Arithmetic properties of (2, 3)-regular overcubic bipartitions

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Abstract

Purpose – Let $\bar{b}_{2,3}(n)$, which enumerates the number of (2, 3)-regular overcubic bipartition of n. The purpose of the paper is to describe some congruences modulo 8 for $\bar{b}_{2,3}(n)$. For example, for each $\alpha \geq 0$ and $n \geq 1$, $\bar{b}_{2,3}(8n+5) \equiv 0 \pmod 8$, $\bar{b}_{2,3}(2\cdot 3^{\alpha+3}n + 4\cdot 3^{\alpha+2}) \equiv 0 \pmod 8$.

Design/methodology/approach – H.C. Chan has studied the congruence properties of cubic partition function a(n), which is defined by $\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}$.

Findings – To establish several congruence modulo 8 for $\bar{b}_{2,3}(n)$, here the author keeps to the classical spirit of q-series techniques in the proofs.

Originality/value – The results established in the work are extension to those proved in ℓ-regular cubic partition pairs.

Keywords Congruences, Dissections, (2, 3)-regular overcubic bipartitions

Paper type Research paper

1. Introduction

A partition λ of a natural number n is a finite non-increasing sequence of positive integer parts λ_i $(1 \leq i \leq m)$ such that

$$n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_m.$$

In this case, we write $|\lambda| = n$. The number of partitions of n is denoted by $p(n)$ and the generating function is given by as follows:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Ramanujan's three famous congruences of $p(n)$ are as follows:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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In [1–3], H.C. Chan has studied the congruence properties of cubic partition function $a(n)$, which is defined by as follows:

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

B. Kim [4] studied its overpartition analog, the overcubic partition function $\bar{a}(n)$, which is defined by as follows:

$$\sum_{n=0}^{\infty} \bar{a}(n)q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty}{(q;q)_\infty (q^2;q^2)_\infty}.$$

In [5], M.D. Hirschhorn obtained the results satisfied by $\bar{a}(n)$, which appeared in Kim's paper [4], and Sellers [6] has proved a number of arithmetic properties of $\bar{a}(n)$ by employing elementary generating function methods. Zhao and Zhong [7] studied cubic partition pairs, which are denoted by $b(n)$, and the generating function is as follows:

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}.$$

Recently, Kim [8] studied congruence properties of $\bar{b}(n)$, which denotes overcubic partition pairs of n, whose generating function is given by as follows:

$$\sum_{n=0}^{\infty} \bar{b}(n)q^n = \frac{(-q;q)_\infty^2 (-q^2;q^2)_\infty^2}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}.$$

More recently, Lin [9] studied various arithmetic properties of $\bar{b}(n)$ modulo 3 and 5. For example, for any $\alpha \geq 2$, $n \geq 0$,

$$\bar{b}(3^\alpha(3n+2)) \equiv 0 \pmod 3,$$

for $\alpha \geq 0$,

$$\bar{b}(380 \cdot 5^\alpha) \equiv 0 \pmod 3.$$

In [10], Naika and Nayaka have established some congruences for ℓ-regular cubic partition pairs. Let $\bar{b}_{2,3}(n)$ denote the number of (2, 3)-regular overcubic bipartitions of n, whose generating function is given by as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(n)q^n = \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty^4 (q^4;q^4)_\infty^4 (q^{24};q^{24})_\infty^2}{(q;q)_\infty^4 (q^6;q^6)_\infty^2 (q^8;q^8)_\infty^2 (q^{12};q^{12})_\infty^4}. \tag{1.1}$$

In this paper, we establish several congruences modulo 8 for $\bar{b}_{2,3}(n)$. These results can be found in Theorems (3.1), and we keep to the classical spirit of q-series techniques in our proofs.

2. Preliminaries

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as follows:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Some special cases of $f(a, b)$ are as follows:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty,$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty.$$

Where the product representation of $f(a, b)$ arises from Jacobi's triple product identity [11, p. 35, Entry 19] as follows:

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

The following dissection formulas to prove our main results.

