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When is (D, K) an S-accr pair?

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Abstract

Purpose – The purpose of this article is to determine necessary and sufficient conditions in order that (D, K) to be an S-accr pair, where D is an integral domain and K is a field which contains D as a subring and S is a multiplicatively closed subset of D.

Design/methodology/approach – The methods used are from the topic multiplicative ideal theory from commutative ring theory.

Findings – Let S be a strongly multiplicatively closed subset of an integral domain D such that the ring of fractions of D with respect to S is not a field. Then it is shown that (D, K) is an S-accr pair if and only if K is algebraic over D and the integral closure of the ring of fractions of D with respect to S in K is a one-dimensional Prüfer domain. Let D, S, K be as above. If each intermediate domain between D and K satisfies S-strong accr*, then it is shown that K is algebraic over D and the integral closure of the ring of fractions of D with respect to S is a Dedekind domain; the separable degree of K over F is finite and K has finite exponent over F, where F is the quotient field of D.

Originality/value – Motivated by the work of some researchers on S-accr, the concept of S-strong accr* is introduced and we determine some necessary conditions in order that (D, K) to be an S-strong accr* pair. This study helps us to understand the behaviour of the rings between D and K.

Keywords Accr pair, S-accr pair, strong accr* pair, S-strong accr* pair, Prüfer domain

Paper type Research paper

1. Introduction

The rings considered in this article are commutative with identity. Modules considered are unitary modules over commutative rings. We use the abbreviation m.c. subset for multiplicatively closed subset. The m.c. subsets considered in this article are assumed that they do not contain the zero element of the ring. This article is motivated by the research work presented in Refs. [1–4]. Let R be a ring and let M be a module over R. Recall from [3, Definition 1] that M is said to *satisfy (accr)* (respectively, *(accr*)*) if for every submodule N of M and every finitely generated (respectively, principal) ideal B of R, the increasing sequence of residuals $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ terminates. We say that a ring *satisfies (accr)* (respectively, *(accr*)*) if it satisfies (accr) (respectively. (accr*)) as a module over itself. Various important properties of Noetherian modules and rings were generalized in Refs. [3, 4] to modules and rings satisfying (accr). It was proved in [3, Theorem 1] that for any R-module M, the properties (accr) and (accr*) are equivalent.

Let M be a module over a ring R. Let S be a m.c. subset of R. We use f.g. for finitely generated. Recall from [2, pp. 409 and 410] that M is said to be *S-finite* if $sM \subseteq F$ for some $s \in S$ and some f.g. submodule F of M. Also, recall from Ref. [2] that M is called *S-Noetherian* if every submodule of M is an S-finite module. We say that R is *S-Noetherian* if it is S-Noetherian as a module over itself. That is, R is S-Noetherian if each ideal of R is S-finite. In Ref. [2],

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D.D. Anderson and T. Dumitrescu stated and proved S-variant of several well-known results on Noetherian rings to S-Noetherian rings (see [2, Corollaries 5, 7 and Propositions 9, 10]).

Let S be a m.c. subset of a ring R and let M be an R-module. In Ref. [1], Hamed Ahmed and Hizem Sana introduced the following definition in order to generalize some known results about Noetherian modules. An increasing sequence of submodules of M, $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ is said to be *S-stationary* if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_n \subseteq N_k$ for all $n \geq k$ [1, Definition 2.1]. A submodule N of M is said to be an *extended submodule* if there exists an ideal I of R such that $N = IM$. In [1, Theorem 2.1], it was shown that an S-finite R-module M is S-Noetherian if every extended submodule of M is S-finite. Also, in Ref. [1], the concept of S-accr modules and S-accr* modules were introduced and investigated. Recall from [1, Definition 3.1] that M is said to *satisfy S-accr* (respectively, *S-accr**) if for every submodule N of M and every f.g. (respectively, principal) ideal B of R, the increasing sequence of submodules of M, $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ is S-stationary. In Ref. [1], several results on (accr) modules were generalized to S-accr modules (see [1, Theorems 3.1, 3.2, and 3.3]). It was shown in [1, Proposition 3.1] that the properties S-accr and S-accr* are equivalent.

Let M be a module over a ring R. Recall from Ref. [5] that M is said to *satisfy strong accr** if for every submodule N of M and every sequence $< r_n >$ of elements of R, the increasing sequence of submodules of M, $(N :_M r_1) \subseteq (N :_M r_1 r_2) \subseteq (N :_M r_1 r_2 r_3) \subseteq \cdots$ terminates. Let S be a m.c. subset of R. We say that M *satisfies S-strong accr** if for every submodule N of M and every sequence $< r_n >$ of elements of R, the increasing sequence of submodules of M, $(N :_M r_1) \subseteq (N :_M r_1 r_2) \subseteq (N :_M r_1 r_2 r_3) \subseteq \cdots$ is S-stationary [5]. The ring R is said to *satisfy strong accr** (respectively, *S-strong accr**) if R satisfies strong accr* (respectively, S-strong accr*) as a module over itself. In Ref. [5], some basic properties of rings and modules satisfying S-strong accr* were proved.

Let M be a module over a ring R. Recall from [6, Exercise 23, p. 295] that M is said to be a *Laskerian R-module* if M is a f.g. R-module and any proper submodule of M is a finite intersection of primary submodules of M. R is said to be a *Laskerian ring* if R is Laskerian as an R-module.

Let N be a $\mathfrak{p}$-primary submodule of an R-module M. N is said to be *strongly primary* if $\mathfrak{p}^k M \subseteq N$ for some $k \geq 1$. Recall from [6, Exercise 28, p. 298] that an R-module M is said to be a *strongly Laskerian R-module* if M is a f.g. R-module and any proper submodule of M is a finite intersection of strongly primary submodules of M. R is said to be a *strongly Laskerian ring* if R is strongly Laskerian as an R-module.

Let S be a m.c. subset of a ring R. Inspired by the research work presented on S-prime ideals of R in Ref. [7], the concept of S-primary ideal of R was introduced and its properties were investigated in Ref. [8]. Recall from Ref. [8] that an ideal $\mathfrak{q}$ of R with $\mathfrak{q} \cap S = \emptyset$ is said to be an *S-primary* ideal of R if there exists $s \in S$ such that for all $a, b \in R$ with $ab \in \mathfrak{q}$, we have either $sa \in \mathfrak{q}$ or $sb \in \sqrt{\mathfrak{q}}$. An S-primary ideal $\mathfrak{q}$ is said to be *S-strongly primary* if there exist $s' \in S$ and $n \in \mathbb{N}$ such that $s'(\sqrt{\mathfrak{q}})^n \subseteq \mathfrak{q}$. (In Ref. [8], an S-strongly primary ideal of a ring was referred to as a strongly S-primary ideal.) Let I be an ideal of R such that $I \cap S = \emptyset$. We say that I is *S-decomposable* (respectively, *S-strongly decomposable*) if I can be expressed as a finite intersection of S-primary (respectively, S-strongly primary) ideals of R. Recall from Ref. [8] that R is said to be *S-Laskerian* (respectively, *S-strongly Laskerian*) if for any ideal I of R, either $I \cap S \neq \emptyset$ or $(I :_R s)$ is S-decomposable (respectively, S-strongly decomposable) for some $s \in S$ (In Ref. [8], an S-strongly Laskerian ring was referred to as a strongly S-Laskerian ring.). Let $f : R \to S^{-1}R$ denote the usual homomorphism of rings given by $f(r) = \frac{r}{1}$. For an ideal I of R, $f^{-1}(S^{-1}I)$ is called the *saturation of I with respect to S* and is denoted by $Sat_S(I)$ or by $S(I)$.

Let P be a property of rings. Whenever, a ring R is a subring of a ring T, we assume that R contains the identity element of T. We denote the collection of all intermediate rings between

R and T by $[R, T]$. We say that (R, T) is a P-pair if A satisfies P for each $A \in [R, T]$. For example, we say that (R, T) is an *accr pair* (respectively, *accr* pair*) if A satisfies (accr) (respectively, (accr*)) for each $A \in [R, T]$. It follows from [3, Theorem 1] that (R, T) is an accr pair if and only if $(R, T)$is an accr* pair. Let S be a m.c. subset of a ring R. We say that (R, T) is an *S-accr pair* (respectively, *S-accr* pair*) if A satisfies S-accr (respectively, S-accr*) for each $A \in [R, T]$. It follows from [1, Proposition 3.1] that (R, T) is an S-accr pair if and only if (R, T) is an S-accr* pair. We use the abbreviation ACCRP (respectively, ACCR*P) for accr pair (respectively, accr* pair). We use the abbreviation S-ACCRP (respectively, S-ACCR*P) for S-accr pair (respectively, S-accr* pair). Similarly, one can define the concept of strong accr* pair (respectively, S-strong accr* pair). We use the abbreviation SACCR*P (respectively, S-SACCR*P) for strong accr* pair (respectively, S-strong accr* pair). We use the abbreviation LP (respectively, SLP) for Laskerian pair (respectively, strongly Laskerian pair). We use the abbreviation S-LP (respectively, S-SLP) for S-Laskerian pair (respectively, S-strongly Laskerian pair). We use the abbreviation NP (respectively, S-NP) for Noetherian pair (respectively, S-Noetherian pair). We know from [8, Corollary 3.9(1)] that any S-Laskerian ring satisfies S-accr. Therefore, it follows that any S-LP is an S-ACCRP. We know from [8, Corollary 3.9(2)] that any S-strongly Laskerian ring satisfies S-strong accr*. Hence, we obtain that any S-SLP is an S-SACCR*P. Let R be a subring of a ring T. In Ref. [9] (respectively [10]), for certain pairs of rings $R \subseteq T$, necessary and sufficient conditions were determined in order that (R, T) to be an LP (respectively, ACCRP). A ring R is said to *satisfy ACCP* if every increasing sequence of principal ideals of R is stationary. Let S be a m.c. subset of an integral domain D. We say that D *satisfies S-ACCP* if every increasing sequence of principal ideals of D is S-stationary [11]. Let T be an integral domain which contains D as a subring. We say that (D, T) is an *S-ACCP pair* if A satisfies S-ACCP for each $A \in [D, T]$. In Ref. [11], for certain pairs of domains $D \subseteq T$, necessary and sufficient conditions were determined in order that (D, T) to be an S-ACCP pair, where S is a m.c. subset of D.

