Multistep-type construction of fixed point for multivalued ρ-quasi-contractive-like maps in modular function spaces

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Abstract

Purpose – In this paper, the explicit multistep, explicit multistep-SP and implicit multistep iterative sequences are introduced in the context of modular function spaces and proven to converge to the fixed point of a multivalued map T such that P_ρ^T, an associate multivalued map, is a ρ-contractive-like mapping.
Design/methodology/approach – The concepts of relative ρ-stability and weak ρ-stability are introduced, and conditions in which these multistep iterations are relatively ρ-stable, weakly ρ-stable and ρ-stable are established for the newly introduced strong ρ-quasi-contractive-like class of maps.
Findings – Noor type, Ishikawa type and Mann type iterative sequences are deduced as corollaries in this study.
Originality/value – The results obtained in this work are complementary to those proved in normed and metric spaces in the literature.

Keywords Multistep iterations, Modular function spaces, Strong ρ-contractions, Relative ρ-stability, Weakly ρ-stability

Paper type Research paper

1. Introduction and preliminary definitions

Modular function spaces are well-known generalizations of both function and sequence variants of many important spaces such as Calderon–Lozanovskii, Kothe, Lebesgue, Lorentz, Musielak–Orlicz, Orlicz and Orlicz–Lorentz spaces. Their applications are also very useful. There is huge interest in quasi-contractive mappings in modular function spaces mainly because of the richness of structure of modular function spaces: apart from being F-spaces in a more general setting, they are equipped with modular equivalents of norm or metric notions and also endowed with convergence in submeasure. It is worthy to mention that modular-type conditions are far more natural as their assumptions can be easily verified than their corresponding metrics or norms, especially when related to fixed-point results and applications to integral-type operators. More so, there are some fixed-point results that can be proved only using the framework of modular function spaces. Thus, results in fixed-point theory in modular function spaces and those in normed and metric spaces are complementary (see, e.g. [1]). Different researchers have proved very useful fixed-points results in the context of modular function spaces (see [1–6] for details).

The following background definitions in [1, 3, 7] are useful in proving the main results in this manuscript:

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Let Ω be a nonempty set and Σ be a nontrivial $\sigma-$ algebra of subsets of Ω. Let $\mathcal{P}$ be a $\delta-$ ring of subsets of Ω such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Assume there exists an increasing sequence $(K_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ such that $\Omega = \cup_{n\in\mathbb{N}} K_n$.

Let $\mathcal{E}$ represent the linear space of all simple functions with supports from $\mathcal{P}$, that is, functions $s = \sum_{k=1}^{n} \alpha_k I_{A_k}$, where $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of real numbers, $(A_k)_{k\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{P}$ and I_A represents the characteristic function of the set A in Ω.

Let $\mathcal{M}_\infty$ represent the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $(g_n) \subset \mathcal{E}$ satisfying $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Definition 1.1. ([7]). Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. ρ is said to be a regular convex function pseudomodular if:

(1) $\rho(0) = 0$;

(2) ρ is monotone, that is, $|f| \le |g|$ on Ω implies $\rho(f) \le \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

(3) ρ is orthogonally subadditive, that is, $\rho(fI_{A\cup B}) \le \rho(fI_A) + \rho(fI_B)$ for any $A, B \in \Omega$ such that $A \cap B \neq \phi$, with $f \in \mathcal{M}_\infty$;

(4) ρ has Fatou's property, that is, $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;

(5) ρ is order continuous in $\mathcal{E}$, that is, $(g_n) \subset \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ for all $\omega \in \Omega$ implies $\rho(g_n) \downarrow 0$.

Concepts similar to those in measure spaces are defined for function pseudomodular ρ: a set $A \in \Sigma$ is said to be *ρ-null* if $\rho(fI_A) = 0\ \forall f \in \mathcal{E}$; a property is said to hold *ρ-almost everywhere (ρ-a.e.)* on Σ if the set for which it does not hold is ρ-null.

The following set is defined:

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : \ |f| < \infty \ \rho - a.e.\},$$

where each $f \in \mathcal{M}_\infty$ is actually an equivalence class of functions equal ρ-a.e. We will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ when no confusion arises.

Definition 1.2. ([1]). Let ρ be a regular function pseudomodular.

(1) ρ is said to be a regular function modular if $\rho(f) = 0$ implies $f = 0$ *ρ-a.e.*

(2) ρ is said to be a regular function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ *ρ-a.e.*

A regular convex function modular ρ satisfies the following properties (see [3])

(1) $\rho(f) = 0$ if $f = 0\rho$ *-a.e.*

(2) $\rho(\alpha f) = \rho(f)$ for every scalar α such that $|\alpha| = 1$, where $f \in \mathcal{M}$.

(3) $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$.

The class of all nonzero regular convex function modulars on Ω is denoted by $\mathfrak{R}$.

Definition 1.3. ([7]). A convex function modular ρ defines the modular function space L_ρ as

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

L_ρ is a normed linear space with respect to

$$|f_\rho| = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

which is known as the Luxemburg norm.

Definition 1.4. ([7]). Let L_ρ be a modular space. The sequence $\{f_n\} \subset L_\rho$ is called:

(1) $\rho-$convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$;

(2) $\rho-$Cauchy, if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

Remark 1.1. $\rho-$convergent sequence implies $\rho-$Cauchy sequence if and only if ρ satisfies the Δ_2- condition given in the definition below. However, ρ does not satisfy the triangle inequality.

Definition 1.5. ([7]). A nonzero regular convex function ρ is said to satisfy the Δ_2- condition, if $\sup_{n\geq 1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}\downarrow\emptyset/$ and $\sup_{n\geq 1}\rho(f_n, D_k) \to 0$ as $k \to \infty$.

Definition 1.6. ([7]). Let L_ρ be a modular space and $D \subset L_\rho$.

