower semicontinuous on } V, \tag{4.16}$$

$$\begin{aligned} &j(\mathbf{u}_1, \mathbf{v}_2) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_1, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ &\leq C_0^2\left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right)\|\mathbf{u}_1 - \mathbf{u}_2\|_V\|\mathbf{v}_1 - \mathbf{v}_2\|_V, \text{ for all } \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V \end{aligned} \tag{4.17}$$

$$\|G_1(\theta_1, \mathbf{w}) - G_1(\theta_2, \mathbf{w})\|_{L^2(\Gamma_3)} \leq C_{G_1}\|\theta_1(t) - \theta_2(t)\|_{\mathcal{Z}}, \text{for all } \theta_1, \theta_2, \mathbf{w} \in \mathcal{Z} \tag{4.18}$$

Proof (Lemma 4.2). We use the assumption (3.26) and inequality (3.7) to see that the functional j defined by (3.39) is a seminorm on V and moreover,

$$|j(\mathbf{u}_1, \mathbf{v}) - j(\mathbf{u}_2, \mathbf{v})| \leq C_0^2(L_\nu + L_\tau)\|\mathbf{u}_1 - \mathbf{u}_2\|_V\|\mathbf{v}\|_V$$

Thus, the seminorm j is continuous on V and, therefore, (4.16) hold.

From the definition of the functional j given by (3.39), we have

$$j(\mathbf{u}_1,\mathbf{v}_2)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_1,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)$$
$$=\int_{\Gamma_3}(p_\nu(u_{1\nu})-p_\nu(u_{2\nu}))(|u_{2\nu}|-|u_{1\nu}|)da$$
$$+\int_{\Gamma_3}(p_\nu(u_{1\nu})-p_\nu(u_{2\nu}))(||\mathbf{v}_{1\tau}||-||\mathbf{v}_{2\tau}||)da, \forall \mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\in V, \tag{4.19}$$

Using (3.39), the last equality becomes

$$j(\mathbf{u}_1,\mathbf{v}_2)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_1,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)$$
$$\leq\int_{\Gamma_3}L_\nu|u_{1\nu}-u_{2\nu}|.||v_{2\nu}|-|v_{1\nu}||da$$
$$+\int_{\Gamma_3}L_\tau|u_{1\nu}-u_{2\nu}\|.|\|\mathbf{v}_{1\tau}\|-\|\mathbf{v}_{2\tau}\||da,\ \text{for all } \mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\in V, \tag{4.20}$$

Next, we use the following inequalities

$$|u_{1\nu}-u_{2\nu}|\leq\|\mathbf{u}_1-\mathbf{u}_2\|,$$
$$||v_{2\nu}|-|v_{1\nu}||\leq|v_{1\nu}-v_{2\nu}|\leq\|\mathbf{v}_1-\mathbf{v}_2\|,$$
$$|\|\mathbf{v}_{1\tau}\|-\|\mathbf{v}_{2\tau}\||\leq\|\mathbf{v}_{1\tau}-\mathbf{v}_{2\tau}\|\leq\|\mathbf{v}_1-\mathbf{v}_2\|, \tag{4.21}$$

The inequality (4.20) becomes

$$j(\mathbf{u}_1,\mathbf{v}_2)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_1,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)$$
$$\leq\int_{\Gamma_3}(L_\nu+L_\tau)\|\mathbf{u}_1-\mathbf{u}_2\|.\|\mathbf{v}_1-\mathbf{v}_2\|da,$$

which implies

$$j(\mathbf{u}_1,\mathbf{v}_2)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_1,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)$$
$$\leq(L_\nu+L_\tau)\|\mathbf{u}_1-\mathbf{u}_2\|_{L^2(\Gamma_3)}.\|\mathbf{v}_1-\mathbf{v}_2\|_{L^2(\Gamma_3)}da,$$

Using (3.7) and (4.1), we conclude

$$j(\mathbf{u}_1,\mathbf{v}_2)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_1,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)$$
$$\leq C_0^2\alpha_0\|\mathbf{u}_1-\mathbf{u}_2\|_V.\|\mathbf{v}_1-\mathbf{v}_2\|_V da.$$

Moreover, the functional G_1 defined in (3.42) by

$$G_1(\theta,\mathbf{w})=-\int_{\Gamma_3}k_e(\theta.\boldsymbol{\nu}-\theta_F.\boldsymbol{\nu})\mathbf{w}da, \text{for all } \theta,\theta_F,\mathbf{w}\in\mathcal{Z},$$