Let D be an integral domain and let S be a m.c. subset of D. Let K be a field which contains D as a subring. The aim of this article is to investigate the conditions under which (D, K) is an S-ACCRP (respectively, S-SACCR*P). In Section 2 of this article, we focus on determining necessary and sufficient conditions in order that (D, K) to be an S-ACCRP. Recall from [7, Definition 2] that a m.c. subset S of a ring R is said to be a *strongly multiplicatively closed* if for any given elements $(s_\alpha)_{\alpha\in\Lambda}$ of S, $\left(\bigcap_{\alpha\in\Lambda} Rs_\alpha\right) \cap S \neq \emptyset$ (equivalently, $\left(\bigcap_{s\in S} Rs\right) \cap S \neq \emptyset$). If S is a strongly m.c. subset of D such that $S^{-1}D$ is not a field, then it is proved in Theorem 2.12 that the statements (1) (D, K) is an S-ACCRP and (2) K is algebraic over D and the integral closure of $S^{-1}D$ in K is a one-dimensional Prüfer domain are equivalent. Let D, S, K be as in the statement of Theorem 2.12. It is shown in Corollary 2.14 that the statements (1) (D, K) is an S-LP and (2) K is algebraic over D and the integral closure of $S^{-1}D$ in K is a Laskerian Prüfer domain are equivalent. Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Let us denote $S^{-1}D$ by F. Let K be an extension field of F. If (D, K) is an S-ACCRP, then it is verified in Lemma 2.17 that $tr.\deg\ K/F \leq 1$, where $tr.\deg\ K/F$ denotes the transcendence degree of K over F. If K is algebraic over F, then it is shown in Proposition 2.19 that (D, K) is an S-SLP and hence, (D, K) is an S-SACCR*P. Several examples are given to illustrate the results proved in this section (see Examples 2.15, 2.16 and 2.20).

Let S be a m.c. subset of an integral domain D. Let K be a field which contains D as a subring. The aim of Section 3 of this article is to discuss some results regarding when (D, K) is an S-SACCR*P. Suppose that D is not a field. Let F denote the quotient field of D. If (D, K) is an SACCR*P, then it is proved in Theorem 3.3 that the following statements hold: (1) K is algebraic over D and the integral closure of D in K is a Dedekind domain and (2) The separable degree of K over F is finite and K has finite exponent over F. Suppose that $S^{-1}D$ is not a field.

If (D, K) is an S-SACCR*P, then it is deduced in Corollary 3.4 that the following statements hold: (1) K is algebraic over D and the integral closure of $S^{-1}D$ in K is a Dedekind domain and (2) The separable degree of K over F is finite and K has finite exponent over F. It is verified in Example 3.5 that the field L, an infinite algebraic extension field of $\mathbb{Q}$, constructed by R. Gilmer [12, Example, p. 520] is such that $(\mathbb{Z}, L)$ is an ACCRP but it is not an SACCR*P. Let S be a countable m.c. subset of an integral domain D such that $S^{-1}D$ is integrally closed but not a field. Let F be the quotient field of D with $char(F) = 0$ (where $char(F)$ denotes the characteristic of F). With the above hypotheses, it is proved in Corollary 3.6 that the following statements (1) (D, K) is an S- SLP; (2) (D, K) is an S-SACCR*P; (3) For any $T \in [D, K]$ and any ideal I of T, there exists $s \in S$ (depending on I) such that $S(I) = (I :_T s)$, K is a finite algebraic extension of F and moreover, $S^{-1}D$ and the integral closure of $S^{-1}D$ in K are Dedekind domains; and (4) (D, K) is an S-NP are equivalent. Let D be an integrally closed domain which is not a field. Let F be the quotient field of D with $char(F) = 0$. Let K be an extension field of F. With the above hypotheses, it is deduced in Corollary 3.7 that the statements (1) (D, K) is an SLP; (2) (D, K) is an SACCR*P; (3) $[K : F] < \infty$, and moreover D and the integral closure of D in K are Dedekind domains; and (4) (D, K) is an NP are equivalent. An integral domain T is provided in Example 3.8 such that the integral closure of T in its quotient field is a Dedekind domain but T does not satisfy strong accr*. Let S be a m.c. subset of an integral domain D. Suppose that $S^{-1}D = F$ is the quotient filed of D. Let K be an extension field of F. If $tr.\deg K/F = 1$ and if (D, K) is an S-SACCR*P, then it is deduced in Corollary 3.9 that the following statements hold. (1) For each $\alpha \in K$ such that α is transcendental over F, the integral closure of $F[\alpha]$ in K is a Dedekind domain and (2) The separable degree of K over $F(\alpha)$ is finite and K has finite exponent over $F(\alpha)$. If F is a perfect field and K is an extension field of F such that $tr.\deg K/F = 1$, then it is shown in Corollary 3.10 that the statements (1) (F, K) is an SLP; (2) (F, K) is an SACCR*P; and (3) (F, K) is an NP are equivalent.

For a ring R, we denote the set of all prime ideals of R by $Spec(R)$ and we denote the set of all maximal ideals of R by $Max(R)$. Whenever a set A is a subset of a set B and $A \neq B$, we denote it by $A \subset B$. For a ring R, we denote the group of units of R by $U(R)$ and we denote the set of all zero-divisors of R by $Z(R)$. The Krull dimension of a ring R is simply referred to as the dimension of R and is denoted by $\dim R$. For concepts and notations from commutative ring theory that are not specified in this article, the reader can refer standard text-books in commutative ring theory (for example [13, 14]).

2. When is (D, K) an S-ACCRP?

As mentioned in the introduction, the m.c. subsets considered in this article are assumed that they do not contain 0. Let S be a m.c. subset of an integral domain D. Let K be a field which contains D as a subring and K is not necessarily the quotient field of D. The aim of this section is to determine necessary and sufficient conditions in order that (D, K) to be an S-ACCRP. In Proposition 2.4, we determine a necessary condition for (D, K) to be an S-ACCRP, where D is an integral domain such that $S^{-1}D$ is not a field. We use Lemma 2.1 in the proof of Proposition 2.4. For a ring R, we denote the polynomial ring in one variable X over R by $R[X]$.

Lemma 2.1. *Let S be a m.c. subset of a ring R. Let r be a non-zero-divisor of R. Let $T = R + (1 + rX)R[X]$. If T satisfies S-accr, then $Rr \cap S \neq \emptyset$.*

Proof. We use some arguments similar to those that were used in the proof of [10, Proposition 1.3]. Let us denote the ideal $(1 + rX)T$ of T by I. By hypothesis, T satisfies S-accr. Hence, the increasing sequence of ideals of T, $(I :_T r) \subseteq (I :_T r^2) \subseteq (I :_T r^3) \subseteq \cdots$ is S-stationary. Therefore, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(I :_T r^n) \subseteq (I :_T r^k)$ for all $n \geq k$. In particular, $s(I :_T r^{k+1}) \subseteq (I :_T r^k)$. Notice that $(1 + rX)X^{k+1} \in T$ is such that $(1 + rX)X^{k+1} \in (I :_T r^{k+1})$. Hence, $s(1 + rX) X^{k+1} \in (I :_T r^k)$. This implies that $s(1 + rX)r^k X^{k+1} = (1 + rX)t$ for some $t \in T$. Since there is no non-zero $y \in R$ such that $y(1+rX)=0$, we obtain from [15, Theorem 2] that $1+rX$ is a non-zero-divisor of $R[X]$ and hence a non-zero divisor of T. Hence, it follows from $s(1+rX)r^kX^{k+1} = (1+rX)t$, we get that $sr^kX^{k+1} \in T$. Notice that $sr^{k-1}X^k = s(1+rX)r^{k-1}X^k - sr^kX^{k+1}$. From $(1+rX)R[X] \subset T$ and $sr^kX^{k+1} \in T$, we obtain that $sr^{k-1}X^k \in T$. This implies that $sX \in T$ if $k=1$. If $k \geq 2$, then from $sr^{k-2}X^{k-1} = s(1+rX)r^{k-2}X^{k-1} - sr^{k-1}X^k$, it follows that $sr^{k-2}X^{k-1} \in T$. Proceeding like this, we obtain that $sX \in T$. Hence, $sX = y + (1+rX)f(X)$ for some $y \in R$ and $f(X) \in R[X]$. It is clear that $f(X) \neq 0$. Since $r \notin Z(R)$ by hypothesis, we get that $\deg((1+rX)f(X)) = 1 + \deg(f(X))$. From $1 = \deg(sX) = \deg(y + (1+rX)f(X))$, it follows that $f(X) \in R$. By comparing the coefficient of X on both sides of $sX = y + (1+rX)f(X)$, it follows that $s = rf(X) \in Rr$. This proves that $s \in Rr \cap S$ and so, $Rr \cap S \neq \emptyset$. □

Corollary 2.2. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. Let K be a field such that D is a subring of K. If K is not algebraic over D, then (D, K) is not an S-ACCRP.*

Proof. By hypothesis, $S^{-1}D$ is not a field. Let $d \in D \backslash \{0\}$ and $s \in S$ be such that $\frac{d}{s} \notin U(S^{-1}D)$. As $S \subseteq U(S^{-1}D)$, it follows that $d \notin U(S^{-1}D)$. We are assuming that K is not algebraic over D. Let $\alpha \in K$ be such that α is transcendental over D. Notice that $T = D + (1+d\alpha)D[\alpha] \in [D, K]$. We claim that T does not satisfy S-accr. Suppose that T satisfies S-accr. Since d is a non-zero-divisor of D, we obtain from Lemma 2.1 that $Dd \cap S \neq \emptyset$. Therefore, there exist $s_1 \in S$ and $d_1 \in D$ such that $s_1 = dd_1$. This implies that $d \in U(S^{-1}D)$ and this is in contradiction to the choice of d. This proves that T does not satisfy S-accr and so, (D, K) is not an S-ACCRP. □

Lemma 2.3. *Let S be a m.c. subset of a valuation domain V. If V satisfies S-accr, then* dim $(S^{-1}V) \leq 1$.

