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Almost Ricci–Yamabe soliton on contact metric manifolds

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Abstract

Purpose – This paper aims to study almost Ricci–Yamabe soliton in the context of certain contact metric manifolds.

Design/methodology/approach – The paper is designed as follows: In Section 3, a complete contact metric manifold with the Reeb vector field ξ as an eigenvector of the Ricci operator admitting almost Ricci–Yamabe soliton is considered. In Section 4, a complete K-contact manifold admits gradient Ricci–Yamabe soliton is studied. Then in Section 5, gradient almost Ricci–Yamabe soliton in non-Sasakian (k, μ)-contact metric manifold is assumed. Moreover, the obtained result is verified by constructing an example.

Findings – We prove that if the metric g admits an almost (α, β)-Ricci–Yamabe soliton with $\alpha \neq 0$ and potential vector field collinear with the Reeb vector field ξ on a complete contact metric manifold with the Reeb vector field ξ as an eigenvector of the Ricci operator, then the manifold is compact Einstein Sasakian and the potential vector field is a constant multiple of the Reeb vector field ξ. For the case of complete K-contact, we found that it is isometric to unit sphere S^{2n+1} and in the case of (k, μ)-contact metric manifold, it is flat in three-dimension and locally isometric to $E^{n+1} \times S^n(4)$ in higher dimension.

Originality/value – All results are novel and generalizations of previously obtained results.

Keywords Ricci soliton, Yamabe soliton, (k, μ)-Contact metric manifold, Ricci–Yamabe soliton, Contact geometry

Paper type Research paper

1. Introduction

The theory of geometric flows plays a significant role in understanding the geometric structure in Riemannian geometry. Hamilton [1] introduced the concept of Ricci flow. A Ricci soliton is a self-similar solution to Ricci flow $\partial_t g(t) = -2S(t)$, where S is the Ricci curvature. Ricci solitons are a generalization of Einstein manifolds. A Ricci soliton on a Riemannian manifold (M, g) is defined by

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1}$$

AMS subject classification — 53C15, 53C25

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where $L_V g$ denotes the Lie derivative of g along a vector field V, λ a constant and arbitrary vector field X, Y on M. If $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$ then the soliton is said to be shrinking, expanding or steady, respectively. A Ricci soliton is said to be a gradient Ricci soliton if $V = \nabla f$, for a smooth function f. For a detailed study on Ricci soliton see Refs. [2, 3] and references therein.

Hamilton [1] introduced a geometric flow that is similar to Ricci flow and called it Yamabe flow. Yamabe solitons correspond to self-similar solutions of the Yamabe flow. A Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general [1]. In dimension $n = 2$, both the solitons are similar. On a Riemannian manifold (M, g) a Yamabe soliton is given by

$$(L_V g)(X, Y) = 2(r - \lambda)g(X, Y), \tag{2}$$

for arbitrary vector fields X, Y on M, λ a scalar and r the scalar curvature of M. If λ is a smooth function, then equations (1) and (2) are called Ricci almost soliton given by Pigola *et al.* [4] and almost Yamabe soliton given by Barbosa and Ribeiro [5], respectively. For a detailed study on Yamabe soliton see Refs. [6–10] and references therein.

Recently, in 2019, Guler and Crasmareanu [11] introduced a new type of geometric flow, a scalar combination of Ricci flow and Yamabe flow under the name Ricci–Yamabe map. In Ref. [11], the authors define the following:

Definition 1. *[11] A Riemannian flow on M is a smooth map:*

$$g : I \subseteq \mathbb{R} \rightarrow Riem(M),$$

where I is a given open interval.

Definition 2. *[11] The map $RY^{(\alpha,\beta,g)} : I \rightarrow T_2^s(M)$ given by:*

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t) g(t),$$

is called the (α, β)-Ricci–Yamabe map of the Riemannian flow (M, g). If

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g(.)$ will be called an (α, β)-Ricci–Yamabe flow.

The Ricci–Yamabe flow can be Riemannian or semi-Riemannian or singular Riemannian flow due to the involvement of scalars α and β [11]. These kinds of choices can be useful when dealing with relativity. The Ricci–Yamabe soliton emerges as the self-similar solutions of the Ricci–Yamabe flow. The notion of Ricci–Yamabe soliton from the Ricci–Yamabe flow can be defined as follows:

Definition 3. *[12] A Riemannian manifold (M^n, g), $n > 2$ is said to admit the Ricci–Yamabe soliton $(g, V, \lambda, \alpha, \beta)$ if*

$$L_V g + 2\alpha S = (2\lambda - \beta r)g, \tag{3}$$

where $\lambda, \alpha, \beta \in \mathbb{R}$. If V is a gradient of some smooth function f on M, then the above notion is called gradient Ricci–Yamabe soliton and then (3) reduces to

$$\nabla^2 f + \alpha S = \left(\lambda - \frac{1}{2}\beta r\right)g, \tag{4}$$

where $\nabla^2 f$ is the Hessian of f.