Thus by the assumption (3.32) and inequality (4.21), we get

$$\|G_1(\theta_1,\mathbf{w})-G_1(\theta_2,\mathbf{w})\|_{L^2(\Gamma_3)}\quad\leq\|k_e\|_{L^\infty(\Gamma_3)}.\Big\|\theta_1(t)-\theta_2(t)\|_{L^2(\Gamma_3)},$$

From the inequality (3.8), we obtain

$$\|G_1(\theta_1,\mathbf{w})-G_1(\theta_2,\mathbf{w})\|_{L^2(\Gamma_3)}\quad\leq C_1\|k_e\|_{L^\infty(\Gamma_3)}\Big\|\theta_1(t)-\theta_2(t)\|_{\mathcal{Z}},$$

Thus, we can write

$$\|G_1(\theta_1,\mathbf{w})-G_1(\theta_2,\mathbf{w})\|_{L^2(\Gamma_3)}\quad\leq C_{G_1}\|\theta_1(t)-\theta_2(t)\|_{\mathcal{Z}}, \text{for all } \theta_1,\theta_2\in\mathcal{Z}.$$

where $C_{G_1}=C_1\|k_e\|_{L^\infty(\Gamma_3)}$. □

We have the following result for Problem $\mathcal{PV}_\eta$.

Lemma 4.3. *Under the hypotheses (3.18)–(3.32), for every* $\eta \in \mathcal{C}(0,T;V)$, *Problem* $\mathcal{PV}_\eta$ *has a unique weak solution* $\{\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta\}$, *such that*

$$\mathbf{u}_\eta \in \mathcal{C}^1(0,T;V), \boldsymbol{\sigma}_\eta \in \mathcal{C}(0,T;\mathcal{H}_1). \tag{4.22}$$

Moreover, if $\{\mathbf{u}_i, \boldsymbol{\sigma}_i\}$ *are the solutions of Problem* $\mathcal{PV}_{\eta_i}$, *corresponding* $\eta = \eta_i \in \mathcal{C}(0,T;V)$ *for* $i = 1, 2$, *then*

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le C \int_0^T \|\eta_1(s) - \eta_2(s)\|_V ds \tag{4.23}$$

Proof [of Lemma (4.3)]. Choosing $\mathbf{v} = \dot{\mathbf{u}}_\eta(t) \pm \xi$ in (4.10), where $\xi \in \mathcal{D}(\Omega)^d$ is arbitrary, we find

$$\left(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\Phi)\right)_{\mathcal{H}} = (\mathbf{f}(t), \Phi)_V$$

Using the definition (3.40) for $\mathbf{f}$, we deduce

$$Div\boldsymbol{\sigma}_\eta(t) + \mathbf{f}_0(t) = 0,\ t \in (0,T), \tag{4.24}$$

With the regularity assumption (3.28) on $\mathbf{f}_0$, we see that $Div\boldsymbol{\sigma}_\eta(t) \in H$. Therefore, $\boldsymbol{\sigma}_\eta(t) \in \mathcal{H}_1$.

Now, we use Riesz Representation Theorem to define the operators A: $V \to V$, B: $V \to V$ and the function $\mathbf{f}_\eta : [0,T] \to V$ by

$$(A\mathbf{u}, \mathbf{v}) = (\mathcal{A}\varepsilon(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \tag{4.25}$$

$$(B\mathbf{u}, \mathbf{v}) = (\mathcal{G}\varepsilon(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \tag{4.26}$$

$$\left(\mathbf{f}_\eta(t), \mathbf{v}\right) = (\mathbf{f}(t), \mathbf{v})_V - (\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}))_V, \tag{4.27}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in [0,T]$.