Proof. If $S^{-1}V$ is a field, then it is clear that dim $(S^{-1}V) = 0 < 1$. Hence, we can assume that $S^{-1}V$ is not a field. Hence, dim $(S^{-1}V) \geq 1$. We are assuming that V satisfies S-accr. Suppose that dim $(S^{-1}V) > 1$. Then it follows that there exists $\mathfrak{P} \in Spec(S^{-1}V)$ such that $height\,\mathfrak{P} > 1$ in $S^{-1}V$. It follows from [13, Proposition 3.11(*iv*)] that there exist non-zero prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2$ of V with $\mathfrak{p}_2 \cap S = \emptyset$ and $\mathfrak{P} = S^{-1}\mathfrak{p}_2$. Let $x \in \mathfrak{p}_2 \backslash\ \mathfrak{p}_1$. Let $y \in \mathfrak{p}_1$ with $y \neq 0$. Let us denote the ideal Vy by I. Notice that for any $n \in \mathbb{N}$, $x^n \notin \mathfrak{p}_1$ and so, $x^n \notin Vy$. Since the set of ideals of V is linearly ordered by inclusion, we get that $y = x^n v_n$ for some non-unit v_n of V. As we are assuming that V satisfies S-accr, the increasing sequence of ideals of V, $(I :_V x) \subseteq (I :_V x^2) \subseteq (I :_V x^3) \subseteq \cdots$ is S-stationary. Hence, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(I :_V x^n) \subseteq (I :_V x^k)$ for all $n \geq k$. In particular, $s(I :_V x^{k+1}) \subseteq (I :_V x^k)$. It is clear that for any $n \in \mathbb{N}$, $(I :_V x^n) = (Vx^n v_n :_V x^n) = Vv_n$. From $y = x^n v_n \in \mathfrak{p}_1$ and $x^n \notin \mathfrak{p}_1$, we get that $v_n \in \mathfrak{p}_1$. From $s(I :_V x^{k+1}) \subseteq (I :_V x^k)$, we obtain that $sVv_{k+1} \subseteq Vv_k$. Hence, $sv_{k+1} = v_k w$ for some $w \in V$. It follows from $y = x^{k+1}v_{k+1} = x^k v_k$ that $v_k = xv_{k+1}$. From $sv_{k+1} = v_k w$, we get that $sv_{k+1} = xv_{k+1}w$ and this implies that $s = xw \in \mathfrak{p}_2$. This is a contradiction, since $\mathfrak{p}_2 \cap S = \emptyset$. Therefore, if V satisfies S-accr, then dim $(S^{-1}V) \leq 1$. □

Proposition 2.4. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. Let K be a field such that D is a subring of K. If (D, K) is an S-ACCRP, then K is algebraic over D and the integral closure of $S^{-1}D$ in K is a one-dimensional Prüfer domain.*

Proof. We are assuming that $(D.K)$ is an S-ACCRP. By hypothesis, $S^{-1}D$ is not a field. Hence, we obtain from Corollary 2.2 that K is algebraic over D. Let F be the quotient field of D. Then F is also the quotient field of $S^{-1}D$. Let $T \in [S^{-1}D, F]$ with $T \neq F$. It is clear that dim $T \geq 1$. If dim $T > 1$, then we obtain from [12, Corollary 19.7(1)] that there exists a valuation domain $V \in [S^{-1}D, F]$ such that dim $V > 1$. As $S^{-1}D \subseteq V$, it follows that $V = S^{-1}V$. By assumption, V satisfies S-accr. Hence, we obtain from Lemma 2.3 that dim $V =$ dim $(S^{-1}V) \leq 1$. This is a contradiction and so, dim $T \leq 1$. Therefore, dim $T = 1$ for each $T \in [S^{-1}D, F]$ with $T \neq F$. Hence, we obtain from [16, Theorem 6] that the integral closure of $S^{-1}D$ in F is a one-dimensional Prüfer domain. Since the field K is an algebraic extension of the field F, using [12, Theorem 22.3], it can be shown as in the proof of [9, Proposition 2.1] that the integral closure of $S^{-1}D$ in K is a one-dimensional Prüfer domain. □

Proposition 2.5. *Let S be a m.c. subset of a ring R. Let I be an ideal of R with $I \cap S = \emptyset$. The following statements are equivalent:*

(1) *There exist $s \in S$ and an ideal J of R such that J is S-decomposable (respectively, S-strongly decomposable) and $sI \subseteq J \subseteq I$.*

(2) *The ideal $S^{-1}I$ of $S^{-1}R$ admits a primary (respectively, strong primary) decomposition in $S^{-1}R$ and there exist $s' \in S$ and primary (respectively, strongly primary) ideals $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_n$ of R such that $S(I) = (I{:}_R s') = \bigcap_{i=1}^{n} \mathfrak{Q}_i$ with $\mathfrak{Q}_i \cap S = \emptyset$ for each $i \in \{1, \ldots, n\}$.*

Proof. (1) ⇒ (2) We are assuming that there exist $s \in S$ and an ideal J of R such that J is S-decomposable (respectively, S-strongly decomposable) and $sI \subseteq J \subseteq I$. Thus, there exist $n \in \mathbb{N}$ and S-primary (respectively, S-strongly primary) ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of R such that $J = \bigcap_{i=1}^{n} \mathfrak{q}_i$. Let $i \in \{1, \ldots, n\}$. It follows from (1) ⇒ (2) of [8, Proposition 2.6] that $S^{-1}\mathfrak{q}_i$ is a primary (respectively, strongly primary) ideal of $S^{-1}R$ and there exists $t_i \in S$ such that $S(\mathfrak{q}_i) = (\mathfrak{q}_i{:}_R t_i)$. It follows from [13, Proposition 3.11(v)] that $S^{-1}J = \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i$. This proves that $S^{-1}J$ admits a primary (respectively. strong primary) decomposition in $S^{-1}R$. As $\frac{s}{1} \in U(S^{-1}R)$, we obtain from $sI \subseteq J \subseteq I$ that $S^{-1}I = S^{-1}J$. Therefore, we get that the ideal $S^{-1}I$ of $S^{-1}R$ admits a primary (respectively, strong primary) decomposition in $S^{-1}R$. Let $t = \prod_{i=1}^{n} t_i$. Then $t \in S$. Let $i \in \{1, \ldots, n\}$. Notice that $(\mathfrak{q}_i{:}_R t_i)$ is a primary (respectively strongly primary) ideal of R with $\sqrt{(\mathfrak{q}_i{:}_R t_i)} \cap S = \emptyset$. It is clear that $t = t_i(\prod_{j \in \{1,\ldots,n\} \setminus \{i\}} t_j)$. Hence, we obtain from [13, Lemma 4.4 (iii)] that $(\mathfrak{q}_i{:}_R t_i) = (\mathfrak{q}_i{:}_R t)$. This shows that $S(\mathfrak{q}_i) = (\mathfrak{q}_i{:}_R t)$ is a primary (respectively. strongly primary) ideal of R. From $J = \bigcap_{i=1}^{n} \mathfrak{q}_i$, it follows that $S(J) = \bigcap_{i=1}^{n} S(\mathfrak{q}_i) = \bigcap_{i=1}^{n} (\mathfrak{q}_i{:}_R t) = ((\bigcap_{i=1}^{n} \mathfrak{q}_i){:}_R t) = (J{:}_R t)$. Let $i \in \{1, \ldots, n\}$. As $s \notin \sqrt{(\mathfrak{q}_i{:}_R t)}$, it follows from [13, Lemma 4.4 (iii)] that $(\mathfrak{q}_i{:}_R t) = (\mathfrak{q}_i{:}_R st)$. Therefore, we obtain that $S(J) = (J{:}_R st)$. From $S^{-1}I = S^{-1}J$ and $J \subseteq I$, we get that $S(I) = S(J) = (J{:}_R st) \subseteq (I{:}_R st) \subseteq S(I)$. Thus with $s' = st$ and $\mathfrak{Q}_i = (\mathfrak{q}_i{:}_R s')$ for each $i \in \{1, \ldots, n\}$, we obtain that $s' \in S$ and $S(I) = (I{:}_R s') = \bigcap_{i=1}^{n} \mathfrak{Q}_i$ is a primary (respectively. strong primary) decomposition of $S(I)$ in R with $\mathfrak{Q}_i \cap S = \emptyset$ for each $i \in \{1, \ldots, n\}$.

(2) ⇒ (1) By assumption, there exist $s' \in S$ and primary (respectively strongly primary) ideals $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_n$ of R such that $S(I) = (I{:}_R s') = \bigcap_{i=1}^{n} \mathfrak{Q}_i$ with $\mathfrak{Q}_i \cap S = \emptyset$ for each $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. Let us denote $s'\mathfrak{Q}_i$ by $\mathfrak{q}_i$. It is clear that $\mathfrak{q}_i \cap S = \emptyset$. As $\mathfrak{Q}_i$ is a primary ideal of R with $\sqrt{\mathfrak{Q}_i} \cap S = \emptyset$, it follows that $(s'\mathfrak{Q}_i{:}_R s') = \mathfrak{Q}_i$. Hence, we obtain from (2) ⇒ (1) of [8, Proposition 2.4] that $\mathfrak{q}_i$ is an S-primary (respectively, S-strongly primary) ideal of R. Let us denote $s'(I{:}_R s')$ by J. Since $\mathfrak{Q}_i$ is a primary ideal of R and $\sqrt{\mathfrak{Q}_i} \cap S = \emptyset$, we obtain that $J = s'(I{:}_R s') = s'(\bigcap_{i=1}^{n} \mathfrak{Q}_i) = \bigcap_{i=1}^{n} s'\mathfrak{Q}_i = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is S-decomposable (respectively. S-strongly decomposable). Notice that $s'I \subseteq s'(I{:}_R s') = J \subseteq I$. □

Corollary 2.6. *Let S be a m.c. subset of a ring R. The following statements are equivalent:*

(1) *R is S -Laskerian (respectively, S -strongly Laskerian).*

(2) *Given an ideal I of R with $I \cap S = \emptyset$, there exist $s \in S$ and an ideal J of R such that J is S -decomposable (respectively,* S-strongly decomposable) with $sI \subseteq J \subseteq I$.

(3) $S^{-1}R$ is Laskerian (respectively. strongly Laskerian) and for any ideal I of R with $I \cap S = \emptyset$, there exists $s' \in S$ such that $S(I) = (I :_R s')$.

Proof. (1) ⇔ (3) This is (1) ⇔ (2) of [8, Proposition 3.2].

(2) ⇒ (3) Let A be any proper ideal of $S^{-1}R$. Then it follows from [13, Proposition 3.11 (*i*) and (*ii*)] that there exists an ideal I of R such that $I \cap S = \emptyset$ and $A = S^{-1}I$. By (2), there exist $s \in S$ and an ideal J of R such that J is S-decomposable (respectively. S-strongly decomposable) with $sI \subseteq J \subseteq I$. Hence, we obtain from (1) ⇒ (2) of Proposition 2.5 that $A = S^{-1}I$ admits a primary (respectively, strong primary) decomposition in $S^{-1}R$ and there exists $s' \in S$ such that $S(I) = (I :_R s')$ admits a primary (respectively, strong primary) decomposition in R. This proves that $S^{-1}R$ is Laskerian (respectively, strongly Laskerian) and given an ideal I of R with $I \cap S = \emptyset$, there exists $s' \in S$ such that $S(I) = (I :_R s')$.

(3) ⇒ (2) Let I be an ideal of R with $I \cap S = \emptyset$. It was shown in the proof of (2) ⇒ (1) of [8, Proposition 3.2] that there exist $n \geq 1$ and primary (respectively, strongly primary) ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of R such that $S(I) = (I :_R s') = \bigcap_{i=1}^{n} \mathfrak{q}_i$ with $\mathfrak{q}_i \cap S = \emptyset$ for each $i \in \{1, \ldots, n\}$. Let $J = s'(I :_R s')$. Then it is already verified in the proof of (2) ⇒ (1) of Proposition 2.5 that J is S-decomposable (respectively, S-strongly decomposable) and $s'I \subseteq J \subseteq I$. □

Recall from Ref. [7] that a m.c. subset S of a ring R is said to be *strongly multiplicatively closed* if $\left(\bigcap_{s \in S} Rs\right) \cap S \neq \emptyset$. In Ref. [17], strongly multiplicatively closed subsets are referred to as m.c. subsets satisfying *maximal multiple condition*. Let S be a strongly m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. In Theorem 2.12, we provide a necessary and sufficient condition in order that (D, K) to be an S-ACCRP, where K is a field which contains D as a subring.