The Ricci–Yamabe soliton is said to be expanding, shrinking or steady if $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$ respectively. Therefore, equation (3) is the Ricci–Yamabe soliton of (α, β)-type which is a combination of Ricci soliton and Yamabe soliton. In particular, (1,0), (0,1), (1, −1) and (1, − 2ρ)-type is the Ricci–Yamabe soliton are Ricci soliton, Yamabe soliton, Einstein soliton and ρ-Einstein soliton, respectively. Therefore, the notion of the Ricci–Yamabe soliton generalizes a large class of soliton-like equations. Using the terminology of Ricci almost soliton, the notion of almost Ricci–Yamabe soliton can be defined as follows:

Definition 4. *A Riemannian manifold* (M^{2n+1}, g), $n \geq 1$ *is said to be admit an almost Ricci–Yamabe soliton* $(g, V, \lambda, \alpha, \beta)$ *if there exist a smooth function* $\lambda : M^{2n+1} \to \mathbb{R}$ *satisfying*

$$L_V g + 2\alpha S = (2\lambda - \beta r)g. \tag{5}$$

Moreover, if $V = \nabla f$, *the gradient of some smooth function in* M^{2n+1}, *then it will be called a gradient almost Ricci–Yamabe soliton.*

Recently, in [12], the author studied the Ricci–Yamabe soliton on almost Kenmotsu manifolds. He shows that a $(k, \mu)'$-almost Kenmotsu manifolds admitting a Ricci–Yamabe soliton or gradient Ricci–Yamabe soliton is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Siddiqi and Akyol [13], introduced the notion of η-Ricci–Yamabe soliton and establish the geometrical bearing on Riemannian submersions in terms of η-Ricci–Yamabe soliton with the potential field and giving the classification of any fiber of Riemannian submersion is an η-Ricci–Yamabe soliton, η-Ricci soliton and η-Yamabe soliton. Ricci–Yamabe soliton in perfect fluid spacetime is analyzed by authors in Ref. [14]. Khatri and Singh [15] studied almost Ricci–Yamabe soliton in different classes of almost Kenmotsu manifolds. In Ref. [16], Ghosh shows that if the metric of a non-Sasakian (k, μ)-contact metric is a gradient Ricci almost soliton, then in Dimension 3, it is flat and in higher dimensions it is locally isometric to $E^{n+1} \times S^n(4)$. Thus a natural question arises. “What happens when the metric of a non-Sasakian (k, μ)-contact metric manifold is a gradient almost Ricci-Yamabe soliton.”

The result of which is shown in section 4. Motivated by the above studies, we study almost Ricci–Yamabe soliton on contact metric manifolds. This paper aims to investigate the properties of almost contact metric manifolds whose metric admits almost Ricci–Yamabe solitons. The classification of K-contact and (κ, μ)-contact admitting almost Ricci–Yamabe soliton is obtained. The significance of studying almost Ricci–Yamabe soliton is that it generalizes a number of previously obtained results by Ghosh [16], Sharma [17] and some well-known results in Ricci soliton, Yamabe soliton and ρ-Einstein soliton within the framework of contact geometry.

The result of which is shown in section 4. Motivated by the above studies, we study almost Ricci–Yamabe soliton on contact metric manifolds. The present paper is organized as follows: After preliminaries in section 2, in section 3 we study almost (α, β)-Ricci–Yamabe solitons with the potential vector field collinear with the Reeb vector field ξ and found interesting results. Next in section 4, the gradient almost Ricci–Yamabe soliton in K-contact metric manifold is analyzed. Moreover in Section 5, the gradient almost Ricci–Yamabe soliton in the framework of (k, μ)-contact metric manifold is investigated and obtained that it is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat if $n = 1$. Finally, an example of a three-dimensional (k, μ)-contact metric manifold is constructed.

2. Preliminaries

In this section, we give some of the basic results and formulas of (k, μ)-contact metric manifold and refer to Refs. [17–20] for more information and details.