It follows from (4.25) and (3.18(a)) that

$$\|A\mathbf{u} - A\mathbf{v}\|_V \le L_{\mathcal{A}} \|\mathbf{u} - \mathbf{v}\|_V, \tag{4.28}$$

Which shows that A: $V \to V$ is Lipschitz continuous. Now, by (4.25) and (3.18(b)) we find

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_V \ge m_{\mathcal{A}} \|\mathbf{u} - \mathbf{v}\|_V^2, \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.29}$$

i.e. that A: $V \to V$ is a strongly monotone operator on V. Moreover, using (4.26) and (3.19(a)) we find

$$\|B\mathbf{u} - B\mathbf{v}\|_V \le L_{\mathcal{B}} \|\mathbf{u} - \mathbf{v}\|_V, \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.30}$$

if (4.1) is satisfied, since A is a strongly monotone and Lipschitz continuous operator on V and B is Lipschitz continuous operator on V, $j(\mathbf{u}, .)$ satisfies conditions (4.16) and (4.17), $\mathbf{u}_0$ satisfies the assumption (3.31), and we note that for any fixed $\eta \in \mathcal{C}(0,T;V)$ we use the definitions 3.44 and (4.27) to show that $\mathbf{f}_\eta \in \mathcal{C}(0,T;V)$ we deduce from classical results for evolutionary elliptic variational inequalities (see for example [27]) that there exists a unique function $\mathbf{u}_\eta \in \mathcal{C}^1(0,T;V)$. Moreover, for $\mathbf{u}_i = \mathbf{u}_{\eta_i}$ solutions of the Problem $\mathcal{PV}_{\eta i}$ for $i = 1$, 2, then

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \le C\left(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \|\eta_1(t) - \eta_2(t)\|_V\right) \tag{4.31}$$

Since

$$\mathbf{u}_i(t) = \int_0^t \dot{\mathbf{u}}_i(s)ds + \mathbf{u}_0, \forall t \in [0, T],$$

We have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq C \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds \tag{4.32}$$

Recent modeling Using (4.31) the inequality (4.32) becomes

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq C\left(\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds\right)$$

Next, we apply Gronwall's inequality to deduce

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq C \int_0^T \|\eta_1(s) - \eta_2(s)\|_V ds \tag{4.33}$$

□

For the Problem $\mathcal{PV}_\chi$we have the following result.

Lemma 4.4. *Under the hypotheses (3.18)–(3.32), for every $\chi \in L^2(0, T; \mathcal{Z}')$, Problem $\mathcal{PV}_\chi$ has a unique weak solution such that*

$$\theta_\chi \in L^2(0, T; \mathcal{Z}) \cap \mathcal{C}\big(0, T; L^2(\Omega)\big), \tag{4.34}$$

Moreover, if θ_i *are the solutions of Problem $\mathcal{PV}_{\chi_i}$, corresponding $\chi = \chi_i \in \mathcal{C}(0, T; \mathcal{Z}')$ for* i = *1, 2, then*

$$\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} \leq C \int_0^T \|\chi_1(s) - \chi_2(s)\|^2_{\mathcal{Z}'} ds \tag{4.35}$$

Proof [*of Lemma (4.4)*]. The inclusion mapping of $(\mathcal{Z}, \|.\|_{\mathcal{Z}})$ into $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We can write the Gelfand triple

$$\mathcal{Z} \subset L^2(\Omega) = \left(L^2(\Omega)\right)' \subset \mathcal{Z}'.$$

The problem (4.12)–(4.13) may be written as

$$\begin{aligned} \dot{\theta}_\chi(t) + K\theta_\chi(t) &= Q(t), \\ \theta_\chi(0) &= \theta_0, \end{aligned}$$

where, $K : \mathcal{Z} \to \mathcal{Z}'$ and $Q : [0, T] \to \mathcal{Z}'$ are defined as

$$(K\boldsymbol{\tau}, \mathbf{w})_{\mathcal{Z}' \times \mathcal{Z}} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \boldsymbol{\tau}}{\partial x_j} \frac{\partial \mathbf{w}}{\partial x_i} dx + \int_{\Gamma_3} \boldsymbol{\tau}.\mathbf{w} da, \tag{4.36}$$

$$(Q, \mathbf{w})_{\mathcal{Z}' \times \mathcal{Z}} = (\chi(t) + q_{th}(t), \mathbf{w})_{\mathcal{Z}' \times \mathcal{Z}}, \tag{4.37}$$

It follows from the definition of the operator K, and (3.15) the assumption (3.25(b)) that

$$\|K\boldsymbol{\tau} - K\mathbf{w}\|_{\mathcal{Z}'} \le L_{\mathcal{K}}\|\boldsymbol{\tau} - \mathbf{w}\|_{\mathcal{Z}}, \forall \boldsymbol{\tau}, \mathbf{w} \in \mathcal{Z}, \tag{4.38}$$

which shows that $K : \mathcal{Z} \to \mathcal{Z}'$ is continuous and so is hemicontinuous