The ρ-distance from $f \in L_\rho$ to the set D is given by:

$$dist_\rho(f, D) = \inf\{\rho(f - h) : h \in D\}.$$

A subset $D \subset L_\rho$ is called:

(1) $\rho-$closed if the $\rho-$limit of a $\rho-$convergent sequence of D always belongs to D;

(2) $\rho-a.e.$ closed if the $\rho-a.e.$ limit of a $\rho-a.e.$ convergent sequence of D always belongs to D;

(3) $\rho-$compact if every sequence in D has a $\rho-$convergent subsequence in D;

(4) $\rho-a.e.$ compact if every sequence in D has a $\rho-a.e.$ convergent subsequence in D;

(5) $\rho-$bounded if $\text{diam}_\rho(D) = \sup\{\rho(f-g) : f, g \in D\} < \infty$.

(6) $\rho-$proximal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f-g) = dist_\rho(f, D)$.

The family of nonempty ρ-bounded ρ-proximal subsets of D is denoted by $P_\rho(D)$, the family of nonempty ρ-closed ρ-bounded subsets of D by $C_\rho(D)$ and the family of ρ-compact subsets of D by $K_\rho(D)$.

Definition 1.7. ([7]). Let L_ρ be a modular space. A function $f \in L_\rho$ is called a fixed point of a multivalued mapping $T : L_\rho \to P_\rho(D)$ if $f \in Tf$. The set of all fixed points of T is represented by $F_\rho(T)$ so that:

$$F_\rho(T) = \{f \in L_\rho : f \in Tf\}.$$

The following set is also defined:

$$P_\rho^T(f) = \{g \in Tf : \rho(f - g) = dist_\rho(f, Tf)\}.$$

Zamfirescu [8] in 1972 proved the following theorem as a generalization of the Banach fixed-point theorem:

Theorem 1.1. ([8]). Let X be a complete metric space and $T : X \to X$ a Zamfirescu operator satisfying:

$$d(Tx, Ty) \leq h\max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)]}{2}\right\}, \tag{1.1}$$

where $0 \leq h < 1$. Then, T has a unique fixed point and the Picard iteration converges to p for any $x_0 \in X$.

Observe that in a Banach space setting, condition (1.1) implies

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|, \ \delta = \max\left\{h, \frac{h}{2-h}\right\} \in [0, 1) \tag{1.2}$$

Osilike [9] used the following contractive definition: for each $x, y \in X$, there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$||Tx - Ty|| \leq \delta\|x - y\| + L||x - Tx||. \tag{1.3}$$

Imoru and Olatinwo [10] proved some stability results using the following general contractive definition: for each $x, y \in X$, there exist $\delta \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(||x - Tx||). \tag{1.4}$$

Observe that (1.4) generalizes (1.3) and (1.2). The map T considered in (1.2)–(1.4) is single-valued. Now, we state the generalizations of (1.2)–(1.4) to multivalued mappings, as conformed to literature. (e.g. see [7]).

Let $H_\rho(\cdot, \cdot)$ be the $\rho-$Hausdorff distance on the family $C_\rho(L_\rho)$ of nonempty ρ-closed ρ-bounded subsets of L_ρ, that is,

$$H_\rho(A, B) = \max\left\{\sup_{f \in A} dist_\rho(f, B), \sup_{g \in B} dist_\rho(g, A)\right\}, \ A, B \in C_\rho(L_\rho).$$

A multivalued map $T : D \to C_\rho(L_\rho)$ is said to be a:

(1) $\rho-$contraction mapping if there exists a constant $\delta \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \ \forall f, g \in D. \tag{1.5}$$

(2) $\rho-$Zamfirescu mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(h - f), \ \forall f, g \in D \ \forall h \in Tf. \tag{1.6}$$

(3) $\rho-$quasi-contractive mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + L\rho(h - f), \ \forall f, g \in D \ \forall h \in Tf, \ L \geq 0. \tag{1.7}$$

(4) $\rho-$quasi-contractive-like mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + \varphi(\rho(h - f)), \ \forall f, g \in D \ \forall h \in Tf. \tag{1.8}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing function with $\varphi(0) = 0$.

Convergence and stability of fixed-point iterative sequences for single mapping T are two very vital concepts in fixed-point theory and applications. Some of the results of colossal value in this work are those in [9–20]. Rhoades and Soltuz [21] introduced the multistep iteration and proved its equivalence with Mann and Ishikawa iterations. Olaleru and Akewe [22] proved convergence of multistep iteration for a pair of mappings (S, T)

We now introduce the following iterative sequences in the framework of modular function spaces and use them to prove new fixed-point theorems.

Let $T : D \to P_\rho(D)$ be a multivalued mapping.

The explicit multistep iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n + \alpha_n v_n^1, \\ g_n^i = (1-\beta_n^i)f_n + \beta_n^i v_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ g_n^{k-1} = (1-\beta_n^{k-1})f_n + \beta_n^{k-1} u_n, \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.9}$$

where $u_n \in P_\rho^T(f_n), v_n^i \in P_\rho^T(g_n^i), i = 1, 2, \ldots, k-1$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, k-1$, are in $[0, 1)$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

The explicit Noor iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n + \alpha_n v_n^1, \\ g_n^1 = (1-\beta_n^1)f_n + \beta_n^1 v_n^2, \\ g_n^2 = (1-\beta_n^2)f_n + \beta_n^2 u_n, \; n = 0, 1, 2, \ldots \end{cases} \tag{1.10}$$

where $u_n \in P_\rho^T(f_n)$, $v_n^1 \in P_\rho^T(g_n^1)$, $v_n^2 \in P_\rho^T(g_n^2)$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^1\}_{n=0}^{\infty}$ and $\{\beta_n^2\}_{n=0}^{\infty}$ are in $[0, 1)$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