A 2n+1-dimensional smooth manifold M is called a contact manifold if it admits a global differential 1-form η (called contact form) such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. A contact manifold induced as almost contact metric structure (η, ξ, ϕ, g), that is, a vector field ξ called the characteristic vector field, a (1,1)-tensor field ϕ and Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

for a vector fields X, Y on M. If in addition, $d\eta(X, Y) = g(X, \phi Y)$ then M is called a contact metric manifold [21]. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{7}$$

From the definition, it pursues that $\phi\xi = 0$ and $\eta \circ \phi = 0$. Then, the manifold $M(\phi, \xi, \eta, g)$ equipped with such a structure is called a contact metric manifold [21, 22].

Given a contact metric manifold M we define a symmetric (1,1)-tensor field h and self adjoint operator l by $h = \frac{1}{2}L_\xi \phi$ and $l = R(., \xi)\xi$, where L denotes Lie differentiation. Then, $h\phi = -\phi h$, $Trh = Tr\,\phi h = 0$, $h\xi = 0$. Also from Ref. [21],

$$g(Q\xi, \xi) = Trl = 2n - |h|^2. \tag{8}$$

A normal contact metric manifold is called a Sasakian manifold. A contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{9}$$

for any vector fields X, Y on M. The vector field ξ is a Killing vector with respect to g if and only if $h = 0$. A contact metric manifold $M(\phi, \xi, \eta, g)$ for which ξ is killing (equivalently $h = 0$ or $Trl = 2n$) is said to be a K-contact metric manifold. On a K-contact manifold, the following formulas are known [21].

$$\nabla_X \xi = -\phi X, \tag{10}$$

$$Q\xi = 2n\xi, \tag{11}$$

$$R(X, \xi)\xi = X - \eta(X)\xi, \tag{12}$$

where ∇ is the operator of covariant differentiation of g, S is the Ricci tensor of type (0,2) such that $S(X, Y) = g(QX, Y)$, where Q is Ricci operator and R is the Riemann curvature tensor of g. A Sasakian manifold is K-contact and the converse is not true except in Dimension 3.

As a generalization of the Sasakian case, Blair *et al.* [18] introduced (k, μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. A full classification of (k, μ)-spaces was given by Boeckx [19].

The (k, μ)-nullity distribution of a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is a distribution

$$\begin{aligned} N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_pM : R(X, Y)Z = k\{g(Y, Z)X \\ -g(X, Z)Y\} + \mu\{g(Y, Z)hX - g(X, Z)hY\}\}, \end{aligned}$$

for any X, Y, $Z \in T_pM$ and real numbers k and μ. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. In particular, if $\mu = 0$, then the notion of (k, μ)-nullity distribution reduces to the notion of k-nullity distribution, introduced by Tanno [23]. If $k = 1$, the structure is Sasakian, and if $k < 1$, the (k, μ)-nullity condition determines the curvature of the manifold completely.

In a (k, μ)-contact metric manifold the following relations hold [18, 20].

$$h^2 = (k - 1)\phi^2, \qquad k \leq 1, \tag{13}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{14}$$

$$S(X,Y) = [2(n-1) - n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ +[2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \tag{15}$$

$$r = 2n(2n-2+k-n\mu). \tag{16}$$

Here, r is the scalar curvature of the manifold.

3. Almost (α, β)-Ricci–Yamabe solitons with $V = \sigma\xi$

Ghosh [16] obtained a result for contact metric manifold with potential vector field collinear with the Reeb vector field. Motivated by this study, we extended it to an almost (α, β)-Ricci–Yamabe soliton. We prove the following:

Theorem 1. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a complete contact metric manifold where the Reeb vector field ξ is an eigenvector of the Ricci operator at each point of M. If g admits an almost (α, β)-Ricci–Yamabe soliton with $\alpha \neq 0$ and non-zero potential vector field collinear with the Reeb vector field ξ, then M is compact Einstein Sasakian and the potential vector field is a constant multiple of the Reeb vector field ξ.*

Proof. Suppose the potential vector field is collinear with the Reeb vector field, i.e. $V = \sigma\xi$, where σ is a non-zero function on M. Differentiating it along arbitrary vector field X gives

$$\nabla_X V = (X\sigma)\xi - \sigma(\phi X + \phi hX). \tag{17}$$

Using this in (5) and simplifying we obtain

$$(X\sigma)\eta(Y) + (Y\sigma)\eta(X) - 2\sigma g(\phi hX, Y) \\ +2\alpha S(X,Y) = (2\lambda - \beta r)g(X,Y). \tag{18}$$