Lemma 2.7. *Let S be a strongly m.c. subset of a ring R. Then there exists $s \in S$ such that $S(I) = (I :_R s)$ for any ideal I of R.*

Proof. By hypothesis, S is a strongly m.c. subset of R. Hence, there exists $s \in S$ such that $s \in Rt$ for all $t \in S$. Let I be any ideal of R. It is clear that $(I :_R s) \subseteq S(I)$. Let $r \in S(I)$. Then there exists $t \in S$ such that $tr \in I$. As $s \in Rt$, it follows that $sr \in I$. This shows that $S(I) \subseteq (I :_R s)$ and so, $S(I) = (I :_R s)$. □

Lemma 2.8. *Let S be a m.c. subset of a ring R. If $S^{-1}R$ satisfies (accr) and if for any ideal I of R, there exists $s \in S$ (depending on I) such that $S(I) = (I :_R s)$, then R satisfies S-accr.*

Proof. It can be proved using arguments similar to those that were used in the proof of [5, Lemma 2.6] that R satisfies S-accr*. We know from [1, Proposition 3.1] that the properties S-accr and S-accr* are equivalent. Therefore, we obtain that R satisfies S-accr. □

Lemma 2.9. *Let S be a m.c. subset of a ring R. If $S^{-1}R$ is Laskerian (respectively, strongly Laskerian) and if for any ideal I of R, there exists $s \in S$ (depending on I) such that $S(I) = (I :_R s)$, then R is S-Laskerian (respectively, S-strongly Laskerian).*

Proof. This follows from (3) ⇒ (1) of Corollary 2.6. □

Corollary 2.10. *Let S be a strongly m.c. subset of a ring R. The following statements are equivalent:*

(1) *R satisfies S-accr.*

(2) *$S^{-1}R$ satisfies (accr).*

Proof. (1) ⇒ (2) We are assuming that R satisfies S-accr. If a ring T satisfies M-accr, where M is any m.c. subset of T, then we know from [1, Example 3.1(3)] that $M^{-1}T$ satisfies (accr*). Hence, we get that $S^{-1}R$ satisfies (accr*). Since the properties (accr) and (accr*) are equivalent by [3, Theorem 1], we obtain that $S^{-1}R$ satisfies (accr).

(2) ⇒ (1) We are assuming that $S^{-1}R$ satisfies (accr). By hypothesis, S is a strongly m.c. subset of R. Hence, we obtain from Lemma 2.7 that there exists $s \in S$ such that $S(I) = (I :_R s)$ for all ideals I of R. It now follows from Lemma 2.8 that R satisfies S-accr. □

Corollary 2.11. *Let S be a strongly m.c. subset of a ring R. The following statements are equivalent:*

1. *R is S-Laskerian (respectively, S-strongly Laskerian).*
2. *$S^{-1}R$ is Laskerian (respectively, strongly Laskerian).*

Proof. (1) ⇒ (2) This follows from (1) ⇒ (3) of Corollary 2.6.

(2) ⇒ (1) With the help of Lemmas 2.7 and 2.9, this can be proved as in the proof of (2) ⇒ (1) of Corollary 2.10. □

Theorem 2.12. *Let S be a strongly m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. Let K be a field which contains D as a subring. The following statements are equivalent:*

1. *(D, K) is an S-ACCRP.*
2. *K is algebraic over D and the integral closure of $S^{-1}D$ in K is a one-dimensional Prüfer domain.*

Proof. (1) ⇒ (2) This follows from Proposition 2.4. The proof of (1) ⇒ (2) does not need the assumption that S is a strongly m.c. subset of D.

(2) ⇒ (1) Let $T \in [D, K]$. We claim that $S^{-1}T$ satisfies (accr). This is clear if $S^{-1}T$ is a field. Hence, we can assume that $S^{-1}T$ is not a field. Therefore, $\dim (S^{-1}T) \geq 1$. Let D_1 denote the integral closure of $S^{-1}D$ in K and let D_2 denote the integral closure of $S^{-1}T$ in K. As $S^{-1}D \subseteq S^{-1}T$, it is clear that D_1 is a subring of D_2. From K is algebraic over D, we obtain that K is the quotient field of D_i for each $i \in \{1, 2\}$. Thus D_2 is an overring of D_1. By assumption, D_1 is a one-dimensional Prüfer domain. Hence, we obtain from [12, Theorem 26.1(1)] that D_2 is a Prüfer domain and $\dim D_2 \leq \dim D_1 = 1$. As D_2 is integral over $S^{-1}T$, we obtain from [12, 11.8] that $\dim (S^{-1}T) = \dim D_2$. From $\dim (S^{-1}T) \geq 1$, we get that $\dim D_2 \geq 1$ and so, $\dim D_2 = 1$. Therefore, it follows that $\dim (S^{-1}T) = 1$. As $S^{-1}T$ is a one-dimensional integral domain, we obtain from [3, Theorem 6] that $S^{-1}T$ satisfies (accr). Since S is a strongly m.c. subset of D, it follows that S is a strongly m.c.subset of T. As $S^{-1}T$ satisfies (accr), we obtain from (2) ⇒ (1) of Corollary 2.10 that T satisfies S-accr. This proves that (D, K) is an S-ACCRP. □

Let D be an integral domain which is not a field. It is clear that $S = U(D)$ is a strongly m.c. subset of D. Let K be a field which contains D as a subring. Notice that $S^{-1}D = D$ and if $T \in [D, K]$, then T satisfies S-accr if and only if T satisfies (accr). Hence, the following Corollary 2.13 is an immediate consequence of Theorem 2.12.

Corollary 2.13. [10, Proposition 2.1] *Let D be an integral domain which is not a field. Let K be a field which contains D as a subring. The following statements are equivalent:*

1. *(D, K) is an ACCRP.*
2. *K is algebraic over D and the integral closure of D* in *K is a one-dimensional Prüfer domain.*

Corollary 2.14. *Let S be a strongly m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. Let K be a field which contains D as a subring. The following statements are equivalent:*

(1) *(D, K) is an S-LP.*

(2) *K is algebraic over D and the integral closure of $S^{-1}D$ in K is a Laskerian Prüfer domain.*

Proof. (1) ⇒ (2) Let $T \in [D, K]$. By hypothesis, T is S-Laskerian. Hence, we obtain from [8, Corollary 3.9(1)] that T satisfies S-accr* and so, we obtain from [1, Proposition 3.1] that T satisfies S-accr. This shows that (D, K) is an S-ACCRP. Therefore, it follows from (1) ⇒ (2) of Theorem 2.12 that K is algebraic over D and the integral closure of $S^{-1}D$ in K is a one-dimensional Prüfer domain. Moreover, for any $T \in [D, K]$, T is S-Laskerian and so, we obtain from (1) ⇒ (3) of Corollary 2.6 that $S^{-1}T$ is Laskerian. If $A \in [S^{-1}D, K]$, then $A = S^{-1}T$ for some $T \in [D, K]$. Hence, we get that A is Laskerian and so, $(S^{-1}D, K)$ is an LP. Therefore, we obtain that the integral closure of $S^{-1}D$ in K is a Laskerian Prüfer domain. Notice that the proof of (1) ⇒ (2) of this proposition does not need the assumption that the m.c. subset S is strongly multiplicatively closed.

(2) ⇒ (1) Let $\overline{S^{-1}D}$ denote the integral closure of $S^{-1}D$ in K. By (2), K is algebraic over D and $\overline{S^{-1}D}$ is a Laskerian Prüfer domain. By hypothesis, $S^{-1}D$ is not a field. Hence, we obtain from the if part of [9, Proposition 2.1] that $(S^{-1}D, K)$ is an LP. Let $T \in [D, K]$. Then $S^{-1}T$ is Laskerian. As S is a strongly m.c. subset of D, we get that S is a strongly m.c. subset of T. Hence, we obtain from (2) ⇒ (1) of Corollary 2.11 that T is S-Laskerian. This proves that (D, K) is an S-LP. □

Example 2.15. Let $\{p_i\}_{i=1}^{\infty}$ be the sequence of positive primes of $\mathbb{Z}$. In [12, Example 42.6] R. Gilmer constructed an infinite algebraic extension F of $\mathbb{Q}$ such that $\mathbb{Z}^*$, the integral closure of $\mathbb{Z}$ in F, is such that $\mathbb{Z}^*$ is an almost Dedekind domain with the property that p_1 belongs to infinitely many maximal ideals of $\mathbb{Z}^*$. Notice that $\dim \mathbb{Z}^* = 1$ and as $\mathbb{Z}^* p_1$ admits an infinite number of prime ideals minimal over it, we get that $\mathbb{Z}^*$ is not Laskerian. Hence, $(\mathbb{Z}, F)$ is not an LP. Since F is algebraic over $\mathbb{Z}$ and the integral closure of $\mathbb{Z}$ in F is a one-dimensional Prüfer domain, we obtain from (2) ⇒ (1) of Corollary 2.13 that $(\mathbb{Z}, F)$ is an ACCRP.

□

In Example 2.16, we provide an example of a domain T and a m.c. subset S of T such that $S^{-1}T$ is a one-dimensional valuation domain but (T, L) is not an S-ACCRP (where L is the quotient field of T) thereby illustrating that (2) ⇒ (1) of Theorem 2.12 can fail to hold if the hypothesis in Theorem 2.12 that S is a strongly m.c. subset is omitted.

Example 2.16. Let K be a field and let $K(X)$ be the field of rational functions in one variable X over K. Let $V = K(X)[[Y]]$ be the power series in one variable Y over $K(X)$. Let $\mathfrak{m} = VY$. Let $D = K[X]_{XK[X]}$ and let $T = D + \mathfrak{m}$. Let $S = T \setminus \mathfrak{m}$. Then S is a m.c. subset of T and (T, L) is not an S-ACCRP, where L is the quotient field of T.