The explicit Ishikawa iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n + \alpha_n v_n^1, \\ g_n^1 = (1-\beta_n^1)f_n + \beta_n^1 u_n, \; n = 0, 1, 2, \ldots \end{cases} \tag{1.11}$$

where $u_n \in P_\rho^T(f_n)$, $v_n^1 \in P_\rho^T(g_n^1)$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^1\}_{n=0}^{\infty}$ are in $[0, 1)$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

The explicit Mann iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n + \alpha_n u_n, \; n = 0, 1, 2, \ldots \end{cases} \tag{1.12}$$

where $u_n \in P_\rho^T(f_n)$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

The explicit multistep-SP iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)g_n^1 + \alpha_n v_n^1 \\ g_n^i = (1-\beta_n^i)g_n^{i+1} + \beta_n^i v_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ g_n^{k-1} = (1-\beta_n^{k-1})f_n + \beta_n^{k-1} u_n, \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.13}$$

where $u_n \in P_\rho^T(f_n), v_n^i \in P_\rho^T(g_n^i), i = 1, 2, \ldots, k-1$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, k-1$, are in $[0, 1)$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

The explicit SP iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)g_n^1 + \alpha_n v_n^1, \\ g_n^1 = (1-\beta_n^1)g_n^2 + \beta_n^1 v_n^2, \\ g_n^2 = (1-\beta_n^2)f_n + \beta_n^2 u_n, \; n = 0, 1, 2, \ldots \end{cases} \tag{1.14}$$

where $u_n \in P_\rho^T(f_n)$, $v_n^1 \in P_\rho^T(g_n^1)$, $v_n^2 \in P_\rho^T(g_n^2)$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^1\}_{n=0}^{\infty}$, and $\{\beta_n^2\}_{n=0}^{\infty}$ are in $[0, 1)$ such that $\sum_0^{\infty} \alpha_n = \infty$.

The implicit multistep iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n^1 + \alpha_n u_{n+1}, \\ f_n^i = (1-\beta_n^i)f_n^{i+1} + \beta_n^i u_n^i, \quad i = 1, 2, \ldots, k-2 \\ f_n^{k-1} = (1-\beta_n^{k-1})f_n + \beta_n^{k-1} u_n^{k-1}, \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.15}$$

where $u_{n+1} \in P_\rho^T(f_n)$, $u_n^i \in P_\rho^T(f_n^i)$, $i = 1, 2, \ldots, k-1$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, k-1$, are in $[0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

It should be noted that the implicit multistep iterative sequence exists if and only if T satisfies the property (I) as follows:

$$(I): \quad \forall h \in D \;\; \forall \beta \in (0, 1) \;\; \exists f \in D \;\; \exists g \in P_\rho^T(f): \quad f = (1-\beta)h + \beta g.$$

The implicit Noor iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n^1 + \alpha_n u_{n+1}, \\ f_n^1 = (1-\beta_n^1)f_n^2 + \beta_n^1 u_n^1, \\ f_n^2 = (1-\beta_n^2)f_n + \beta_n^2 u_n^2, \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.16}$$

where $u_{n+1} \in P_\rho^T(f_{n+1})$, $u_n^1 \in P_\rho^T(f_n^1)$, $u_n^2 \in P_\rho^T(f_n^2)$, and the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^1\}_{n=0}^{\infty}$, and $\{\beta_n^2\}_{n=0}^{\infty}$ are in $[0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Ishikawa iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n^1 + \alpha_n u_{n+1}, \\ f_n^1 = (1-\beta_n^1)f_n + \beta_n^1 u_n^1, \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.17}$$

where $u_{n+1} \in P_\rho^T(f_{n+1})$, $u_n^1 \in P_\rho^T(f_n^1)$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$, $\{\beta_n^1\}_{n=0}^{\infty} \subset [0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Mann iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 & \in & D \\ f_{n+1} & = & (1-\alpha_n)f_n + \alpha_n u_{n+1},\ n = 0,\ 1,\ 2,\ \ldots \end{cases} \tag{1.18}$$

where $u_{n+1} \in P_\rho^T(f_{n+1})$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0,\ 1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

The following Lemmas will be needed in proving the main results.

Lemma 1.1. ([3]). Let $T : D \to P_\rho(D)$ be a multivalued mapping and $P_\rho^T(f) = \{g \in Tf : \rho(f-g) = dist_\rho(f,\ Tf)\}$. Then the following are equivalent:

(1) $f \in F_\rho(T)$, that is, $f \in Tf$.

(2) $P_\rho^T(f) = \{f\}$.

(3) $f \in F(P_\rho^T(f))$, that is, $f \in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T(f))$ where $F(P_\rho^T(f))$ represent the set of fixed points of $P_\rho^T(f)$.

Lemma 1.2. (see [13]). Let δ be a real number satisfying $0 \le \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ two sequences of positive or zero numbers, less than 1, such that $\lim_{n\to\infty}\varepsilon_n = 0$ and $\sum_{n=0}^{\infty}\tau_n = \infty$. Then any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying any of the following properties converges to 0:

(1) $u_{n+1} \le \delta u_n + \varepsilon_n$ for all $n = 0,\ 1,\ 2,\ \ldots$

(2) $u_{n+1} \le (1-\tau_n)u_n$ for all $n = 0,\ 1,\ 2,\ \ldots$

(3) $u_{n+1} \le \varepsilon_n + (1-\tau_n)u_n$ for all $n = 0,\ 1,\ 2,\ \ldots$ if in addition, $\{\tau_n\}_{n=0}^{\infty}$ is bounded away from 0.

2. Convergence results

2.1 Strong convergence results for explicit multistep iterative sequences in modular function spaces

Theorem 2.1. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition (1.8). Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the explicit multistep iterative sequence (1.9), where the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty} \subset [0,\ 1)$, $(i = 1,\ 2,\ \ldots,\ k-1)$ are such that $\sum_0^{\infty}\alpha_n = \infty$. Then the explicit multistep iterative sequence (1.9) converges strongly to the fixed point of T.