Taking $X = Y = \xi$ in (18) yields

$$\xi\sigma + 2\alpha Trl = 2\lambda - \beta r. \tag{19}$$

Replacing Y by ξ in (18) gives

$$D\sigma + (\xi\sigma)\xi + 2\alpha Q\xi = (2\lambda - \beta r)\xi. \tag{20}$$

Suppose that the Reeb vector field ξ is an eigenvector of the Ricci operator at each point of M, then $Q\xi = (Trl)\xi$. Using this in the forgoing equation along with (19) gives, $D\sigma = (\xi\sigma)\xi$. Differentiating it along with vector field X yields

$$\nabla_X D\sigma = X(\xi\sigma)\xi - (\xi\sigma)(\phi X + \phi hX). \tag{21}$$

Making use of Poincare lemma in (21), we obtain

$$X(\xi\sigma)\eta(Y) - Y(\xi\sigma)\eta(X) + 2(\xi\sigma)d\eta(X,Y) = 0. \tag{22}$$

Choosing $X, Y \perp \xi$ and using the fact that $d\eta \neq 0$ in (22), we see that $\xi\sigma = 0$. Hence, $D\sigma = 0$ i.e. σ is a constant. Then (18) becomes,

$$2\alpha QY + 2\sigma h\phi Y = (2\lambda - \beta r)Y. \tag{23}$$

Contracting (23) and using the fact that $Trh\phi = 0$, we get

$$[2\alpha + (2n+1)\beta]r = 2(2n+1)\lambda. \tag{24}$$

Differentiating (23) along arbitrary vector field X gives

$$2\alpha(\nabla_X Q)Y + 2\sigma(\nabla_X h\phi)Y = 2(X\lambda)Y - \beta(Xr)Y. \tag{25}$$

Contracting (25) and using the fact that in contact metric manifold, $div(h\phi)Y = g(Q\xi, Y) - 2n\eta(Y)$, in the forgoing equation result in the following:

$$(\alpha + \beta)(Yr) + 2\sigma[Trl - 2n]\eta(Y) - 2(Y\lambda) = 0. \tag{26}$$

Taking $Y \perp \xi$ and using (24) in (26) gives $\alpha = 0$ or $Yr = 0$. Assuming $\alpha \neq 0$ and replacing Y by $\phi^2 Y$ shows $Dr = (\xi r)\xi$. Differentiating along arbitrary vector field X gives, $\nabla_X Dr = X(\xi r)\xi - (\xi r)(\phi X + \phi hX)$. Applying Poincare lemma, the forgoing equation yields

$$X(\xi r)\eta(Y) - Y(\xi r)\eta(X) - (\xi r)d\eta(X, Y) = 0. \tag{27}$$

choosing $X, Y \perp \xi$, it follows that $\xi r = 0$. Hence, $Dr = 0$ i.e. r is constant. Then (24) implies λ is constant and consequently from (19), Trl is constant. In view of (26) we get $Trl = 2n$, i.e. $h = 0$. Hence manifold is K-contact and then from (23), it is Einstein provided $\alpha \neq 0$. Suppose M is complete, then making use of results in Sharma [17] and Boyer and Galicki [24], we see that the manifold is compact Einstein Sasakian. This completes the proof. □

From (19) we get, $2\alpha Trl = (2\lambda - \beta r)$. Using this in (20) gives

$$2\alpha[Q\xi - (Trl)\xi] + D\sigma + (\xi\sigma)\xi = 0. \tag{28}$$

making use of result by Perrone [25] and (28), we can state the following:

Corollary 2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold such that g represents an almost (α, β)-Ricci–Yamabe soliton with $\alpha \neq 0$. Then M is an H-contact metric manifold if and only if the potential vector field is a constant multiple of the Reeb vector field ξ.*

In consequence of Theorem 1, considering a particular case when potential vector field V is the Reeb vector field ξ, we can easily prove the following:

Corollary 3. *There does not exist almost Ricci–Yamabe soliton with $\alpha \neq 0$ in a non-Sasakian (k, μ)-contact metric manifold whose potential vector field is the Reeb vector field ξ.*

4. Almost Ricci–Yamabe soliton on *K*-contact manifold

In [17], Sharma proved that if a compact K-contact metric is a gradient Ricci soliton then it is Einstein Sasakian. Extending this for gradient Ricci almost soliton, Ghosh [16] proved that compact K-contact metric is Einstein Sasakian and isometric to a unit sphere S^{2n+1}. However, this result is also true if one relaxes the hypothesis compactness to completeness (see Ref. [26]). In this section, we consider the gradient almost Ricci–Yamabe soliton and extend these results and prove.