Proof. It is well-known that $V = K(X)[[Y]]$ is a discrete valuation ring (for example, refer [18, p. 322]). Notice that $\mathfrak{m} = VY$ is the only non-zero prime ideal of V. It is clear that $V = K(X) + \mathfrak{m}$. We know from [13, Example (2), p. 94] that $D = K[X]_{XK[X]}$ is a discrete valuation ring. Since D is a valuation domain with quotient field $K(X)$, it follows from [19, Theorem 2.1(*h*)] that $T = D + \mathfrak{m}$ is a valuation domain and as $\dim D = \dim V = 1$, we obtain from [19, Theorem 2.1(*f*)] that $\dim T = 2$. Since $\frac{T}{\mathfrak{m}} \cong D$ as rings and D is an integral domain, we get that $\mathfrak{m} \in Spec(T)$. Let $S = T \setminus \mathfrak{m}$. From $\mathfrak{m} \in Spec(T)$, it follows that S is a m.c. subset of T. Notice that $S^{-1}T = T_{\mathfrak{m}}$. We claim that $T_{\mathfrak{m}} = V$. It is clear that $T_{\mathfrak{m}} \subseteq V$. Let $v \in V$. As $V = K(X) + \mathfrak{m}$ and $K(X)$ is the quotient field of D, it follows that $v = \frac{a}{b} + m$ for some $a, b \in D$ with $b \neq 0$ and $m \in \mathfrak{m}$. This implies that $v = \frac{a+bm}{b}$. From $a + bm \in T$ and $b \in T \setminus \mathfrak{m}$, we get that

$v \in T_{\mathfrak{m}}$. This proves that $V \subseteq T_{\mathfrak{m}}$ and so, $V = T_{\mathfrak{m}}$. Hence, $S^{-1}T$ is a one-dimensional valuation domain. We know from [13, Proposition 5.18 (*iii*)] that $S^{-1}T$ is integrally closed. Thus if L is the quotient field of T, then $S^{-1}T$ is the integral closure of $S^{-1}T$ in L and it is a one-dimensional valuation domain. We claim that T does not satisfy S-accr. Suppose that T satisfies S-accr. Let us denote the ideal TY by I. As T satisfies S-accr by assumption, the increasing sequence of ideals of T, $(I :_T X) \subseteq (I :_T X^2) \subseteq (I :_T X^3) \subseteq \cdots$ is S-stationary. Hence, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(I :_T X^n) \subseteq (I :_T X^k)$ for all $n \geq k$. Let $r \in \mathbb{N}$. Notice that $Y = \frac{Y}{X^r} X^r$ and hence, it follows that $(I :_T X^r) = \left(T\left(\frac{Y}{X^r}\right)X^r :_T X^r\right) = T\frac{Y}{X^r}$. From $s(I :_T X^n) \subseteq (I :_T X^k)$ for all $n \geq k$, we obtain that $sT\frac{Y}{X^n} \subseteq T\frac{Y}{X^k}$. This implies that $s \in \bigcap_{r=1}^{\infty} TX^r$. It is clear that $\bigcap_{r=1}^{\infty} TX^r = (\bigcap_{r=1}^{\infty} DX^r) + \mathfrak{m}$. Since D is a discrete valuation ring, we get that $\bigcap_{r=1}^{\infty} DX^r = (0)$. Hence, we obtain that $s \in \bigcap_{r=1}^{\infty} TX^r = \mathfrak{m}$. This is impossible, since $S = T \backslash \mathfrak{m}$. Therefore, we obtain that T does not satisfy S-accr and so, (T, L) is not an S-ACCRP. □

Let F be a field and let K be an extension field of F. Let S be a m.c. subset of F. As each element of S is a unit in F, it follows that (F, K) is an S-ACCRP (respectively, S-LP) if and only if (F, K) is an ACCRP (respectively, LP). The reader can refer [10, p. 320] (respectively, [9, pp. 94 and 95]) for the solution to the problem of when (F, K) is an ACCRP (respectively, LP).

Let K be an extension field of a field F. We denote the transcendence degree of K over F by the notation $tr.\deg\ K/F$.

Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Then $S^{-1}D$ is necessarily the quotient field of D. Let K be a field which contains D as a subring. Let us denote $S^{-1}D$ by F. It is clear that K is an extension field of F. If (D, K) is an S-ACCRP, then we verify in Lemma 2.17 that $tr.\deg\ K/F \leq 1$. If K is algebraic over F, then we verify in Proposition 2.19 that (D, K) is an S-SACCR*P. We use Lemma 2.18 in the proof of Proposition 2.19.

Lemma 2.17. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Let K be an extension field of $S^{-1}D$. If (D, K) is an S-ACCRP, then $tr.\deg\ K/S^{-1}D \leq 1$. In particular, if (D, K) is an S-LP, then $tr.\deg\ K/S^{-1}D \leq 1$.*

Proof. Let us denote $S^{-1}D$ by F. Then F is necessarily the quotient field of D. Suppose that $tr.\deg\ K/F > 1$. Then it is possible to find $X, Y \in K$ such that $\{X, Y\}$ is algebraically independent over F. Observe that S is a m.c. subset of $F[X]$ and $S \subseteq U(F) = U(F[X])$. As (D, K) is an S-ACCRP by hypothesis, it follows that $(F[X], F(X, Y))$ is an S-ACCRP. From $S \subseteq U(F[X])$, we get that $(F[X], F(X, Y))$ is an ACCRP. Since $F[X]$ is not a field, we obtain from (1) ⇒ (2) of Corollary 2.13 that $F(X, Y)$ is algebraic over $F[X]$. This is a contradiction and so, we obtain that $tr.\deg\ K/S^{-1}D \leq 1$.

Assume that (D, K) is an S-LP. We know from [8, Corollary 3.9(1)] and [1, Proposition 3.1] that any S-Laskerian ring satisfies S-accr. Hence, (D, K) is an S-ACCRP and therefore, we obtain that $tr.\deg\ K/S^{-1}D \leq 1$.

□

Lemma 2.18. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Then D is S-strongly Laskerian and so, D satisfies S- strong accr*.*

Proof. As $S^{-1}D$ is a field, we get that $S^{-1}D$ is strongly Laskerian. Let I be any ideal of D with $I \cap S = \emptyset$. Then $S^{-1}I = (0)$ and so, $I = (0)$. It is clear that $S(I) = (0) = ((0) :_D s)$ for any $s \in S$. Hence, we obtain from (3) ⇒ (1) of Corollary 2.6 that D is S-strongly Laskerian. We know from [8, Corollary 3.9(2)] that any S-strongly Laskerian ring satisfies S-strong accr*. Hence, we obtain that D satisfies S-strong accr*. □

Proposition 2.19. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Let K be an extension field of $S^{-1}D$ such that K is algebraic over D. Then (D, K) is an S-SLP and so, (D, K) is an* S-SACCR*P.

Proof. Let $T \in [D, K]$. Then S is a m.c. subset of T. If T is a field, then it is clear that T is S-strongly Laskerian. Suppose that T is not a field. By hypothesis, K is algebraic over D and as $S^{-1}D$ is a field, it follows that K is integral over $S^{-1}D$. Notice that $S^{-1}T \in [S^{-1}D, K]$. As $S^{-1}T$ is integral over $S^{-1}D$, we obtain from [13, Proposition 5.7] that $S^{-1}T$ is a field. It now follows from Lemma 2.18 that T is S-strongly Laskerian. This proves that (D, K) is an S-SLP. As any S-strongly Laskerian ring satisfies S-strong accr* by [8, Corollary 3.9(2)], we obtain that (D, K) is an S-SACCR*P. □

In Example 2.20, we provide an example of a domain T and a m.c. subset S of T such that (T, L) is an S-SACCR*P but (T, L) is not an ACCRP, where L is the quotient field of T.

Example 2.20. Let $V, T, \mathfrak{m}$ be as in the statement of Example 2.16. Let $S = \{Y^n | n \in \mathbb{N} \cup \{0\}\}$. Then S is a m.c. subset of T, (T, L) is an S-SACCR*P but (T, L) is not an ACCRP, where L is the quotient field of T.

Proof. In the notation of Example 2.16, $\mathfrak{m} = VY$ and $\mathfrak{m}$ is the only non-zero prime ideal of V. The integral domain $T = D + \mathfrak{m}$ is such that $\mathfrak{m}$ is an ideal of both T and V. Now, $S = \{Y^n | n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of T. Notice that $S^{-1}T = S^{-1}V = L$, where L is the quotient field of T. It now follows from Proposition 2.19 that (T, L) is an S-SLP and so, (T, L) is an S-SACCR*P. It is clear that if a ring satisfies S-strong accr*, then it satisfies S-accr*. We know from [1, Proposition 3.1] that the properties S-accr and S-accr* are equivalent. Therefore, we obtain that (T, L) is an S-ACCRP. As T is a two-dimensional valuation domain, it follows from Lemma 2.3 that T does not satisfy (accr). Indeed, it is already observed in the proof of Example 2.16 that T does not satisfy S_1-accr, where $S_1 = T \backslash \mathfrak{m}$. Thus (T, L) is an S-SACCR*P but (T, L) is not an ACCRP. □

Proposition 2.21. *Let S be a strongly m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Let us denote $S^{-1}D$ by F. Let K be an extension field of F such that $tr.\deg K/F = 1$. Then the following statements are equivalent:*

1. *(D, K) is an S-ACCRP.*
2. *For each $\alpha \in K$ such that α is transcendental over F, the integral closure of $F[\alpha]$ in K is a one-dimensional Prüfer domain.*

Proof. (1) ⇒ (2) We are assuming that $(D.K)$ is an S-ACCRP. Let $\alpha \in K$ be such that α is transcendental over F. Notice that $F[\alpha]$ is not a field. Now, $(F[\alpha], K)$ is an S-ACCRP. As $S \subseteq U(F[\alpha])$, it follows that $(F[\alpha], K)$ is indeed an ACCRP. Therefore, we obtain from (1) ⇒ (2) of Corollary 2.13 that the integral closure of $F[\alpha]$ in K is a one-dimensional Prüfer domain. It is clear that (1) ⇒ (2) of this proposition does not need the assumption that the m.c. subset S of D is a strongly m.c. subset of D.

(2) ⇒ (1) Let $T \in [D, K]$. Let L denote the quotient field of T. By hypothesis, $tr.\deg K/F = 1$. Hence, either L is algebraic over F or $tr.\deg L/F = 1$. If L is algebraic over F, then it follows from Proposition 2.19 that (D, L) is an S-SLP and as $T \in [D, L]$, we get that T satisfies S-strong accr* and so, T satisfies S-accr. Suppose that $tr.\deg L/F = 1$. Let $t \in T$ be such that t is transcendental over F. Notice that $T \in [D[t], K]$. It is clear that S is a strongly m.c. subset of $D[t]$ and $S^{-1}(D[t]) = F[t]$ is not a field. From $tr.\deg K/F = 1$, it follows that K is algebraic over $D[t]$. By assumption, the integral closure of $S^{-1}(D[t])$ in K is a one-dimensional Prüfer domain. Hence, we obtain from (2) ⇒ (1) of Theorem 2.12 that $(D[t], K)$ is an S-ACCRP. Since $T \in [D[t], K]$, we get that T satisfies S-accr. This proves that $(D.K)$ is an S-ACCRP. □

Proceeding as in the proof of Proposition 2.21, the following Proposition 2.22 can be proved with the help of Corollary 2.14 and Proposition 2.19.