Now, by (4.36) and (3.25(c)), we find

$$(K\boldsymbol{\tau} - K\mathbf{w}, \boldsymbol{\tau} - \mathbf{w})_{\mathcal{Z}' \times \mathcal{Z}} \ge m_{\mathcal{K}}\|\boldsymbol{\tau} - \mathbf{w}\|^2_{\mathcal{Z}}, \forall \boldsymbol{\tau}, \mathbf{w} \in \mathcal{Z}, \tag{4.39}$$

Which shows that is K a strongly monotone operator. Choosing $\mathbf{w} = \mathbf{0}_{\mathcal{Z}}$ in (4.39), we obtain

$$\begin{aligned}(K\boldsymbol{\tau}, \boldsymbol{\tau})_{\mathcal{Z}' \times \mathcal{Z}} &\ge m_{\mathcal{K}}\|\boldsymbol{\tau}\|^2_{\mathcal{Z}} - \|K\mathbf{0}_{\mathcal{Z}}\|_{\mathcal{Z}'}\|\boldsymbol{\tau}\|_{\mathcal{Z}} \\ &\ge \frac{1}{2}m_{\mathcal{K}}\|\boldsymbol{\tau}\|^2_{\mathcal{Z}} - \frac{1}{2m_{\mathcal{K}}}\|K\mathbf{0}_{\mathcal{Z}}\|_{\mathcal{Z}'}, \forall \boldsymbol{\tau} \in \mathcal{Z},\end{aligned}$$

Thus, K satisfies condition (3.16) with $\omega = \frac{m_{\mathcal{K}}}{2}$ and $\zeta = -\frac{1}{2m_{\mathcal{K}}}\|K\mathbf{0}_{\mathcal{Z}}\|_{\mathcal{Z}'}$.

Next, by (4.38) we deduce that

$$\|K\boldsymbol{\tau}\|_{\mathcal{Z}'} \le L_{\mathcal{K}}\|\boldsymbol{\tau}\|_{\mathcal{Z}} + \|K\mathbf{0}_{\mathcal{Z}}\|_{\mathcal{Z}'}, \forall \boldsymbol{\tau} \in \mathcal{Z}.$$

This inequality implies that K satisfies condition (3.17).

Moreover, for $\chi(t) \in L^2(0, T; \mathcal{Z}')$ and $q_{th}(t) \in L^2(0, T; L^2(\Omega))$ which implies $Q \in L^2(0, T; \mathcal{Z}')$ and $\theta_0 \in L^2(\Omega)$.

It follows now from Theorem 3.2 that there exists a unique function $\theta_\chi \in L^2(0, T; \mathcal{Z}) \cap C(0, T; L^2(\Omega))$, which satisfies the Problem $\mathcal{PV}_\chi$.

Now, to provide the estimate (4.35), we take the substitution $\chi = \chi_1$ and $\chi = \chi_2$ in (4.12) and subtracting the two obtained equations, we deduce by choosing $\mathbf{w} = \theta_1(t) - \theta_2(t)$ as test function.

$$\begin{aligned}&\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{\mathcal{Z}' \times \mathcal{Z}} + (K\theta_1(t) - K\theta_2(t), \theta_1(t) - \theta_2(t))_{\mathcal{Z}' \times \mathcal{Z}} \\ &= (\chi_1(t) - \chi_2(t), \theta_1(t) - \theta_2(t))_{\mathcal{Z}' \times \mathcal{Z}},\end{aligned}$$

Then integrating the last property over $(0, t)$, using (3.15),(4.38) and (4.39), we deduce (4.35). □

For the last Problem $\mathcal{PV}_\varphi$ we have the following result.