Theorem 4. *If a K-contact manifold $M^{(2n+1)}(\phi, \xi, \eta, g)$ admits a gradient almost Ricci–Yamabe soliton with $\alpha \neq 0$ and $4\alpha n + \beta r \geq 2\lambda$, then it is Einstein with constant scalar curvature $r = 2n(2n+1)$. Further, if M is complete, then it is compact Sasakian and isometric to a unit sphere S^{2n+1}.*

Proof. A gradient almost Ricci–Yamabe soliton is given by

$$\nabla_X Df + 2\alpha QX = (2\lambda - \beta r)X. \tag{29}$$

Taking covarient differentiation of (29) along arbitrary vector field Y yields

$$\begin{aligned}&\nabla_Y\nabla_X Df + 2\alpha(\nabla_Y Q)X + 2\alpha Q(\nabla_Y X)\\ &= 2(Y\lambda)X - \beta(Yr)X + (2\lambda - \beta r)g(Y,X).\end{aligned} \tag{30}$$

Since $R(X, Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df$, then in consequence of (30) we get

$$\begin{aligned}R(X,Y)Df &= 2[(X\lambda)Y - (Y\lambda)X] - \beta[(Xr)Y - (Yr)X]\\ &- 2\alpha[(\nabla_X Q)Y - (\nabla_Y Q)X].\end{aligned} \tag{31}$$

Differentiating (11) along vector field Y and using (12) gives

$$(\nabla_X Q)\xi = Q\phi X - 2n\phi X. \tag{32}$$

Taking inner product of (31) with ξ and replacing Y by ξ and using the fact that $g(R(X, Y)Df, \xi) = -g(R(X, Y)\xi, Df)$ along with (12) and (32), Eq. (31) reduces to $X(f + 2\lambda - \beta r) = \xi(f + 2\lambda - \beta r)\eta(X)$, which can be written as $d(f + 2\lambda - \beta r) = \xi(f + 2\lambda - \beta r)\eta$. Then operating the last equation by d and using Poincare lemma, i.e. $d^2 = 0$ we get $d\xi(f + 2\lambda - \beta r) \wedge \eta + \xi(f + 2\lambda - \beta r)d\eta = 0$. Taking the wedge product of forgoing equation with η and using the fact that $\eta \wedge \eta = 0$ yields $\xi(f + 2\lambda - \beta r)d\eta \wedge \eta = 0$. Therefore $\xi(f + 2\lambda - \beta r) = 0$ on M as $d\eta$ is non-vanishing everywhere on M, consequently, $D(f + 2\lambda - \beta r) = 0$. Hence, $f + 2\lambda - \beta r$ is constant on M.

Taking Lie differentiation of (29) along ξ and noting $\mathcal{L}_\xi Q = 0$ (as ξ is Killing) we obtain

$$\mathcal{L}_\xi(\nabla_X Df) + 2\alpha Q(\mathcal{L}_\xi X) = 2(\xi\lambda)X - \beta(\xi r)X + (2\lambda - \beta r)\mathcal{L}_\xi X. \tag{33}$$

Lie differentiating Df along ξ and using (10) yields

$$\mathcal{L}_\xi Df = [\xi, Df] = \nabla_\xi Df - \nabla_{Df}\xi = (2\lambda - \beta r)\xi - 4n\alpha\xi + \phi Df. \tag{34}$$

Differentiating covariently (34) along vector field Y and using (10) we obtain

$$\nabla_Y\mathcal{L}_\xi Df = 2(Y\lambda)\xi - \beta(Yr)\xi + 4n\alpha\phi Y + (\nabla_Y\phi)Df - 2\alpha\phi QY \tag{35}$$

According to Yano [27], we have the commutative formula

$$\mathcal{L}_V\nabla_Y X - \nabla_Y\mathcal{L}_V X - \nabla_{[V,Y]}X = (\mathcal{L}_V\nabla)(Y,X). \tag{36}$$

Setting $V = \xi$ and $X = Df$ in (36) and noting $\mathcal{L}_\xi\nabla = 0$ and using (33)-(35) yields

$$\begin{aligned}&[2(\xi\lambda) - \beta(\xi r)]g(X,Y) - Y(2\lambda - \beta r)\eta(X) - 4n\alpha g(\phi Y, X)\\ &\quad + g((\nabla_Y\phi)X, Df) + 2\alpha g(\phi QY, X) = 0.\end{aligned} \tag{37}$$