Proposition 2.22. *Let S be a strongly m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Let us denote the field $S^{-1}D$ by F. Let K be an extension field of F such that* $tr.\deg K/F = 1$. *The following statements are equivalent:*

(1) *(D, K) is an S-LP.*

(2) *For each $\alpha \in K$ such that α is transcendental over F, the integral closure of $F[\alpha]$ in K is a Laskerian Prüfer domain.*

3. When is (D, K) an S-SACCR*P?

Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. Let K be a field which contains D as a subring. The aim of this section is to determine when (D, K) is an S-SACCR*P. Let S be a m.c. subset of a ring R such that R satisfies S-strong accr*. Then it is clear that R satisfies S-accr* and hence, it follows from [1, Proposition 3.1] that R satisfies S-accr. Thus if (D, K) is an S-SACCR*P, then (D, K) is an S-ACCRP. In Corollary 3.4, we determine some necessary conditions for (D, K) to be an S-SACCR*P. First, we state and prove some preliminary results that are useful for solving some of the problems considered in this section.

Let R be a ring. It is well-known that the set of all nilpotent elements of R forms an ideal of R and is called the *nilradical* of R [13, Proposition 1.7]. We denote the nilradical of R by $nil(R)$. Recall from [20, p. 466] that a sequence $< x_n >$ of elements of R is said to be *T-nilpotent* if there exists $k \in \mathbb{N}$ such that $\prod_{i=1}^{k} x_i = 0$.

Lemma 3.1. *Let S be a m.c. subset of a ring R. If R satisfies S-strong accr*, then for any ideal I of R and for any sequence $< x_n >$ of elements of $\sqrt{I}$, there exist $s \in S$ and $k \in \mathbb{N}$ (depending on I and the sequence $< x_n >$) such that $s\prod_{j=1}^{n} x_j \in I$ for all $n \geq k$.*

Proof. Let I be any ideal of R and let $< x_n >$ be any sequence of elements of $\sqrt{I}$. We consider the following cases.

Case(1): $I \cap S \neq \emptyset$.

Let $s \in I \cap S$. Then $s\prod_{j=1}^{n} x_j \in I$ for all $n \geq 1$.

Case(2): $I \cap S = \emptyset$.

As $I \cap S = \emptyset$, we obtain from [13, Proposition 3.11 (*ii*)] that $S^{-1}I \neq S^{-1}R$. As R satisfies S-strong accr* by hypothesis, it follows from [5, Lemma 2.4] that $S^{-1}R$ satisfies strong accr*. That is, $S^{-1}R$ satisfies (C) in the notation of [21]. From $S^{-1}R$ satisfies (C), it follows that $\frac{S^{-1}R}{S^{-1}I}$ satisfies (C). Hence, we obtain from [21, Lemma 1.2] that $nil\left(\frac{S^{-1}R}{S^{-1}I}\right)$ is T-nilpotent. Notice that $nil\left(\frac{S^{-1}R}{S^{-1}I}\right) = \frac{\sqrt{S^{-1}I}}{S^{-1}I}$. We know from [13, Proposition 3.11 (v)] that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$. It is clear that for each $n \in \mathbb{N}$, $\frac{x_n}{1} + S^{-1}I \in nil\left(\frac{S^{-1}R}{S^{-1}I}\right)$. From $nil\left(\frac{S^{-1}R}{S^{-1}I}\right)$ is T-nilpotent, we get that there exists $k \in \mathbb{N}$ such that $\prod_{j=1}^{k}\left(\frac{x_j}{1} + S^{-1}I\right) = 0 + S^{-1}I$. This implies that $\prod_{j=1}^{k}\frac{x_j}{1} \in S^{-1}I$. Hence, there exists $s \in S$ such that $s\prod_{j=1}^{k} x_j \in I$. Let $n \geq k$. Then it is clear that $s\prod_{j=1}^{n} x_j \in I$. □

Let S be a m.c. subset of a ring R such that R is S-Laskerian. In Lemma 3.2, we determine necessary and sufficient conditions in order that R to satisfy S-strong accr*.

Lemma 3.2. *Let S be a m.c. subset of a ring R such that R is S-Laskerian. The following statements are equivalent:*

(1) *R satisfies S-strong accr*.*

(2) *For any ideal I of R and for any sequence $< x_n >$ of elements of $\sqrt{I}$, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s\prod_{j=1}^{n} x_j \in I$ for all $n \geq k$.*

(3) *For any primary ideal $\mathfrak{q}$ of R and for any sequence $< x_n >$ of elements of $\sqrt{\mathfrak{q}}$, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s\prod_{j=1}^{n} x_j \in \mathfrak{q}$ for all $n \geq k$.*

Proof. (1) $\Rightarrow$ (2) This follows from Lemma 3.1.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let I be any ideal of R and $< r_n >$ be any sequence of elements of R. We verify that the increasing sequence of ideals of R, $(I:_R r_1) \subseteq (I:_R r_1r_2) \subseteq (I:_R r_1r_2r_3) \subseteq \cdots$ is S-stationary. First, we verify the above assertion in the case $I = \mathfrak{q}$ is a primary ideal of R. Let $\mathfrak{q}$ be $\mathfrak{p}$-primary. Observe that $\mathfrak{p} = \sqrt{\mathfrak{q}}$. We consider the following cases:

Case(i): There exists $k \in \mathbb{N}$ such that $r_i \notin \mathfrak{p}$ for each $i \in \mathbb{N}$ with $i \geq k$.

In such a case, for any $n \in \mathbb{N}$ with $n \geq k$, $\prod_{i=k}^{n} r_i \notin \mathfrak{p}$. Hence, we obtain from [13, Lemma 4.4 (*iii*)] that $(\mathfrak{q}:_R \prod_{i=k}^{n} r_i) = \mathfrak{q}$ for all $n \geq k$. Therefore, for all $n \geq k$, $(\mathfrak{q}:_R \prod_{i=1}^{n} r_i) = \mathfrak{q}$ in the case $k = 1$. If $k \geq 2$, then for all $n \geq k$, $(\mathfrak{q}:_R \prod_{i=1}^{n} r_i) = (\mathfrak{q}:_R \prod_{i=1}^{k-1} r_i)$.

Case(ii): There exist positive integers $k_1 < k_2 < k_3 < \cdots$ such that $r_{k_j} \in \mathfrak{p}$ for each $j \geq 1$.

By (3), there exist $s \in S$ and $j_0 \in \mathbb{N}$ such that $s\prod_{i=1}^{j} r_{k_i} \in \mathfrak{q}$ for all $j \geq j_0$. Hence, for all $n \geq k_{j_0}$, $s(\mathfrak{q}:_R \prod_{i=1}^{n} r_i) \subseteq Rs \subseteq (\mathfrak{q}:_R \prod_{i=1}^{k_{j_0}} r_i)$.

This shows that for any primary ideal $\mathfrak{q}$ of R and for any sequence $< r_n >$ of elements of R, the increasing sequence of ideals of R, $(\mathfrak{q}:_R r_1) \subseteq (\mathfrak{q}:_R r_1r_2) \subseteq (\mathfrak{q}:_R r_1r_2r_3) \subseteq \cdots$ is S-stationary.

Let I be any ideal of R and let $< r_n >$ be any sequence of elements of R. Suppose that $I \cap S \neq \emptyset$. Let $s \in I \cap S$. Then for all $n \geq 1$, $s(I:_R \prod_{i=1}^{n} r_i) \subseteq Rs \subseteq I \subseteq (I:_R r_1)$. Suppose that $I \cap S = \emptyset$. Since R is S-Laskerian by hypothesis, we obtain from the proof of (2) $\Rightarrow$ (3) of Corollary 2.6 that there exists $s' \in S$ such that $(I:_R s')$ admits a primary decomposition in R. Hence, there exist $t \in \mathbb{N}$ and primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ of R such that $(I:_R s') = \bigcap_{i=1}^{t} \mathfrak{q}_i$. Let $i \in \{1, \ldots, t\}$. Since $\mathfrak{q}_i$ is a primary ideal of R, it follows as shown above that there exist $s_i \in S$ and $k_i \in \mathbb{N}$ such that $s_i(\mathfrak{q}_i:_R \prod_{j=1}^{n} r_j) \subseteq (\mathfrak{q}_i:_R \prod_{j=1}^{k_i} r_j)$ for all $n \geq k_i$. Let $s = \prod_{i=1}^{t} s_i$ and let $k = \max(k_1, \ldots, k_t)$. Now, $s \in S$ and for all $n \geq k$, $s((I:_R s'):_R \prod_{j=1}^{n} r_j) = s((\bigcap_{i=1}^{t} \mathfrak{q}_i):_R \prod_{j=1}^{n} r_j) \subseteq \bigcap_{i=1}^{t} s_i(\mathfrak{q}_i:_R \prod_{j=1}^{n} r_j)$ $\subseteq \bigcap_{i=1}^{t}(\mathfrak{q}_i:_R \prod_{j=1}^{k} r_j) = ((\bigcap_{i=1}^{t} \mathfrak{q}_i):_R \prod_{j=1}^{k} r_j) = ((I:_R s'):_R \prod_{j=1}^{k} r_j)$. This implies that for all $n \geq k$, $ss'(I:_R \prod_{i=1}^{n} r_i) \subseteq (I:_R \prod_{i=1}^{k} r_i)$.

This shows that for any ideal I of R and for any sequence $< r_n >$ of elements of R, the increasing sequence of ideals of R, $(I:_R r_1) \subseteq (I:_R r_1r_2) \subseteq (I:_R r_1r_2r_3) \subseteq \cdots$ is S-stationary. Therefore, we obtain that R satisfies S-strong accr*. □

Theorem 3.3. *Let D be an integral domain which is not a field. Let F be the quotient field of D. Let K be an extension field of F. If* (*D, K*) *is an* SACCR*P, *then the following hold:*

(1) *K is algebraic over F and the integral closure of D* in *K is a Dedekind domain.*

(2) *The separable degree of K over F is finite and K has finite exponent over F.*

Proof. (1) If a ring T satisfies strong accr*, then T satisfies (accr*) and hence by [3, Theorem 1], we get that T satisfies (accr). We are assuming that (D, K) is an SACCR*P. Therefore, (D, K) is an ACCRP. Hence, we obtain from (1) $\Rightarrow$ (2) of Corollary 2.13 that K is an algebraic extension of F and the integral closure of D in K is a one-dimensional Prüfer domain. Let $\bar{D}$ denote the integral closure of D in K. Notice that $\bar{D}$ is a Prüfer domain, $\dim\bar{D} = 1$, and it satisfies strong accr*. That is, the Prüfer domain $\bar{D}$ satisfies (C) in the notation of [21]. Hence, we obtain from [21, Proposition 2.3] that $\bar{D}$ is Noetherian. Therefore, each non-zero fractional ideal of $\bar{D}$ is f.g. and as $\bar{D}$ being a Prüfer domain, it follows that each non-zero fractional ideal of $\bar{D}$ is invertible. Hence, we obtain from [13, Theorem 9.8] that $\bar{D}$ is a Dedekind domain. Let $S = \{1\}$. It follows from (2) $\Rightarrow$ (1) of Corollary 2.14 that (D, K) is an S-LP. As $S = \{1\}$, we get that (D, K) is an LP.