Lemma 4.5. *Under the hypotheses* (3.18)–(3.32)*, for every* $\eta \in C(0, T; V)$*, Problem* $\mathcal{PV}_\varphi$ *has a unique weak solution* $\{\varphi_\eta, \mathbf{D}_\eta\}$ *such that*

$$\varphi_\eta \in C(0, T; W), \; \mathbf{D}_\eta \in C(0, T; \mathcal{W}), \tag{4.40}$$

Moreover, if $\{\varphi_i, \mathbf{D}_i\}$ *are the solutions of problem* $\mathcal{PV}_{\eta_i}$*, corresponding* $\eta = \eta_i \in C(0, T; V)$ *for* $i = 1, 2$*, then*

$$\|\varphi_1(t) - \varphi_2(t)\|_W \le C(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V), \tag{4.41}$$

Proof [of Lemma (4.5)]. First, for the functional $G_2 : W \times W \to \mathbb{R}$ defined in (3.43):

Let $\varphi_1, \varphi_2 \in W$, we find that

$$\|G_2(\varphi_1, \phi) - G_2(\varphi_2, \phi)\|_{L^2(\Gamma_3)} = \int_{\Gamma_3} H(\varphi_1)\phi da - \int_{\Gamma_3} H(\varphi_2)\phi da,$$

We use the definition on the functional H given in (1.5) to obtain

$$\|G_2(\varphi_1,\phi)-G_2(\varphi_2,\phi)\|_{L^2(\Gamma_3)}=\int_{\Gamma_3}(\varphi_1-\varphi_0)\phi da-\int_{\Gamma_3}(\varphi_2-\varphi_0)\phi da,$$

which implies

$$\|G_2(\varphi_1,\phi)-G_2(\varphi_2,\phi)\|_{L^2(\Gamma_3)}=\|\varphi_1-\varphi_2\|^2_{L^2(\Gamma_3)}$$

Using the inequality (3.8), we get

$$\|G_2(\varphi_1,\phi)-G_2(\varphi_2,\phi)\|_{L^2(\Gamma_3)}\leq C_1\|\varphi_1-\varphi_2\|^2_W \tag{4.42}$$

We use Riesz representation theorem to define the operator F: $W\rightarrow W$ by

$$(F\varphi,\phi)_W=(\boldsymbol{B}\nabla\varphi(t),\nabla\phi)_H-\left(\mathcal{E}\varepsilon\left(\mathbf{u}_\eta(t)\right),\nabla\phi\right)_W+G_2(\varphi,\phi),\forall\varphi,\phi\in W, \tag{4.43}$$

Let φ_1, $\varphi_2\in W$. Using the assumption (3.23) and (3.27), we find that

$$(F\varphi_1-F\varphi_2,\phi_1-\phi_2)_W\geq(m_B+m_H)\|\varphi_1-\varphi_2\|^2_W,\forall\varphi,\phi\in W. \tag{4.44}$$

On the other hand, using the assumptions (3.20), (3.23) and the inequality (4.42), we have

$$(F\varphi_1-F\varphi_2,\phi)_W\leq(C_\mathcal{E}+C_1)\|\varphi_1-\varphi_2\|^2_W\|\phi\|_W,\forall\varphi,\phi\in W,$$

where $C_\mathcal{E}$ and C_1 are a positives constants. Thus,

$$\|F\varphi_1-F\varphi_2\|_W\leq(C_\mathcal{E}+C_1)\|\varphi_1-\varphi_2\|_W. \tag{4.45}$$

Thus, by (4.44) and (4.45) we conclude that $F(t)$ is a strongly monotone and Lipschitz continuous operator on W and, therefore, there exists a unique element $\varphi_\eta\in W$ such that

$$F(t)\varphi_\eta(t)=q_e+\mathcal{P}\theta;\forall\varphi_\eta\in W. \tag{4.46}$$

Let $\eta_1,\eta_2\in\mathcal{C}(0,T;V)$. Using the last equality, we get

$$\|F\varphi_1-F\varphi_2\|_W\leq\|q_{e1}-q_{e2}\|_W+L_\mathcal{P}\|\theta_1-\theta_2\|_\mathcal{Z}$$

Moreover, we use the assumption (3.30), to obtain

$$\|F\varphi_1-F\varphi_2\|_W\leq\|q_{e1}-q_{e2}\|_W+L_\mathcal{P}L_\theta\|\varphi_1-\varphi_2\|_W,$$

We conclude that $\varphi_\eta(t)$ is a solution of $\mathcal{PV}_\varphi$. It follows from (3.20), (3.23) and (4.15) that

$$\begin{aligned}(m_B+m_H)\|\varphi_1-\varphi_2\|^2_W\leq{}&C_\mathcal{E}\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\|\varphi_1-\varphi_2\|_W\\&+\|q_{e1}-q_{e2}\|_W\|\varphi_1-\varphi_2\|_W\\&+\int_{\Gamma_3}|(\varphi_1-\varphi_0)-(\varphi_2-\varphi_0)|.|\varphi_1-\varphi_2|da\\&+L_\mathcal{P}\|\theta_1-\theta_2\|_\mathcal{Z}\|\varphi_1-\varphi_2\|_W,\end{aligned}$$