Lemma 2.1. *For each prime p and $n \geq 1$,*

$$(q; q)_\infty^{p^n} \equiv (q^p; q^p)_\infty^{p^{n-1}} \pmod{p^n}. \tag{2.1}$$

Lemma 2.2. *The following 2-dissections holds:*

$$(q; q)_\infty^2 = \frac{(q^2; q^2)_\infty (q^8; q^8)_\infty^5}{(q^4; q^4)_\infty^2 (q^{16}; q^{16})_\infty^2} - 2q \frac{(q^2; q^2)_\infty (q^{16}; q^{16})_\infty^2}{(q^8; q^8)_\infty}, \tag{2.2}$$

$$\frac{1}{(q; q)_\infty^2} = \frac{(q^8; q^8)_\infty^5}{(q^2; q^2)_\infty^5 (q^{16}; q^{16})_\infty^2} + 2q \frac{(q^4; q^4)_\infty^2 (q^{16}; q^{16})_\infty^2}{(q^2; q^2)_\infty^5 (q^8; q^8)_\infty}. \tag{2.3}$$

Lemma (2.2) is a consequence of dissection formulas of Ramanujan, which is collected in Berndt's book [11, p. 40, Entry 25].

Lemma 2.3. *The following 2-dissections holds:*

$$\frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} = \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} + q \frac{(q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty}, \tag{2.4}$$

$$\frac{(q; q)_\infty^3}{(q^3; q^3)_\infty} = \frac{(q^4; q^4)_\infty^3}{(q^{12}; q^{12})_\infty} - 3q \frac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty (q^6; q^6)_\infty^2}, \tag{2.5}$$

$$\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3} = \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^3}{(q^2; q^2)_\infty^9 (q^{12}; q^{12})_\infty^2} + 3q \frac{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty^7}. \tag{2.6}$$

Hirschhorn, Garvan and Borwein [12] proved (2.4) and (2.5). For proof of (2.6), see [13].

Lemma 2.4. *The following 2-dissections holds:*

$$\frac{1}{(q;q)_\infty(q^3;q^3)_\infty} = \frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^4(q^{24};q^{24})_\infty^2} + q\frac{(q^4;q^4)_\infty^5(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^6;q^6)_\infty^2(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty} \tag{2.7}$$

$$(q;q)_\infty(q^3;q^3)_\infty = \frac{(q^2;q^2)_\infty(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^4}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{24};q^{24})_\infty^2} - q\frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^2}. \tag{2.8}$$

Eqn (2.7) was proved by Baruah and Ojah [14]. Replacing q by $-q$ in (2.7) and using the fact that $(-q;-q)_\infty = \frac{f_2^3}{f_1f_4}$, we get (2.8).

Lemma 2.5. *The following 3-dissection hold:*

$$(q;q)_\infty(q^2;q^2)_\infty = \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^4}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2} - q(q^9;q^9)_\infty(q^{18};q^{18})_\infty - 2q^2\frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}. \tag{2.9}$$

One can see this identity in [15].

Lemma 2.6. [11, p. 345, Entry 1 (iv)]. *We have the following 3-dissection*

$$(q;q)_\infty^3 = (q^9;q^9)_\infty^3(\zeta^{-1} - 3q + 4q^3\zeta^2), \tag{2.10}$$

where

$$\zeta = \frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^3}{(q^6;q^6)_\infty(q^9;q^9)_\infty^3}. \tag{2.11}$$