Proof. Let $f \in F_\rho(T)$; from Lemma 1.1, $P_\rho^T(f) = \{f\}$ and $F_\rho(T) = F(P_\rho^T(f))$.

Using the explicit multistep iterative sequence (1.9) and the convexity of ρ, we obtain the following estimate:

$$\begin{aligned} \rho(f_{n+1}-f) &= \rho[(1-\alpha_n)f_n + \alpha_n v_n^1 - f] \\ &= \rho\left[(1-\alpha_n)(f_n-f) + \alpha_n\left(v_n^1-f\right)\right] \\ &\le (1-\alpha_n)\rho(f_n-f) + \alpha_n\rho\left(v_n^1-f\right). \end{aligned} \tag{2.1}$$

$v_n^1 \in P_\rho^T(g_n^1)$ and $P_\rho^T(f) = \{f\}$ imply that:

$$\rho(v_n^1 - f) = dist_\rho(v_n^1, P_\rho^T(f)) \leq H_\rho(P_\rho^T(g_n^1), P_\rho^T(f)),$$

which combined with (2.1) yields:

$$\rho(f_{n+1} - f) \leq (1 - \alpha_n)\rho(f_n - f) + \alpha_n H_\rho(P_\rho^T(g_n^1), P_\rho^T(f)). \tag{2.2}$$

In (1.8), letting $g = g_n^1$ and noting that $P_\rho^T(f) = \{f\}$ and $\varphi(0) = 0$, we have:

$$\begin{aligned} H_\rho(P_\rho^T(g_n^1), P_\rho^T(f)) &\leq \delta\rho(g_n^1 - f) + \varphi(f - h) \quad \forall h \in P_\rho^T(f) \\ &\leq \delta\rho(g_n^1 - f). \end{aligned} \tag{2.3}$$

Substituting (2.3) in (2.2), we obtain

$$\rho(f_{n+1} - f) \leq (1 - \alpha_n)\rho(f_n - f) + \delta\alpha_n\rho(g_n^1 - f). \tag{2.4}$$

Similarly, from (1.9) and the convexity of ρ,

$$\begin{aligned} \rho(g_n^1 - f) &= \rho[(1 - \beta_n^1)f_n + \beta_n^1 v_n^2 - f] \\ &= \rho[(1 - \beta_n^1)(f_n - f) + \beta_n^1(v_n^2 - f)] \\ &\leq (1 - \beta_n^1)\rho(f_n - f) + \beta_n^1\rho(v_n^2 - f). \end{aligned} \tag{2.5}$$

$v_n^2 \in P_\rho^T(g_n^2)$ and $P_\rho^T(f) = \{f\}$ imply that:

$$\rho(v_n^2 - f) = dist_\rho(v_n^2, P_\rho^T(f)) \leq H_\rho(P_\rho^T(g_n^2), P_\rho^T(f)),$$

which combined with (2.5) yields:

$$\rho(g_n^1 - f) \leq (1 - \beta_n^1)\rho(f_n - f) + \beta_n^1 H_\rho(P_\rho^T(g_n^2), P_\rho^T(f)). \tag{2.6}$$

In (1.8), letting $g = g_n^2$ and noting that $P_\rho^T(f) = \{f\}$ and $\varphi(0) = 0$, we get:

$$\rho(g_n^1 - f) \leq (1 - \beta_n^1)\rho(f_n - f) + \delta\beta_n^1\rho(g_n^2 - f). \tag{2.7}$$

Similarly, an application of (1.8) and (1.9) gives

$$\rho(g_n^2 - f) \leq (1 - \beta_n^2)\rho(f_n - f) + \delta\beta_n^2\rho(g_n^3 - f). \tag{2.8}$$

Also, an application of (1.8) and (1.9) gives

$$\rho(g_n^3 - f) \leq (1 - \beta_n^3)\rho(f_n - f) + \delta\beta_n^3\rho(g_n^4 - f). \tag{2.9}$$

Substituting (2.9) in (2.8), (2.8) in (2.7) and (2.7) in (2.4), and simplifying, we obtain

$$\begin{aligned} \rho(f_{n+1} - f) \leq [1 - (1 - \delta)\alpha_n - (1 - \delta)\delta\alpha_n\beta_n^1 - (1 - \delta)\delta^2\alpha_n\beta_n^1\beta_n^2 \\ -(1 - \delta)\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3] + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\rho(g_n^4 - f). \end{aligned} \tag{2.10}$$

Continuing this process, an application of (1.8) and (1.9) gives

$$\rho(g_n^{k-2} - f) \leq (1 - \beta_n^{k-2})\rho(f_n - f) + \delta\beta_n^{k-2}\rho(g_n^{k-1} - f). \tag{2.11}$$

and

$$\rho\left(g_n^{k-1}-f\right)\leq\left(1-\beta_n^{k-1}\right)\rho(f_n-f)+\delta\beta_n^{k-1}\rho(f_n-f). \tag{2.12}$$

Substituting (2.12) and (2.11) in (2.10) inductively and simplifying, we obtain

$$\begin{aligned}\rho(f_{n+1}-f)\leq&\left[1-(1-\delta)\alpha_n-\sum_{i=1}^{k-1}(1-\delta)\delta^i\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^i\right.\\&\left.+\delta^k\alpha_n\beta_n^1\beta_n^2\beta_n^3\beta_n^4\ldots\beta_n^{k-1}\right]\rho(f_n-f)\\\leq&[1-(1-\delta)\alpha_n]\rho(f_n-f).\end{aligned} \tag{2.13}$$

From (2.13), we inductively obtain

$$\rho(f_{n+1}-f)\leq\left[\prod_{m=0}^{n}(1-(1-\delta)\alpha_m)\right]\rho(f_0-f). \tag{2.14}$$