Replacing X by ϕX and Y by ϕY along with well-known formula

$$(\nabla_Y\phi)X + \left(\nabla_{\phi Y}\phi\right)\phi X = 2g(Y,X)\xi - \eta(X)(Y + \eta(Y)\xi)$$

we get

$$\begin{aligned}&2\xi(f + 2\lambda - \beta r)g(X,Y) - Y(f + 2\lambda - \beta r)\eta(X)\\ &- \xi(f + 2\lambda - \beta r)\eta(X)\eta(Y) + 2\alpha g(Q\phi Y, X)\\ &+ 2\alpha g(\phi QY, X) - 8n\alpha g(\phi Y, X) = 0.\end{aligned} \tag{38}$$

Suppose $\alpha \neq 0$. Since $f + 2\lambda - \beta r$ is constant Eq. (38) reduces to

$$Q\phi X + \phi QX = 4n\phi X, \tag{39}$$

for any vector field X on M. Taking an inner product of (31) along with $f + 2\lambda - \beta r = constant$ yields

$$g((\nabla_Y Q)X - (\nabla_X Q)Y, Df) = 0. \tag{40}$$

Let $\{e_i, \phi e_i, \xi; i = 1, 2, \ldots n\}$ be an orthonormal ϕ − basis of M such that $Qe_i = \sigma_i e_i$. Using this in (39) we get $Q\phi e_i = (4n - \sigma_i)\phi e_i$. Then the scalar curvature is given by

$$r = g(Q\xi, \xi) + \sum_{i=1}^{n}[g(Qe_i . e_i) + g(Q\phi e_i, \phi e_i)] = 2n(2n + 1).$$

Replacing X by ξ in (40) and using (32) yields $Q\phi Df - 2n\phi Df = 0$. In consequence of this in (39), it reduces to $\phi QDf = 2n\phi Df$. Operating last equation with ϕ and using (11) gives $QDf = 2nDf$. Then taking covariant derivative results in

$$(\nabla_X Q)Df - 2\alpha Q^2 X + (2\lambda - \beta r + 4n\alpha)QX - 2n(2\lambda - \beta r)X = 0. \tag{41}$$

Since $r = 2n(2n + 1)$ is constant, then $divQ = \frac{1}{2}dr = 0$. Making use of this and contracting (41) we obtain $\|Q\|^2 = 2nr$. In consequence of this with $r = 2n(2n + 1)$, we can easily see that $\|Q - \frac{r}{2n+1}I\|^2 = 0$ i.e., length of the symmetric tensor $Q - \frac{r}{2n+1}I$ vanishes, we must have $QX = 2nX$. Thus M is Einstein with Einstein constant $2n$. Suppose M is complete, then by the result of Sharma [17] we can conclude that M is compact. Applying Boyer–Galicki [24] we conclude that it is Sasakian. Also, Eq. (29) can be rewritten as $\nabla_X Df = -\rho X$, where $\rho = 4\alpha n + \beta r - 2\lambda$, then by Obata's theorem [28] it is isometric to a unit sphere S^{2n+1}. This completes the proof. □

Corollary 5. *If a Sasakian manifold $M^{(2n+1)}(\phi, \xi, \eta, g)$ admits a gradient almost Ricci–Yamabe soliton with $\alpha \neq 0$ and $4\alpha n + \beta r \geq 2\lambda$, then it is Einstein with constant scalar curvature $r = 2n(2n + 1)$. Further, if M is complete, then it is compact and isometric to a unit sphere S^{2n+1}.*

5. Almost Ricci–Yamabe soliton on (k, μ)-contact metric manifold

In [16], Ghosh proved that if the metric of a non-Sasakian (κ, μ)-contact metric manifold admits a gradient Ricci almost soliton, then it is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$. Following his work, we explore the gradient almost Ricci–Yamabe soliton on non-Sasakian (κ, μ)-contact metric manifold and obtain the following:

Theorem 6. *If a non-Sasakian (k, μ)-contact metric manifold $M^{(2n+1)}(\phi, \xi, \eta, g)$ admits a gradient almost Ricci–Yamabe soliton with $\alpha \neq 0$, then M^3 is flat and the soliton vector field is homothetic, and for $n > 1$, M is locally isometric to $E^{n+1} \times S^n(4)$ and the soliton vector field is tangential to the Euclidean factor E^{n+1}.*

Proof. Making use of $R(X, Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df$ and (29), we get

$$R(X, Y)Df = 2\alpha[(\nabla_Y Q)X - (\nabla_X Q)Y] + 2[(X\lambda)Y - (Y\lambda)X]. \tag{42}$$