(2) Let L be the maximal separable subfield of K over F. We claim that $[L: F] < \infty$. Suppose that $[L: F]$ is not finite. Let $\alpha_1 \in L\backslash F$. Let $[F(\alpha_1): F] = n_1$. Then $n_1 > 1$. Notice that L is an infinite algebraic and separable extension of $F(\alpha_1)$. Hence, we obtain from [22, Lemma 1, p. 194] that there exists $\alpha_2 \in L$ such that $[F(\alpha_1, \alpha_2): F(\alpha_1)] = n_2 > n_1$. Since the separable algebraic extension L over F is assumed to be an infinite extension, by repeated use of [22, Lemma 1, p. 194], it is possible to find positive integers $1 < n_1 < n_2 < n_3 < \cdots$ and an infinite sequence $< \alpha_k >$ of elements from L such that $[F(\alpha_1): F] = n_1$ and for each $k \geq 2$, $[F(\alpha_1, \alpha_2, \ldots, \alpha_k): F(\alpha_1, \ldots, \alpha_{k-1})] = n_k$.

The remaining part of the proof is suggested by the proof of [23, Lemma 3] and the proof of [9, Proposition 2.12]. Notice that as F is the quotient field of D, there exists $y_1 \in D\backslash\{0\}$ such that the irreducible polynomial of $y_1\alpha_1$ over F has coefficients in D and its degree is n_1. It is clear that $y_1\alpha_1 \in F(\alpha_1)$ is integral over D. Set $z_1 = y_1\alpha_1$. It is clear that $D[z_1]$ is a free D-module with basis $\{1, z_1, \ldots, z_1^{n_1-1}\}$. By hypothesis, D is not a field. Hence, it is possible to find a non-zero element $d \in D$ such that $d \notin U(D)$. Let us denote the ring $D[dz_1]$ by D_1. It is clear that $F(\alpha_1)$ is the quotient field of D_1, D_1 is a free D-module with basis $\{1, dz_1, \ldots, (dz_1)^{n_1-1}\}$, and D_1 is an integral extension of D. Observe that the irreducible polynomial of α_2 over $F(\alpha_1)$ is of degree n_2 and it is possible to find $y_2 \in D_1\backslash\{0\}$ such that the irreducible polynomial of $y_2\alpha_2$ over $F(\alpha_1)$ has coefficients in D_1. Set $z_2 = y_2\alpha_2$. It is clear that z_2 is integral over D_1 and $D_1[z_2]$ is a free D_1-module with basis $\{1, z_2, \ldots, z_2^{n_2-1}\}$. Let us denote $D_1[dz_2]$ by D_2. Notice that $F(\alpha_1, \alpha_2)$ is the quotient field of D_2, D_2 is a free D_1-module with basis $\{1, dz_2, \ldots, (dz_2)^{n_2-1}\}$, and D_2 is integral over D_1. Proceeding like this, we obtain a strictly increasing sequence of subrings of L, $D_1 \subset D_2 \subset D_3 \subset \cdots$ such that for each $k \geq 1$, $D_k = D_{k-1}[dz_k]$ (with $D_0 = D$) is a free D_{k-1}- module with basis $\{1, dz_k, \ldots, (dz_k)^{n_k-1}\}$, and D_k is integral over D_{k-1}. Also, $F(\alpha_1, \ldots, \alpha_k)$ is the the quotient field of D_k for each $k \in \mathbb{N}$ and by the choice of z_k, it is clear that z_k is integral over D_{k-1} for each $k \geq 1$. Let us denote the ring $\bigcup_{k=1}^{\infty} D_k$ by T. Since D_1 is integral over D and D_2 is integral over D_1, it follows from [13, Corollary 5.4] that D_2 is integral over D. Let $k \geq 2$. Assume it is verified that D_k is integral over D. As D_{k+1} is integral over D_k, we obtain from [13, Corollary 5.4] that D_{k+1} is integral over D. This proves that $T = \bigcup_{k=1}^{\infty} D_k$ is integral over D. Also, observe that z_k is integral over D for each $k \geq 1$. Notice that $\bigcup_{k=1}^{\infty} F(\alpha_1, \ldots, \alpha_k)$ is the quotient field of T. Now, $T \in [D, L]$ and as L is a subfield of K and (D, K) is an SACCR*P by hypothesis, it follows that T satisfies strong accr*. Let $I = Td + \sum_{k=1}^{\infty} Tdz_k$. We assert that $\sqrt{I} = \sqrt{Td}$. As $Td \subseteq I$, it is clear that $\sqrt{Td} \subseteq \sqrt{I}$. Let $\mathfrak{p} \in Spec(T)$ be such that $d \in \mathfrak{p}$. Let us denote the ring $\bigcup_{k=1}^{\infty} D[z_k]$ by T_1. Notice that T is a subring of T_1. From z_k is integral over D for each $k \geq 1$, it follows that T_1 is integral over T. Now, it follows from [13, Theorem 5.10] that there exists $\mathfrak{q} \in Spec(T_1)$ such that $\mathfrak{q} \cap T = \mathfrak{p}$. Notice that $d \in \mathfrak{q}$ and as $z_k \in T_1$ for each $k \geq 1$, it follows that $IT_1 = T_1d$. Therefore, $IT_1 \subseteq \mathfrak{q}$. Hence, $I \subseteq IT_1 \cap T \subseteq \mathfrak{q} \cap T = \mathfrak{p}$. Let $V(Td) = \{\mathfrak{p} \in Spec(T) \mid \mathfrak{p} \supseteq Td\}$. We know from [13, Proposition 1.14] that $\sqrt{Td} = \bigcap_{\mathfrak{p} \in V(Td)} \mathfrak{p}$ and so, we get that $I \subseteq \sqrt{Td}$. It follows from [13, Exercise 1.13 (*ii*), page 9] that $\sqrt{\sqrt{Td}} = \sqrt{Td}$. Hence, we obtain that $\sqrt{I} \subseteq \sqrt{\sqrt{Td}} = \sqrt{Td}$. Therefore, $\sqrt{I} = \sqrt{Td}$. Now, T satisfies strong accr* and $< dz_n >$ is a sequence of elements of T such that $dz_n \in \sqrt{I} = \sqrt{Td}$ for each $n \in \mathbb{N}$. Hence, by applying Lemma 3.1 with $S = \{1\}$, we obtain that there exists $k \in \mathbb{N}$ such that $\prod_{i=1}^{n} dz_i \in Td$ for each $n \geq k$. Thus $\prod_{i=1}^{k} dz_i \in Td \cap D_k$. Observe that D_j is a free D_{j-1}-module with basis $\{1, dz_j, \ldots, (dz_j)^{n_j-1}\}$ for each $j \geq 1$ (where, $D_0 = D$) and from $d \in D$, it follows that $D_jd \cap D_{j-1} = D_{j-1}d$ for each $j \geq 1$. Hence, we obtain from $\prod_{i=1}^{k} dz_i \in Td \cap D_k$ that $\prod_{i=1}^{k} dz_i \in D_k d$. If $k = 1$, then we get that $dz_1 \in D_1 d$. Observe that $D_1 d = Dd + Dd(dz_1) + \cdots + Dd(dz_1)^{n_1-1}$. Therefore, $dz_1 \in Dd + Dd(dz_1) + \cdots + Dd(dz_1)^{n_1-1}$ —— (1). Hence, by comparing the coefficient of dz_1 on both sides of (1), it follows that $1 \in Dd$. This is impossible, since d is not a unit of D. Suppose that $k \geq 2$. Notice that $\prod_{i=1}^{k-1} dz_i \in D_{k-1}$. It is clear that $(\prod_{i=1}^{k-1} dz_i)dz_k \in D_k d = D_{k-1}d + D_{k-1}d(dz_k) + \cdots + D_{k-1}d(dz_k)^{n_k-1}$—— (2). Hence, by comparing the coefficient of dz_k on both sides of (2), it follows that $\prod_{i=1}^{k-1} dz_i \in D_{k-1}d$. Proceeding like this, we get that $dz_1 \in D_1 d$ and this is already verified to be impossible. Therefore, T does not satisfy strong accr*. This is in contradiction to the assumption that (D, K) is an SACCR*P. Therefore, the separable degree of K over F must be finite.

If $char(F) = 0$, then K is separable over F and so, $L = K$, where L is the maximal separable subfield of K over F. Suppose that $char(F) = p > 0$. We claim that $K^{p^n} \subseteq L$ for some $n \geq 1$. Suppose that $K^{p^n} \not\subseteq L$ for each $n \geq 1$. Then it is possible to find a sequence $< \beta_k >$ of elements of K and positive integers $1 < n_1 < n_2 < n_3 < \cdots$ such that n_1 is least with the property that $\beta_1^{p^{n_1}} \in L$ and for each $k \geq 2$, n_k is least with the property that $\beta_k^{p^{n_k}} \in L(\beta_1, \ldots, \beta_{k-1})$. Notice that $[L(\beta_1) : L] = p^{n_1}$ and for each $k \geq 2$, $[L(\beta_1, \ldots, \beta_k) : L(\beta_1, \ldots, \beta_{k-1})] = p^{n_k}$. Since $\bar{D}$ is the integral closure of D in K, it follows that $\bar{D} \cap L$ is the integral closure of D in L. It is convenient to denote $\bar{D} \cap L$ by E. As L is algebraic over F, it follows that L is the quotient field of E. Observe that the irreducible polynomial of β_1 over L is $X^{p^{n_1}} - \beta_1^{p^{n_1}}$. Since L is the quotient field of E, there exists $v_1 \in E \backslash \{0\}$ such that the irreducible polynomial of $v_1\beta_1$ has coefficients in E. Set $w_1 = v_1\beta_1$. It is clear that w_1 is integral over E and $E[w_1]$ is a free E-module with basis $\{1, w_1, \ldots, w_1^{p^{n_1}-1}\}$. Since E is integral over D and D is not a field, we obtain from [13, Proposition 5.7] that E is not a field. Let $a \in E \backslash \{0\}$ be such that $a \notin U(E)$. Let us denote the ring $E[aw_1]$ by A_1. Notice that A_1 is a free E-module with basis $\{1, aw_1, \ldots, (aw_1)^{p^{n_1}-1}\}$. Since $L(\beta_1)$ is the quotient field of A_1, it is possible to find $v_2 \in A_1 \backslash \{0\}$ such that the irreducible polynomial of $v_2\beta_2$ has coefficients in A_1. Set $w_2 = v_2\beta_2$. It is clear that w_2 is integral over A_1 and $A_1[w_2]$ is a free A_1-module with basis $\{1, w_2, \ldots, w_2^{p^{n_2}-1}\}$. Let us denote the ring $A_1[aw_2]$ by A_2. Observe that A_2 is a free A_1-module with basis $\{1, aw_2, \ldots, (aw_2)^{p^{n_2}-1}\}$, A_2 is integral over A_1 and $L(\beta_1, \beta_2)$ is the quotient field of A_2. Proceeding like this, it is possible to find a strictly increasing sequence of subrings $A_1 \subset A_2 \subset A_3 \subset \cdots$ of K such that for each $k \geq 1$, $A_k = A_{k-1}[aw_k]$ is a free A_{k-1}-module with basis $\{1, aw_k, \ldots, (aw_k)^{p^{n_k}-1}\}$ (where we set $A_0 = E$). Let us denote the ring $\bigcup_{k=1}^{\infty} A_k$ by A. It is clear that $A \in [E, K]$ and it can be shown as in the previous paragraph that A does not satisfy strong accr*. This is in contradiction to the assumption that (D, K) is an SACCR*P. Therefore, there exists $n \geq 1$ such that $K^{p^n} \subseteq L$.