Using that fact that $\delta\in[0,\,1)$ $\{\alpha_n\}_{n=0}^{\infty}\subset[0,\,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$, then from (2.14), we obtain

$$\lim_{n\to\infty}\rho(f_{n+1}-f)\leq\lim_{n\to\infty}\prod_{m=0}^{n}[1-(1-\delta)\alpha_m]\rho(f_0-f)=0. \tag{2.15}$$

Therefore, $\{f_n\}$ ρ-converges to $f\in F_\rho(T)$. The proof is complete. ■

Since the explicit Noor (1.10), explicit Ishikawa (1.11), explicit Mann (1.12) iterative sequences are special cases of the explicit multistep iterative sequence (1.9) (see [22] for details), then Theorem 2.1 leads to the following corollary:

Corollary 2.1. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T:D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition (1.8). Suppose that $F_\rho(T)\neq\emptyset/$. Let $f_0\in D$ and $\{f_n\}\subset D$ be defined by the explicit Noor (1.10), the explicit Ishikawa (1.11) and the explicit Mann (1.12) iterative sequences respectively, where the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^1\}_{n=0}^{\infty}$, $\{\beta_n^2\}_{n=0}^{\infty}\subset[0,1)$ are such that $\sum_0^{\infty}\alpha_n=\infty$. Then:

(1) the explicit Noor iterative sequence (1.10) converges strongly to the fixed point of T.

(2) the explicit Ishikawa iterative sequence (1.11) converges strongly to the fixed point of T.

(3) the explicit Mann iterative sequence (1.12) converges strongly to the fixed point of T.

2.2 Strong convergence results for explicit multistep-SP iterative sequences in modular function spaces

Theorem 2.2. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T:D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition (1.8). Suppose that $F_\rho(T)\neq\emptyset/$. Let $f_0\in D$ and $\{f_n\}\subset D$ be defined by the explicit multistep-SP iterative sequence (1.13), where the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}\subset[0,\,1)$, $(i=1,\,2,\,\ldots,\,k-1)$ are

such that $\sum_0^\infty \alpha_n = \infty$. Then the explicit multistep-SP iterative sequence (1.13) ρ-converges to a fixed point of T.

Proof. Let $f \in F_\rho(T)$. From Lemma 1.1, we have that $P_\rho^T(f) = \{f\}$ and $F_\rho(T) = F(P_\rho^T(f))$. Using the explicit multistep-SP iterative sequence (1.13) and the convexity of ρ, we obtain the following estimate:

$$\begin{aligned}\rho(f_{n+1} - f) &= \rho[(1-\alpha_n)g_n^1 + \alpha_n v_n^1 - f] \\ &= \rho\left[(1-\alpha_n)\left(g_n^1 - f\right) + \alpha_n\left(v_n^1 - f\right)\right] \\ &\leq (1-\alpha_n)\rho\left(g_n^1 - f\right) + \alpha_n\rho\left(v_n^1 - f\right).\end{aligned} \tag{2.16}$$

Since $v_n^1 \in P_\rho^T(g_n^1)$ and $P_\rho^T(f) = \{f\}$, we have

$$\rho\left(v_n^1 - f\right) = dist_\rho\left(v_n^1, P_\rho^T(f)\right) \leq H_\rho\left(P_\rho^T(g_n^1), P_\rho^T(f)\right),$$

which combined with (2.16) yields:

$$\rho(f_{n+1} - f) \leq (1-\alpha_n)\rho\left(g_n^1 - f\right) + \alpha_n H_\rho\left(P_\rho^T(g_n^1), P_\rho^T(f)\right). \tag{2.17}$$

In (1.8), letting $g = g_n^1$ and noting that $P_\rho^T(f) = \{f\}$ and $\varphi(0) = 0$, we get

$$H_\rho\left(P_\rho^T(g_n^1), P_\rho^T(f)\right) \leq \delta\rho\left(g_n^1 - f\right) + \varphi(0) = \delta\rho\left(g_n^1 - f\right). \tag{2.18}$$

Substituting (2.18) in (2.17), we obtain

$$\begin{aligned}\rho(f_{n+1} - f) &\leq (1-\alpha_n)\rho\left(g_n^1 - f\right) + \delta\alpha_n\rho\left(g_n^1 - f\right) \\ &= [1 - (1-\delta)\alpha_n]\rho\left(g_n^1 - f\right).\end{aligned} \tag{2.19}$$

Next, from (1.13) and the convexity of ρ,

$$\begin{aligned}\rho\left(g_n^1 - f\right) &= \rho\left[(1-\beta_n^1)g_n^2 + \beta_n^1 v_n^2 - f\right] \\ &= \rho\left[\left(1-\beta_n^1\right)\left(g_n^2 - f\right) + \beta_n^1\left(v_n^2 - f\right)\right] \\ &\leq \left(1-\beta_n^1\right)\rho\left(g_n^2 - f\right) + \beta_n^1\rho\left(v_n^2 - f\right).\end{aligned} \tag{2.20}$$

Since $v_n^2 \in P_\rho^T(g_n^2)$ and $P_\rho^T(f) = \{f\}$, we have

$$\rho\left(v_n^2 - f\right) = dist_\rho\left(v_n^2, P_\rho^T(f)\right) \leq H_\rho\left(P_\rho^T(g_n^2), P_\rho^T(f)\right),$$

which combined with (2.20) yields:

$$\rho\left(g_n^1 - f\right) \leq \left(1-\beta_n^1\right)\rho\left(g_n^2 - f\right) + \beta_n^1 H_\rho\left(P_\rho^T(g_n^2), P_\rho^T(f)\right). \tag{2.21}$$

Using (1.8) with $g = g_n^2$ in (2.21) and noting that $\varphi(0) = 0$ and $P_\rho^T(f) = \{f\}$, then we get the following:

$$\begin{aligned}\rho\left(g_n^1 - f\right) &\leq \left(1-\beta_n^1\right)\rho\left(g_n^2 - f\right) + \delta\beta_n^1\rho\left(g_n^2 - f\right) \\ &= \left[1 - (1-\delta)\beta_n^1\right]\rho\left(g_n^2 - f\right).\end{aligned} \tag{2.22}$$