3. Congruences modulo 8 for $\bar{b}_{2,3}(n)$

Theorem 3.1. *For each $\alpha \geq 0$ and $n \geq 1$, we have*

$$\bar{b}_{2,3}(6n+5) \equiv 0 \pmod 8, \tag{3.1}$$

$$\overline{b}_{2,3}(8n+5) \equiv 0 \pmod{8}, \tag{3.2}$$

$$\overline{b}_{2,3}(12n+7) \equiv 0 \pmod{8}, \tag{3.3}$$

$$\overline{b}_{2,3}(18n+15) \equiv 0 \pmod{8}, \tag{3.4}$$

$$\overline{b}_{2,3}(36n+21) \equiv 0 \pmod{8}, \tag{3.5}$$

$$\overline{b}_{2,3}(72n+38) \equiv 0 \pmod{8}, \tag{3.6}$$

$$\overline{b}_{2,3}(72n+51) \equiv 0 \pmod{8}, \tag{3.7}$$

$$\overline{b}_{2,3}(72n+57) \equiv 0 \pmod{8}, \tag{3.8}$$

$$\overline{b}_{2,3}(216n+99) \equiv 0 \pmod{8}, \tag{3.9}$$

$$\overline{b}_{2,3}(216n+171) \equiv 0 \pmod{8}, \tag{3.10}$$

$$\overline{b}_{2,3}(8\cdot 9^{\alpha+2}n+57\cdot 9^{\alpha+1}) \equiv 0 \pmod{8}, \tag{3.11}$$

$$\overline{b}_{2,3}(8\cdot 9^{\alpha+2}n+35\cdot 9^{\alpha+2}) \equiv 0 \pmod{8}, \tag{3.12}$$

$$\overline{b}_{2,3}(2\cdot 3^{\alpha+3}n+4\cdot 3^{\alpha+2}) \equiv 0 \pmod{8}, \tag{3.13}$$

$$\overline{b}_{2,3}(72n+3) \equiv b_{2,3}(24n+1) \pmod{8}, \tag{3.14}$$

$$\overline{b}_{2,3}(36n+3) \equiv b_{2,3}(12n+1) \pmod{8}. \tag{3.15}$$

Proof. Employing (2.4) and (2.5) in (1.1), we have

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(n)q^n = \frac{(q^4;q^4)_\infty^{13}(q^6;q^6)_\infty^3(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^9(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^7}$$

$$+4q\frac{(q^4;q^4)_\infty^9(q^6;q^6)_\infty(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^7(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^3} \tag{3.16}$$

$$+3q^2\frac{(q^4;q^4)_\infty^5(q^{12};q^{12})_\infty(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty(q^8;q^8)_\infty^2}, \tag{3.17}$$

which implies the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(2n+1)q^n = 4\frac{(q^2;q^2)_\infty^9(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2}{(q;q)_\infty^7(q^4;q^4)_\infty^2(q^6;q^6)_\infty^3}. \tag{3.18}$$

Invoking (2.1) in (3.18), we obtain the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(2n+1)q^n \equiv 4(q;q)_\infty (q^3;q^3)_\infty (q^2;q^2)_\infty (q^6;q^6)_\infty \pmod 8. \tag{3.19}$$

Substituting (2.8) into (3.19), we get the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(2n+1)q^n \equiv 4\frac{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2 (q^{12};q^{12})_\infty^4}{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty^2} + 4q\frac{(q^6;q^6)_\infty^2 (q^4;q^4)_\infty^4 (q^{24};q^{24})_\infty^2}{(q^8;q^8)_\infty^2 (q^{12};q^{12})_\infty^2} \pmod 8. \tag{3.20}$$

Extracting the terms in which powers of q are congruent to 1 modulo 2 from (3.20), we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(4n+3)q^n \equiv 4\frac{(q^3;q^3)_\infty^2 (q^2;q^2)_\infty^4 (q^{12};q^{12})_\infty^2}{(q^4;q^4)_\infty^2 (q^6;q^6)_\infty^2} \pmod 8. \tag{3.21}$$

Invoking (2.1) in (3.21), we obtain as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(4n+3)q^n \equiv 4(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^2 \pmod 8. \tag{3.22}$$

Extracting the terms involving q^{3n} from (3.22), replacing q^3 by q, we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(12n+3)q^n \equiv 4(q;q)_\infty^2 (q^2;q^2)_\infty^2 \pmod 8. \tag{3.23}$$

Employing (2.2) into (3.23), we find the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(12n+3)q^n \equiv 4\frac{(q^2;q^2)_\infty^3 (q^8;q^8)_\infty^5}{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2} \pmod 8. \tag{3.24}$$

Extracting the terms involving q^{2n} from (3.24), replacing q^2 by q, we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+3)q^n \equiv 4\frac{(q;q)_\infty^3 (q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2} \pmod 8. \tag{3.25}$$

Invoking (2.1) in (3.25), we get the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+3)q^n \equiv 4\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \pmod 8. \tag{3.26}$$

Ramanujan recorded the following identity in his third note book; for proof, one can see [11, p. 49].