Using (3.2) and (3.30), we get

$$\begin{aligned}(m_B+m_H)\|\varphi_1-\varphi_2\|^2_W\leq{}&C_\mathcal{E}\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\|\varphi_1-\varphi_2\|_W+C_1\|\varphi_1-\varphi_2\|^2_W\\&+\|q_{e1}-q_{e2}\|_W\|\varphi_1-\varphi_2\|_W+L_\mathcal{P}L_\theta\|\varphi_1-\varphi_2\|^2_\mathcal{Z},\end{aligned}$$

which implies

$$\|\varphi_1-\varphi_2\|_W \le C\big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|q_{e1}(t)-q_{e2}(t)\|_W\big), \tag{4.47}$$

Then, for every $\mathbf{u}_\eta\in\mathcal{C}^1(0,T;V)$, the previous inequality and the regularity of q_e imply that $\varphi_\eta\in\mathcal{C}(0,T;W)$.

We now use (3.41) and definition of the divergence operator div to see that

$$\big(div\mathbf{D}_\eta,\phi\big)_H=(q_e,\phi)_H,\forall\phi\in H^1(\Omega), \tag{4.48}$$

This shows that $\mathbf{D}_\eta\in\ \mathcal{C}(0,T;\mathcal{W})$.

Let $\eta_1,\eta_2\in\mathcal{C}(0,T;V)$ and let $\mathbf{u}_i\in\mathcal{C}^1(0,T;V)$, for $i=1,2$, We use (4.15) and arguments similar to those used in the proof of (4.47) to obtain (4.41) □

Finally, as a consequence of these results and using the properties of the operators $\mathcal{E}$, $\mathcal{M}$ and the function ψ for $t\in[0,T]$, we consider the element

$$\Lambda(\boldsymbol{\eta},\chi)(t)=(\Lambda_1(\boldsymbol{\eta},\chi)(t),\Lambda_2(\boldsymbol{\eta},\chi)(t))\in V\times L^2(\mathcal{Z}'), \tag{4.49}$$

defined by

$$\Lambda_1(\boldsymbol{\eta},\chi)(t)=\mathcal{E}^*\nabla\varphi_\eta(t)-\mathcal{M}\theta_\chi,\forall t\in[0,T], \tag{4.50}$$

$$\Lambda_2(\boldsymbol{\eta},\chi)(t)=G_1(\theta,\mathbf{w})+\psi(\mathcal{M}\theta(t),\mathbf{u}(t)),\forall t\in[0,T], \tag{4.51}$$

We have the following result.

Lemma 4.6. *Let* (4.1) *be satisfied. Then for* $(\eta,\chi)\in C(0,T;V\times L^2(\mathcal{Z}'))$, *the function* $\Lambda(\eta,\chi):[0,T]\to V\times L^2(\mathcal{Z}')$ *is continuous, and there is a unique element* $(\eta^*,\chi^*)\in C(0,T;V\times L^2(\mathcal{Z}'))$. *Such that* $\Lambda(\eta^*,\chi^*)=(\eta^*,\chi^*)$

Proof [of Lemma 4.6]. Let $(\eta,\chi)\in C(0,T;V\times L^2(\mathcal{Z}'))$, and $t_1,t_2\in[0,T]$. Using the assumptions (3.19)–(3.22) and (3.24), we have

$$\begin{aligned}&\|\Lambda(\eta_1,\chi_1)(t)-\Lambda(\eta_2,\chi_2)(t)\|_{V\times L^2(\mathcal{Z}')}\\ &\le C_{\mathcal{E}}\|\varphi_1(t)-\varphi_2(t)\|_W+\big(L_{\mathcal{M}}+L_{\mathcal{M}}L_\psi\big)\|\theta_1-\theta_2\|_{L^2(\Omega)}\\ &\ +L_\psi\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|G_1(\theta_1,\mathbf{w}_1)-G_1(\theta_2,\mathbf{w}_2)\|_{L^2(\Gamma_3)})\end{aligned} \tag{4.52}$$