Similarly, an application of (1.8) and (1.13) gives

$$\rho\left(g_n^2 - f\right) \leq \left(1-\beta_n^2\right)\rho\left(g_n^3 - f\right) + \delta\beta_n^2\rho\left(g_n^3 - f\right) \tag{2.23}$$

$$= \left[1-(1-\delta)\beta_n^2\right]\rho\left(g_n^3-f\right).$$

Also, an application of (1.8) and (1.13) gives

$$\rho\left(g_n^3-f\right) \leq \left(1-\beta_n^3\right)\rho\left(g_n^4-f\right)+\delta\beta_n^3\rho\left(g_n^4-f\right) \tag{2.24}$$
$$= \left[1-(1-\delta)\beta_n^3\right]\rho\left(g_n^4-f\right).$$

Continuing this process, an application of (1.8) and (1.13) gives

$$\rho\left(g_n^{k-2}-f\right) \leq \left(1-\beta_n^{k-2}\right)\rho\left(g_n^{k-1}-f\right)+\delta\beta_n^{k-2}\rho\left(g_n^{k-1}-f\right) \tag{2.25}$$
$$= \left[1-(1-\delta)\beta_n^{k-2}\right]\rho\left(g_n^{k-1}-f\right).$$

and

$$\rho\left(g_n^{k-1}-f\right) \leq \left(1-\beta_n^{k-1}\right)\rho\left(f_n-f\right)+\delta\beta_n^{k-1}\rho\left(f_n-f\right) \tag{2.26}$$
$$= \left[1-(1-\delta)\beta_n^{k-1}\right]\rho\left(f_n-f\right).$$

Substituting (2.22)–(2.26) in (2.19) inductively and simplifying, we obtain

$$\rho\left(f_{n+1}-f\right) \leq \left(\left[1-(1-\delta)\alpha_n\right]\prod_{i=1}^{k\,1}\left[1-(1-\delta)\beta_n^i\right]\right)\rho\left(f_n-f\right) \tag{2.27}$$
$$\leq \left[1-(1-\delta)\alpha_n\right]\rho\left(f_n-f\right).$$

From (2.27), we inductively obtain

$$\rho\left(f_{n+1}-f\right) \leq \prod_{m=0}^{n}\left[1-(1-\delta)\alpha_m\right]\rho\left(f_0-f\right). \tag{2.28}$$

Using that fact that $\delta \in [0,\,1)$ $\{\alpha_n\}_{n=0}^{\infty} \subset [0,\,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$, then from (2.28), we obtain

$$\lim_{n\to\infty}\rho\left(f_{n+1}-f\right) \leq \lim_{n\to\infty}\prod_{m=0}^{n}\left[1-(1-\delta)\alpha_m\right]\rho\left(f_0-f\right) = 0. \tag{2.29}$$

Therefore, $\lim_{n\to\infty}\rho\left(f_n-f\right)=0$, where $f \in F_\rho(T)$. The proof is complete. ■

Theorem 2.2 leads to the following corollary:

Corollary 2.2. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition (1.8). Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the explicit SP iterative sequence (1.14), with the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^1\}_{n=0}^{\infty}$, $\{\beta_n^2\}_{n=0}^{\infty} \subset [0,\,1)$ such that $\sum_0^{\infty}\alpha_n = \infty$. Then, the explicit SP iterative sequence (1.14) ρ-converges strongly to a fixed point of T.

2.3 Strong convergence results for implicit multistep iterative sequences in modular function spaces

Theorem 2.3. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping satisfying property (I) and such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition (1.8). Suppose that $F_\rho(T) \neq \emptyset/$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the implicit multistep iterative sequence (1.15), where the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty} \subset [0,\,1)$

$(i = 1, 2, \ldots, k-1)$ are such that $\sum_0^\infty \alpha_n = \infty$. Then, the implicit multistep iterative sequence (1.15) ρ-converges strongly to a fixed point of T.

Proof. Let $f \in F_\rho(T)$. From Lemma 1.1, we have that $P_\rho^T(f) = \{f\}$ and $F_\rho(T) = F(P_\rho^T(f))$.

Using implicit multistep iterative sequence (1.15) and the convexity of ρ, we obtain the following estimate:

$$\begin{aligned}\rho(f_{n+1} - f) &= \rho[(1-\alpha_n)f_n^1 + \alpha_n u_{n+1} - f] \\ &= \rho\left[(1-\alpha_n)\left(f_n^1 - f\right) + \alpha_n(u_{n+1} - f)\right] \\ &\leq (1-\alpha_n)\rho\left(f_n^1 - f\right) + \alpha_n\rho(u_{n+1} - f)\end{aligned} \tag{2.30}$$

Since $u_{n+1} \in P_\rho^T(f_{n+1})$ and $P_\rho^T(f) = \{f\}$,

$$\rho(u_{n+1} - f) \leq dist_\rho\left(u_{n+1}, P_\rho^T(f)\right) \leq H_\rho\left(P_\rho^T(f_{n+1}), P_\rho^T(f)\right),$$

which combined with (2.30) gives

$$\rho(f_{n+1} - f) \leq (1-\alpha_n)\rho\left(f_n^1 - f\right) + \alpha_n H_\rho\left(P_\rho^T(f_{n+1}), P_\rho^T(f)\right). \tag{2.31}$$

In (1.8), by letting $g = f_{n+1}$ and noting that $\varphi(0) = 0$ and $P_\rho^T(f) = \{f\}$, we get:

$$H_\rho\left(P_\rho^T(f_{n+1}), P_\rho^T(f)\right) \leq \delta\rho(f_{n+1} - f) + \varphi_\rho(0) = \delta\rho(f_{n+1} - f). \tag{2.32}$$