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty} + q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}. \tag{3.27}$$

Substituting (3.27) into (3.26), we obtain the generating function as follows

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+3)q^n \equiv 4\frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty} + 4q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty} \pmod 8. \tag{3.28}$$

Congruence (3.7) follows from (3.28).

Extracting the terms in which powers of q are congruent to 1 modulo 3 from (3.28), we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(72n+27)q^n \equiv 4\frac{(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty} \pmod 8. \tag{3.29}$$

The results (3.9) and (3.10) follow from (3.29).

From (3.29), we obtain the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(216n+27)q^n \equiv 4\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \pmod 8. \tag{3.30}$$

Using the congruences (3.30) and (3.26), we can see that

$$\overline{b}_{2,3}(216n+27) \equiv \overline{b}_{2,3}(24n+3) \pmod 8.$$

By mathematical induction on α, we find that

$$\overline{b}_{2,3}(216\cdot 9^\alpha n + 27\cdot 9^\alpha) \equiv \overline{b}_{2,3}(24n+3) \pmod 8. \tag{3.31}$$

Using (3.7) in (3.31), we get (3.12).

Extracting the terms involving q^{3n} from (3.28), replacing q^3 by q, we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(72n+3)q^n \equiv 4\frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^6;q^6)_\infty} \pmod 8. \tag{3.32}$$

Invoking (2.1) in (3.32), we get the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(72n+3)q^n \equiv 4(q;q)_\infty \pmod 8. \tag{3.33}$$

From (3.20), we can see that

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(4n+1)q^n \equiv 4\frac{(q;q)_\infty^2 (q^4;q^4)_\infty^2 (q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^2 (q^{12};q^{12})_\infty^2} \pmod 8. \tag{3.34}$$

Invoking (2.1) in (3.34), we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(4n+1)q^n \equiv 4(q^2;q^2)_\infty (q^4;q^4)_\infty \pmod 8. \tag{3.35}$$

Congruence (3.2) follows from (3.34).

Extracting the terms involving q^{2n} from (3.35), replacing q^2 by q, we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(8n+1)q^n \equiv 4(q;q)_\infty (q^2;q^2)_\infty \pmod 8. \tag{3.36}$$

Employing (2.9) into (3.36), we obtain the generating function as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{b}_{2,3}(8n+1)q^n \equiv\ & 4\frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^4}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty^2} \\ & + 4q(q^9;q^9)_\infty (q^{18};q^{18})_\infty \pmod 8.\end{aligned} \tag{3.37}$$

Extracting the terms in which powers of q are congruent to 1 modulo 3 from (3.37), we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+9)q^n \equiv 4(q^3;q^3)_\infty (q^6;q^6)_\infty \pmod 8. \tag{3.38}$$

The results (3.6) and (3.8) follow from (3.38).

From (3.38), we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+9)q^n \equiv 4(q;q)_\infty (q^2;q^2)_\infty \pmod 8. \tag{3.39}$$

Using the congruences (3.39) and (3.36), we can see that

$$\overline{b}_{2,3}(72n+9) \equiv \overline{b}_{2,3}(8n+1) \pmod 8.$$

By mathematical induction on α, we obtain the generating function as follows:

$$\overline{b}_{2,3}\left(8\cdot 9^{\alpha+1}n + 9^{\alpha+1}\right) \equiv \overline{b}_{2,3}(8n+1) \pmod 8. \tag{3.40}$$

Using (3.8) in (3.40), we get (3.11).

Extracting the terms involving q^{3n} from (3.37), replacing q^3 by q, we have the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+1)q^n \equiv 4\frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^4}{(q;q)_\infty (q^6;q^6)_\infty^2} \pmod 8. \tag{3.41}$$

Invoking (2.1) in (3.41), we get the generating function as follows:

$$\sum_{n=0}^{\infty} \overline{b}_{2,3}(24n+1)q^n \equiv 4(q;q)_\infty \pmod 8. \tag{3.42}$$

Using the congruences (3.33) and (3.42), we obtain (3.14).