The last inequality and (4.18), implies

$$\begin{aligned}&\|\Lambda(\eta_1,\chi_1)(t)-\Lambda(\eta_2,\chi_2)(t)\|_{V\times L^2(\mathcal{Z}')}\\ &\le C_{\mathcal{E}}\|\varphi_1(t)-\varphi_2(t)\|_W+\big(L_{\mathcal{M}}+L_{\mathcal{M}}L_\psi+C_{G_1}\big)\|\theta_1-\theta_2\|_{L^2(\Omega)}\\ &\ +L_\psi\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}&\|\Lambda(\eta_1,\chi_1)(t)-\Lambda(\eta_2,\chi_2)(t)\|^2_{V\times L^2(\mathcal{Z}')}\\ &\le C\Big(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\theta_1-\theta_2\|^2_{L^2(\Omega)}+\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V\Big)\end{aligned} \tag{4.53}$$

For the electric potential field, we use (4.33) and (4.41), we obtain

$$\|\varphi_1(t)-\varphi_2(t)\|^2_W\le C\int_0^T\|\eta_1(s)-\eta_2(s)\|^2_V ds \tag{4.54}$$

For the displacement, we use (4.23) to get

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2 \le C\int_0^T \|\eta_1(s)-\eta_2(s)\|_V^2 ds, \tag{4.55}$$

Moreover, using the inequality (4.35) obtained in Lemma 4.4 for the temperature.

Applying Young's, Hölder's inequalities, the increases (4.35), (4.54) and (4.55), then the inequality (4.53) becomes

$$\begin{aligned}\|\Lambda(\eta_1,\chi_1)(t) & -\Lambda(\eta_2,\chi_2)(t)\|_{V\times L^2(\mathcal{Z}')}^2 \\ & \le C\int_0^T \|(\eta_1,\chi_1)(s)-(\eta_2,\chi_2)(s)\|_{V\times L^2(\mathcal{Z}')}^2 ds,\end{aligned} \tag{4.56}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathcal{C}(0,T;V\times L^2(\mathcal{Z}'))$, and so Λ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof [of Theorem (4.1)]. **Existence**

Let $(\eta^*,\chi^*)\in\mathcal{C}(0,T;V\times L^2(\mathcal{Z}'))$ be the fixed point of Λ defined by (4.49)–(4.51) and denote

$$\begin{aligned}&\mathbf{u}_* = \mathbf{u}_{\underset{\eta}{*}},\, \theta_* = \theta_{\underset{\chi}{*}},\, \varphi_* = \varphi_{\underset{\eta}{*}},\\ &\boldsymbol{\sigma}_* = \mathcal{A}\Big(\varepsilon\Big(\dot{\mathbf{u}}_*\Big)\Big) + \mathcal{B}\varepsilon\Big(\mathbf{u}_*\Big) + \mathcal{E}^*\nabla\varphi_* - \mathcal{M}\theta_*,\\ &\mathbf{D}_* = \mathcal{E}\varepsilon\Big(\mathbf{u}_*\Big) - \boldsymbol{B}\nabla\varphi_* - \mathcal{P}\theta_*\,.\end{aligned}$$

Let $\{\mathbf{u}_*,\boldsymbol{\sigma}_*\},\theta^*$ and $\Big\{\varphi_*,\mathbf{D}_*\Big\}$ be the solutions of the problems $\mathcal{PV}_{\underset{\eta}{*}}, \mathcal{PV}_{\underset{\chi}{*}}$ and $\mathcal{PV}_{\underset{\varphi}{*}}$ respectively, obtained in Lemmas 4.3, 4.4 and 4.5. The equalities $\Lambda_1(\eta^*,\chi^*)=\eta^*$ and $\Lambda_2(\eta^*,\chi^*)=\chi^*$ combined with 4.49–4.51 show that 3.33–3.38 are satisfied. Next, the regularity 4.2–4.6 follows from Lemmas 4.3, 4.4 and 4.5. □

Uniqueness

Proof. The uniqueness part of solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.49)–(4.51) and the unique solvability of the problems $\mathcal{PV}_\eta$, $\mathcal{PV}_\chi$ and $\mathcal{PV}_\varphi$ which completes the proof. □

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Corresponding author
Abdelmoumene Djabi can be contacted at: djabi.zaki@yahoo.fr