Substituting (2.32) in (2.31), we obtain

$$\rho(f_{n+1} - f) \leq (1-\alpha_n)\rho\left(f_n^1 - f\right) + \delta\alpha_n\rho(f_{n+1} - f)$$

That is,

$$\rho(f_{n+1} - f) \leq \left[\frac{1-\alpha_n}{1-\delta\alpha_n}\right]\rho\left(f_n^1 - f\right). \tag{2.33}$$

Next, from (1.15) and the convexity of ρ, we have

$$\begin{aligned}\rho\left(f_n^1 - f\right) &= \rho\left[\left(1-\beta_n^1 f_n^2 + \beta_n^1\right)u_n^1 - f\right] \\ &= \rho\left[\left(1-\beta_n^1\right)\left(f_n^2 - f\right) + \beta_n^1\left(u_n^1 - f\right)\right] \\ &= \left(1-\beta_n^1\right)\rho\left(f_n^2 - f\right) + \beta_n^1\rho\left(u_n^1 - f\right).\end{aligned} \tag{2.34}$$

Since $u_n^1 \in P_\rho^T(f_n^1)$ and $P_\rho^T(f) = \{f\}$,

$$\rho\left(u_n^1, f\right) = dist_\rho\left(u_n^1, P_\rho^T(f)\right) \leq H_\rho\left(P_\rho^T(f_n^1), P_\rho^T(f)\right),$$

which combined with (2.34) gives:

$$\rho\left(f_n^1 - f\right) \leq \left(1-\beta_n^1\right)\rho\left(f_n^2 - f\right) + \beta_n^1 H_\rho\left(P_\rho^T(f_n^1), P_\rho^T(f)\right). \tag{2.35}$$

By letting $g = f_n^1$ in (1.8) and noting that $\varphi(0) = 0$ and $P_\rho^T(f) = \{f\}$, we get:

$$H_\rho\left(P_\rho^T(f_n^1), P_\rho^T(f)\right) \leq \delta\rho\left(f_n^1 - f\right) + \varphi_\rho(0) = \delta\rho\left(f_n^1 - f\right) \tag{2.36}$$

Substituting (2.36) in (2.35), we obtain

$$\rho\left(f_n^1 - f\right) \leq \left(1-\beta_n^1\right)\rho\left(f_n^2 - f\right) + \delta\beta_n^1\rho\left(f_n^1 - f\right)$$

That is,

$$\rho\left(f_n^1 - f\right) \leq \left[\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right]\rho\left(f_n^2 - f\right). \tag{2.37}$$

Similarly, an application of (1.8) and (1.15) gives

$$\rho\left(f_n^2 - f\right) \leq \left[\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right]\rho\left(f_n^3 - f\right). \tag{2.38}$$

$$\rho\left(f_n^3 - f\right) \leq \left[\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right]\rho\left(f_n^4 - f\right). \tag{2.39}$$

$$\vdots$$

$$\rho\left(f_n^{k-2} - f\right) \leq \left[\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right]\rho\left(f_n^{k-1} - f\right). \tag{2.40}$$

$$\rho\left(f_n^{k-1} - f\right) \leq \left[\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right]\rho(f_n - f). \tag{2.41}$$

Substituting (2.37)–(2.40) in (2.33) inductively and simplifying, we obtain

$$\rho(f_{n+1} - f) \leq \left[\frac{1-\alpha_n}{1-\delta\alpha_n}\right]\left[\prod_{i=1}^{k}\frac{1-\beta_n^i}{1-\delta\beta_n^i}\right]\rho(f_n - f). \tag{2.42}$$

Observe that

$$\frac{1-\alpha_n}{1-\delta\alpha_n} \leq 1 - \alpha_n + \delta\alpha_n,\ \left[\frac{1-\beta_n^i}{1-\delta\beta_n^i}\right] \leq 1 - \beta_n^i + \delta\beta_n^i,\ i = 1,\ \ldots,\ k-1 \tag{2.43}$$

Substituting (2.43) in (2.42) and simplifying, we obtain

$$\rho(f_{n+1} - f) \leq [1 - (1-\delta)\alpha_n]\rho(f_n - f). \tag{2.44}$$

From (2.44), we inductively obtain

$$\rho(f_{n+1} - f) \leq \prod_{m=0}^{n}[1 - (1-\delta)\alpha_m]\rho(f_0 - f). \tag{2.45}$$

Using that fact that $\delta \in [0,\ 1)$ $\{\alpha_n\}_{n=0}^{\infty} \subset [0,\ 1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$, then from (2.45), we obtain

$$\lim_{n\to\infty}\rho(f_{n+1} - f) \leq \lim_{n\to\infty}\prod_{m=0}^{n}[1 - (1-\delta)\alpha_m]\rho(f_0 - f) = 0. \tag{2.46}$$

Therefore, $\lim_{n\to\infty}\rho(f_n - f) = 0$, with $f \in F_\rho(T)$. The proof is complete. ■

Theorem 2.3 leads to the following corollary:

Corollary 2.3. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping satisfying property (I), such

that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition (1.8). Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the implicit Noor (1.16), implicit Ishikawa (1.17) and implicit Mann (1.18) iterative sequences respectively, where the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^1\}_{n=0}^{\infty}$, $\{\beta_n^2\}_{n=0}^{\infty} \subset [0, 1)$ are such that $\sum_0^{\infty} \alpha_n = \infty$. Then:

(1) the implicit Noor iterative sequence (1.16) converges strongly to the fixed point of T.

(2) the implicit Ishikawa iterative sequence (1.17) converges strongly to the fixed point of T.

(3) the implicit Mann iterative sequence (1.18) converges strongly to the fixed point of T.

3. Stability results for strong ρ-quasi-contractive-like maps

In this section, conditions for some stability types of the explicit and implicit multistep iterative sequences are stated and backed by proofs in the framework of modular function spaces.