From (3.19), it can be rewritten as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(2n+1)q^n \equiv 4(q;q)_\infty (q^2;q^2)_\infty (q^3;q^3)_\infty^3 \pmod 8. \tag{3.43}$$

Employing (2.9) into (3.43), we obtain the generating function as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(2n+1)q^n \equiv 4\frac{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty (q^9;q^9)_\infty^4}{(q^{18};q^{18})_\infty^2} + 4q(q^3;q^3)_\infty^3 (q^9;q^9)_\infty (q^{18};q^{18})_\infty \pmod 8. \tag{3.44}$$

Congruence (3.1) follows from (3.44).

Extracting the terms in which powers of q are congruent to 1 modulo 3 from (3.44), we have the generating function as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(6n+3)q^n \equiv 4(q;q)_\infty^3 (q^3;q^3)_\infty (q^6;q^6)_\infty \pmod 8, \tag{3.45}$$

which implies as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(6n+3)q^n \equiv 4(q;q)_\infty^3 (q^3;q^3)_\infty^3 \pmod 8. \tag{3.46}$$

Substituting (2.10) into (3.46), we obtain the generating function as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(6n+3)q^n \equiv 4\frac{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty (q^9;q^9)_\infty^6}{(q^{18};q^{18})_\infty^3} + 4q(q^3;q^3)_\infty^3 (q^9;q^9)_\infty^3 \pmod 8. \tag{3.47}$$

Congruence (3.4) follows from (3.47).

Extracting the terms in which powers of q are congruent to 1 modulo 3 from (3.47), we get the generating function as follows:

$$\sum_{n=0}^{\infty} \bar{b}_{2,3}(18n+9)q^n \equiv 4(q;q)_\infty^3 (q^3;q^3)_\infty^3 \pmod 8. \tag{3.48}$$

Using the congruences (3.48) and (3.46), we find that

$$\bar{b}_{2,3}(18n+9) \equiv \bar{b}_{2,3}(6n+3) \pmod 8.$$

By mathematical induction on α, we obtain the generating function as follows:

$$\bar{b}_{2,3}\left(2\cdot 3^{\alpha+2}n + 3^{\alpha+2}\right) \equiv \bar{b}_{2,3}(6n+3) \pmod 8. \tag{3.49}$$

Using (3.4) in (3.49), we get (3.13).

From (3.47), we have the generating function as follows:

$$\sum_{n=0}^{\infty}\overline{b}_{2,3}(18n+3)q^n \equiv 4\frac{(q;q)_\infty^2(q^2;q^2)_\infty(q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^3} \pmod 8. \tag{3.50}$$

Invoking (2.1) in (3.50), we find that

$$\sum_{n=0}^{\infty}\overline{b}_{2,3}(18n+3)q^n \equiv 4(q^2;q^2)_\infty^2 \pmod 8. \tag{3.51}$$

Congruence (3.5) follows from (3.51).

Extracting the terms involving q^{3n} from (3.43), replacing q^3 by q, we have the generating function as follows:

$$\sum_{n=0}^{\infty}\overline{b}_{2,3}(6n+1)q^n \equiv 4\frac{(q;q)_\infty^2(q^2;q^2)_\infty(q^3;q^3)_\infty^4}{(q^6;q^6)_\infty^2} \pmod 8. \tag{3.52}$$

Invoking (2.1) in (3.52), we obtain the generating function as follows:

$$\sum_{n=0}^{\infty}\overline{b}_{2,3}(6n+1)q^n \equiv 4(q^2;q^2)_\infty^2 \pmod 8. \tag{3.53}$$

Congruence (3.3) easily follows from (3.53).

From (3.51) and (3.53), we have the generating function as follows:

$$\sum_{n=0}^{\infty}\overline{b}_{2,3}(36n+3)q^n \equiv 4(q;q)_\infty^2 \pmod 8 \tag{3.54}$$

and

$$\sum_{n=0}^{\infty}\overline{b}_{2,3}(12n+1)q^n \equiv 4(q;q)_\infty^2 \pmod 8. \tag{3.55}$$

Using the congruences (3.54) and (3.55), we get internal congruence (3.15).

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