Taking covariant derivative of (15) and using it in (42) yields

$$\begin{aligned} R(X,Y)Df &= 2\alpha\{[2(n-1)+\mu][2(1-k)g(Y,\phi X)\xi + \\ \eta(X)\{h(\phi Y+\phi hY\} &- \eta(Y)\{h(\phi X+\phi hX\} + \mu\eta(X)\phi hY \\ -\mu\eta(Y)\phi hX] &+ [2(1-n)+n(2k+\mu)]\{2g(Y,\phi X)\xi - \\ (\phi Y+\phi hY)\eta(X) &+ (\phi X+\phi hX)\eta(Y)\}\} + 2[(X\lambda)Y-(Y\lambda)X]. \end{aligned} \tag{43}$$

Taking the inner product of (43) with ξ gives

$$\begin{aligned} g(R(X,Y)Df,\xi) &= 4\alpha(\mu+2k-k\mu+n\mu)g(Y,\phi X) \\ &+ 2[(X\lambda)Y-(Y\lambda)X]. \end{aligned} \tag{44}$$

Taking the inner product of (14) with Df, we get

$$\begin{aligned} g(R(X,Y)\xi,Df) &= k[\eta(Y)g(X,Df)-\eta(X)g(Y,Df)] \\ &+\mu[\eta(Y)g(hX,Df)-\eta(X)g(hY,Df)]. \end{aligned} \tag{45}$$

Combining (44) and (45) we get

$$\begin{aligned} k[\eta(Y)g(X,Df)-\eta(X)g(Y,Df)] \\ +\mu[\eta(Y)g(hX,Df)-\eta(X)g(hY,Df)] \\ +4\alpha(\mu+2k-k\mu+n\mu)g(Y,\phi X) \\ +2[(X\lambda)\eta(Y)-(Y\lambda)\eta(X)] = 0. \end{aligned} \tag{46}$$

Taking $X = \phi X$ and $Y = \phi Y$ and using the fact that $R(\phi X, \phi Y)\xi = 0$, Eq. (46) for $\alpha \neq 0$ reduces to

$$k = \frac{\mu(1+n)}{\mu-2}. \tag{47}$$

Replacing $Y = \xi$ in (46) gives

$$(k+\mu h)Df + 2(D\lambda) - [k(\xi f)+2(\xi\lambda)]\xi = 0. \tag{48}$$

In consequence of (15), replacing X by Df and simplifying we obtain

$$QDf = -4n(D\lambda). \tag{49}$$

Making use of (49) in (48) gives

$$2n(k+\mu h)Df - QDf - 2n[k(\xi f)+2(\xi\lambda)]\xi = 0. \tag{50}$$

Taking an inner product of (50) with ξ we get, $k(\xi f) + 2(\xi\lambda) = 0$ and using this in forgoing equation

$$2n(k+\mu h)Df = QDf. \tag{51}$$

Differentiating (51) and simplifying, we obtain

$$\left(2n\mu^2 - \mu[2(n-1)+\mu]\right)\phi hDf - 2n\mu h(2\lambda-\beta r-4n\alpha k)\xi = 0. \tag{52}$$

Taking inner product of (52) with ξ gives, $\mu h(2\lambda - \beta r - 4n\alpha k) = 0$, and using it in (52)

$$\left(2n\mu^2 - \mu[2(n-1)+\mu]\right)\phi hDf = 0. \tag{53}$$

Operating h in the above equation and using (13), we get

$$(k-1)\mu[2(n-1)+\mu-2n\mu]\phi Df=0. \tag{54}$$

We get the following cases:

Case-I: For $\mu=0$. In consequence, equation (47) gives $k=0$. Hence, $R(X, Y)\xi=0$.

Now in Blair [29] proved that a $(2n+1)$-dimensional contact metric manifold satisfying $R(X, Y)\xi=0$ is locally isometric to $E^{n+1}\times S^n(4)$ for $n>1$ and flat if $n=1$.

Therefore, we conclude that the manifold under consideration is locally isometric to $E^{n+1}\times S^n(4)$ for $n>1$ and flat if $n=1$.

Case-II: For $\phi Df=0$. Operating ϕ on both sides gives $Df=(\xi f)\xi$. Differentiating along arbitrary vector field X gives

$$\nabla_X Df=X(\xi f)\xi-(\xi f)(\phi X+\phi hX). \tag{55}$$

Applying the Poincare lemma in the above equation yields

$$X(\xi f)\eta(Y)-Y(\xi f)\eta(X)+(\xi f)d\eta(X,Y)=0. \tag{56}$$

Taking $X, Y\perp\xi$ and since $d\eta$ is nowhere vanishing on M, it follows $\xi f=0$. Hence $Df=0$ i.e. f is constant. Then from (29) we see that M is Einstein (i.e., $2\alpha QY=(2\lambda-\beta r)Y$). Taking a trace of the last equation yields $2\alpha r=(2n+1)(2\lambda-\beta r)$. Also, replacing Y by ξ in the second last equation and using the previous equation results in $QY=2nkY$. Consequently, the scalar curvature is $r=2nk(2n+1)$. Now proceeding similarly as in Theorem 4.1 of Ghosh [16], we also find that for $n=1$, M is locally flat (as $\mu=0$ and $k=0$ consequently $R(X, Y)\xi=0$), using $\mu=2(1-n)$ in (47), we see that $k=n-\frac{1}{n}>1$, a contraction. Since M^3 is flat and λ is constant in view of (29), we see that the vector field is homothetic.