Thus if (D, K) is an SACCR*P, then K is algebraic over D, the integral closure of D in K is a Dedekind domain, the separable degree of K over F is finite and K has finite exponent over F, where F is the quotient field of D. □

Corollary 3.4. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is not a field. Let F be the quotient field of D. Let K be an extension field of F. If (D, K) is an S-SACCR*P, then the following hold:*

(1) *K is algebraic over F and the integral closure of $S^{-1}D$ in K is a Dedekind domain.*

(2) *The separable degree of K over F is finite and K has finite exponent over F.*

Proof. Let $A \in [S^{-1}D, K]$. Notice that $A = S^{-1}T$ for some $T \in [D, K]$. By hypothesis, T satisfies S-strong accr*. Hence, we obtain from [5, Lemma 2.4] that $S^{-1}T$ satisfies strong accr*. This shows that $(S^{-1}D, K)$ is an SACCR*P. By hypothesis, $S^{-1}D$ is not a field and it is clear that F is the quotient field of $S^{-1}D$. Hence, (1) and (2) of this corollary follow from (1) and (2) of Theorem 3.3. □

In Example 3.5, we provide an infinite algebraic extension field L of $\mathbb{Q}$ such that $(\mathbb{Z}, L)$ is an ACCRP but $(\mathbb{Z}, L)$ is not an SACCR*P.

Example 3.5. In [12, Example, p. 520], R. Gilmer showed that it is possible to find a sequence $\{t_i\}_{i=1}^{\infty}$ of algebraic integers such that the integral closure of $\mathbb{Z}$ in $L = \mathbb{Q}(\{t_i\}_{i=1}^{\infty})$ is a Dedekind domain. Since any Dedekind domain is a one-dimensional Prüfer domain, we obtain from (2) ⇒ (1) of Corollary 2.13 that $(\mathbb{Z}, L)$ is an ACCRP. Since $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$ and L is an infinite separable extension field of $\mathbb{Q}$, we obtain from Theorem 3.3(2) that $(\mathbb{Z}, L)$ is not an SACCR*P. □

Let S be a countable m.c. subset of an integral domain D such that $S^{-1}D$ is integrally closed but it is not a field. Let F be the quotient field of D. Let $char(F) = 0$. Let K be an extension field of F. We verify in Corollary 3.6 that (D, K) is an S-SACCR*P if and only if (D, K) is an S- NP.

Corollary 3.6. *Let S be a countable m.c. subset of an integral domain D such that $S^{-1}D$ is not a field and $S^{-1}D$ is integrally closed. Let F be the quotient field of D and let char(F) = 0. Let K be an extension field of F. The following statements are equivalent:*

(1) *(D, K) is an S-SLP.*

(2) *(D, K) is an S-*SACCR**P.*

(3) *For any $T \in [D, K]$ and any ideal I of T, there exists $s \in S$ (depending on I) such that $S(I) = (I :_T s)$ and K is a finite algebraic extension of F. Moreover, $S^{-1}D$ and the integral closure of $S^{-1}D$ in K are Dedekind domains.*

(4) *(D, K) is an S-NP.*

Proof. (1) ⇒ (2) Let $T \in [D, K]$. Then by assumption, T is S-strongly Laskerian. Hence, we obtain from [8, Corollary 3.9(2)] that T satisfies S-strong accr*. This shows that (D, K) is an S-SACCR*P.

(2) ⇒ (3) We are assuming that (D, K) is an S-SACCR*P. Let $T \in [D, K]$. As S is a countable m.c. subset of T and T satisfies S-strong accr*, we obtain from (*i*) ⇒ (*ii*) of [5, Theorem 2.7] that for any ideal I of T, there exists $s \in S$ (depending on I) such that $S(I) = (I :_T s)$. As (D, K) is an S-SACCR*P, we obtain from Corollary 3.4(1) that K is algebraic over F and the integral closure of $S^{-1}D$ in K is a Dedekind domain. Notice that (D, F) is an S-SACCR*P. By hypothesis, $S^{-1}D$ is integrally closed. Hence, $S^{-1}D$ is the integral closure of $S^{-1}D$ in F. Therefore, we obtain from Corollary 3.4(1) that $S^{-1}D$ is a Dedekind domain. By hypothesis, $char(F) = 0$. Hence, K is a separable extension of F. Therefore, we obtain from Corollary 3.4(2) that $[K : F] < \infty$.

(3) ⇒ (4) Now, as $S^{-1}D$ is a Dedekind domain, $S^{-1}D$ is Noetherian and dim $S^{-1}D = 1$. Since $[K : F] < \infty$, it follows from Krull-Akizuki Theorem [24, Theorem 11.7] that $(S^{-1}D, K)$ is an NP. Let $T \in [D, K]$. Then $S^{-1}T \in [S^{-1}D, K]$ and so, $S^{-1}T$ is Noetherian. By (3), for any ideal I of T, there exists $s \in S$ (depending on I) such that $S(I) = (I :_T s)$. Hence, we obtain from [2, Proposition 2.2(*f*)] that T is S-Noetherian. This proves that (D, K) is an S-NP.

(4) ⇒ (1) We are assuming that (D, K) is an S-NP. Let $T \in [D, K]$. As T is S-Noetherian, we obtain from [8, Corollary 3.3] that T is S-strongly Laskerian. Therefore, (D, K) is an S-SLP. □

Applying Corollary 3.6 with $S = \{1\}$, we obtain the following corollary.

Corollary 3.7. *Let D be an integrally closed domain which is not a field. Let F be the quotient field of D with char(F) = 0. Let K be an extension field of F. The following statements are equivalent:*

(1) *(D, K) is an SLP.*

(2) *(D, K) is an* SACCR*P.

(3) *K is a finite algebraic extension of F. Moreover, D and the integral closure of D in K are Dedekind domains.*

(4) *(D, K) is an NP.*

Example 3.8 mentioned below provides an integral domain T such that the integral closure of T in its quotient field is a Dedekind domain but T does not satisfy strong accr*.

Example 3.8. Let $L = \mathbb{Q}(\{t_i\}_{i=1}^{\infty})$ be as mentioned in Example 3.5. The field L was constructed by R. Gilmer (see [12, Example, p. 520]). Notice that L is an infinite algebraic extension of $\mathbb{Q}$. It was already verified in [12, Example, p. 520] that any integrally closed domain between $\mathbb{Z}$ and L is either a field or a Dedekind domain. Since L is an infinite separable extension of $\mathbb{Q}$, proceeding as in the proof of Theorem 3.3(2), it is possible to find a subring T of L such that T does not satisfy strong accr*. It follows from [8, Corollary 3.9(2)] that T is not strongly Laskerian. Observe that the integral closure of T in its quotient field is a Dedekind domain.

As the integral closure of $\mathbb{Z}$ in L is a Dedekind domain (and hence, a one-dimensional Prüfer domain), it follows from (2) ⇒ (1) of Corollary 2.13 that $(\mathbb{Z}, L)$ is an ACCRP (indeed, it follows from [9, Proposition 2.1] that $(\mathbb{Z}, L)$ is an LP). It is noted in the previous paragraph that $T \in [\mathbb{Z}, L]$ is such that T does not satisfy strong accr* and hence, $(\mathbb{Z}, L)$ is not an SACCR*P. Thus the ring T is Laskerian and it does not satisfy strong accr*. As T is not strongly Laskerian, we get that $(\mathbb{Z}, L)$ is not an SLP. □

Corollary 3.9. *Let S be a m.c. subset of an integral domain D such that $S^{-1}D$ is a field. Let us denote $S^{-1}D$ by F. Let K be an extension field of F such that tr.*deg $K/F = 1$. *Let $\alpha \in K$ be transcendental over F. If (D, K) is an S*-SACCR*P, *then the following hold.*

(1) *The integral closure of $F[\alpha]$ in K is a Dedekind domain.*

(2) *The separable degree of K over $F(\alpha)$ is finite and K has finite exponent over $F(\alpha)$.*

Proof. We are assuming that (D, K) is an S-SACCR*P. Let $\alpha \in K$ be such that α is transcendental over F. As $S \subseteq U(F) = U(F[\alpha])$, it follows that $S^{-1}(F[\alpha]) = F[\alpha]$ is not a field. By hypothesis, *tr.*deg $K/F = 1$ and so, K is algebraic over $F(\alpha)$. As $(F[\alpha], K)$ is an SACCR*P, we obtain from Theorem 3.3(1) that the integral closure of $F[\alpha]$ in K is a Dedekind domain. This proves (1).

From Theorem 3.3(2), we get that the separable degree of K over $F(\alpha)$ is finite and K has finite exponent over $F(\alpha)$. This proves (2). □

Recall from [22, p. 190] that a field F is said to be *perfect* if either *char*$(F) = 0$ or if *char*$(F) = p > 0$, then $F^p = F$.

Corollary 3.10. *Let K be an extension field of a perfect field F such that tr.*deg $K/F = 1$. *Then the following statements are equivalent:*

(1) *(F, K) is an SLP.*

(2) *(F, K) is an* SACCR*P.

(3) *(F, K) is an NP.*

Proof. (1) ⇒ (2) This is clear, since we know from [8, Corollary 3.9(2)] that any strongly Laskerian ring satisfies strong accr*.

(2) ⇒ (3) Let $T \in [F, K]$. If T is algebraic over F, then it follows from [13, Proposition 5.7] that T is a field. Suppose that T is not algebraic over F. Let $t \in T$ be such that t is transcendental over F. Since $(F[t], K)$ is an SACCR*P, it follows from Theorem 3.3(2) that the separable degree of K over $F(t)$ is finite and K has finite exponent over $F(t)$. By hypothesis, F is a perfect field. Hence, it can be shown as in the proof of [9, Corollary 2.16] that $[K : F(t)] < \infty$. Now, $A = F[t]$ is a Noetherian domain and dim $A = 1$ (indeed, A is a principal ideal domain). Notice that K is a finite algebraic extension of the quotient field of A and hence, we obtain from Krull-Akizuki Theorem [24, Theorem 11.7] that $(F[t], K)$ is an NP. As $T \in [F[t], K]$, we get that T is Noetherian. This shows that (F, K) is an NP.

(3) ⇒ (1) This is clear, since any Noetherian ring is strongly Laskerian. □

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