Case-III: For $2(n-1)+\mu-2n\mu=0$ implies $\mu=\frac{2(1-n)}{1-2n}$.

Using this value of μ in the expression of k in (47), we get $k=\frac{1}{n}-n$.

Replacing X by Df in (15) then inserting it in (51) yields

$$[2(1-n)+n(2k+\mu)](Df-(\xi f)\xi)+[2n\mu-2(n-1)-\mu]hDf=0. \tag{57}$$

Inserting $\mu=\frac{2(1-n)}{1-2n}$ and $k=\frac{1}{n}-n$ in (57), we obtain $Df=(\xi f)\xi$. Then proceeding similarly as in Case-II we obtain a similar conclusion. Since $QX=2nkX$, taking covarient differentiation gives $\nabla Q=0$ and consequently (42) reduces to

$$R(X,Y)Df=2[(X\lambda)Y-(Y\lambda)X].$$

Since $R(X, Y)\xi=0$ and taking inner product of forgoing equation with ξ and replacing Y by ξ gives $X\lambda=(\xi\lambda)\eta(X)$. Similarly as above we can easily see that λ is constant and consequently $R(X, Y)Df=0$ i.e., Df is tangent to the flat factor E^{n+1}. This completes the proof. □

Finally, we construct an example for verifying the obtained result. In Ref. [30], De and Mandal constructed a 3-dimensional example of a generalized (κ, μ)-contact metric manifold with $\kappa=1-(\sigma(z))^2$ and $\mu=2(1+\sigma(z))$. Let $k: I\subset\mathbb{R}\to\mathbb{R}$ be a smooth function defined on an open interval I such that $k(z)\leq 1$ for any $z\in I$. Set $\sigma(z)=\sqrt{1-k(z)}\geq 0$ and let $\{e_1, e_2, e_3\}$ be three linearly independent vector fields on $M=\mathbb{R}^2\times I\subset\mathbb{R}^3$ given by

$$
\begin{aligned}
e_1 &= \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \\
e_3 &= (2y + f(z))\frac{\partial}{\partial x} + \left(2\sigma(z)x - \frac{\sigma'(z)}{1+2\sigma(z)}y + h(z)\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z},
\end{aligned}
$$

where $f(z)$ and $h(z)$ are arbitrary functions of z. Let g be the Riemannian metric defined by

$$
g(e_i, e_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j; i, j = 1, 2, 3. \end{cases}
$$

Let η be the 1-form defined by $\eta(X) = g(X, e_1)$ for all vector field X on M. Also, let ϕ be the (1,1)-tensor field define as

$$
\phi e_1 = 0, \qquad \phi e_2 = e_3, \qquad \phi e_3 = -e_2.
$$

Clearly, $M(\phi, \xi = e_1, \eta, g)$ formed a contact metric manifold. The non-vanishing components of the curvature tensor are given by Ref. [30]:

$$
R(e_2, e_1)e_1 = (1 + \sigma(z))^2 e_2, \qquad R(e_3, e_1)e_1 = \left(1 - 2\sigma(z) - 3\sigma^2(z)\right)e_3.
$$

The non-vanishing components of the Ricci tensor and scalar curvature are as follows:

$$
\begin{aligned}
S(e_1, e_1) &= 2\left(1 - \sigma^2(z)\right), \quad S(e_2, e_2) = (1 + \sigma(z))^2, \\
S(e_3, e_3) &= \left(1 - 2\sigma(z) - 3\sigma^2(z)\right), \quad r = 4\left(1 - \sigma^2(z)\right).
\end{aligned}
$$

Now let us take $k \in \mathbb{R}$ as a constant. Then the manifold M becomes (κ, μ)-contact metric manifold. Moreover, we see that for $k \neq 1$, M is a non-Sasakian manifold. Let the potential vector field $V = e_1$, then solving (5) gives $\alpha\sigma(\sigma + 1) = 0$, which implies $\sigma = 0$, a contradiction. Thus, Corollary 3 is verified.

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