The first important result on $T-$ stable single mappings was proved by Ostrowski [18] for Picard iteration. Berinde [13], presented useful explanation on how to obtain the stability of various iterative sequences. Okeke and Khan [7] gave a similar version of stability results for multivalued mapping in modular function spaces.

In this paper, we introduce two other versions of ρ-stability and attempt to relate them with the concept of ρ-stability in literature.

Definition 3.1. Let D be a nonempty $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping with $F_\rho(T) \neq \varnothing$. Suppose that a fixed-point iterative sequence defined by

$$f_{n+1} = F(T, f_n) \tag{3.1}$$

with initial guess $f_0 \in D$ and F is a given function, converges to a fixed point f of T. Let $\{h_n\}_{n=0}^{\infty}$ be an arbitrary sequence in D. The fixed-point iterative sequence is said to be:

(1) ρ-stable with respect to T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} h_n = f, \text{ where } \varepsilon_n = \rho(h_{n+1} - F(T, h_n)). \tag{3.2}$$

(2) relatively ρ-stable with respect to T if and only if

$$\lim_{n\to\infty} \delta_n = 0 \Rightarrow \lim_{n\to\infty} h_n = f, \text{ where } \delta_n = \rho(h_{n+1} - f) - \rho(F(T, h_n) - f). \tag{3.3}$$

(3) weakly ρ-stable with respect to T if and only if

$$\sup_{\lambda\in(0,1]} \lambda\rho\left(\frac{h_{n+1} - F(T, h_n)}{\lambda}\right) \to 0 \Rightarrow \inf_{\lambda\in[1,\infty)} \lambda\rho\left(\frac{h_n - f}{\lambda}\right) \to 0. \tag{3.4}$$

The term "relatively" in (2) is employed because the premise of the convergence of $\{h_n\}$ to f is hinged to the fact that $\rho(h_{n+1} - f)$ and $\rho(F(T, h_n) - f)$ get closer to each other as n increases. It is not known if this concept is directly related to ρ-stability as defined in [7]. If ρ satisfies the triangular inequality (an unwanted condition in this paper), the relation between relatively ρ-stability and ρ-stability is as follows: (1) a relative ρ-stable fixed-point iteration is ρ-stable if

$\delta_n > 0$ for n sufficiently big since $|\delta_n| \leq \varepsilon_n$; (2) a ρ-stable fixed-point iteration is relatively ρ-stable if for n sufficiently big, $\delta_n < 0$ and $|\delta_n| \leq \varepsilon_n$.

However, a ρ-stable fixed-point iteration is weakly ρ-stable, hence the term "weakly."

In this sequel, we also introduce the following concepts of strong quasi-contractions particular to modular function spaces and compatible in some sense to the newly introduced stability notions.

Definition 3.2. Let $H_\rho(\cdot, \cdot)$ be the ρ-Hausdorff distance on the family $C_\rho(L_\rho)$ of nonempty ρ-closed ρ-bounded subsets of L_ρ, that is,

$$H_\rho(A, B) = \max\left\{\sup_{f\in A} dist_\rho(f, B), \sup_{g\in B} dist_\rho(g, A)\right\}, A, B \in C_\rho(L_\rho).$$

A multivalued map $T : D \to C_\rho(L_\rho)$ is said to be an:

(1) m-strong $\rho-$contraction mapping, where $m \in \mathbb{N}$, if there exists a constant $\delta \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \leq m\delta\rho\left(\frac{f-g}{m}\right), \forall f, g \in D; \tag{3.5}$$

(If $\delta = 1$ in (3.5), T is said to be an m-strong ρ-nonexpansive mapping)

(2) m-strong $\rho-$quasi-contractive mapping, where $m \in \mathbb{N}$, if

$$H_\rho(Tf, Tg) \leq m\delta\rho\left(\frac{f-g}{m}\right) + L\rho(h-f), \forall f, g \in D \quad \forall h \in Tf, L \geq 0; \tag{3.6}$$

(If $\delta = 1$ in (3.6), T is said to be an m-strong ρ-quasi-contractive mapping)

(3) m-strong $\rho-$quasi-contractive-like mapping, where $m \in \mathbb{N}$, if

$$H_\rho(Tf, Tg) \leq m\delta\rho\left(\frac{f-g}{m}\right) + \varphi(\rho(h-f)), \forall f, g \in D \ \forall h \in Tf. \tag{3.7}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing function with $\varphi(0) = 0$. (If $\delta = 1$ in (3.7), T is said to be a m-strong ρ-quasi-contractive-like mapping).

Given any $m \in \mathbb{N}$, an m-strong ρ-contraction (resp. ρ-quasi-contractive mapping, or a ρ-quasi-contractive-like mapping) is a ρ-contraction (resp. ρ-quasi-contractive mapping, or a ρ-quasi-contractive-like mapping), thus, the convergence results in the previous section hold for m-strong ρ-quasi-contractive-like mappings. The converse is trivial when $m = 1$.

3.1 Stability results for explicit multistep iterative sequences in modular function spaces

Theorem 3.1. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is an m-strong ρ-quasi-contractive-like mapping, satisfying contractive-like condition (3.7), where $m \in \mathbb{N}$. Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the explicit multistep iterative sequence (1.9), where the sequences $\{\alpha_n\}_{n=0}^\infty, \{\beta_n^i\}_{n=0}^\infty \subset [0, 1)$ $(i = 1, 2, \ldots, k-1)$ are such that $\{\alpha_n\}_{n=0}^\infty$ is bounded away from 0. Then, (1.9) is:

(1) relatively ρ-stable with respect to T if $m = 1$;

(2) weakly ρ-stable with respect to T if $m > 1$.

(3) ρ-stable with respect to T if $m > 1$ and $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$ where $f \in F_\rho(T)$ (in this case, P_ρ^T is a ρ-quasi-contractive-like map).

Proof. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty} \subset [0, 1)$ $(i = 1, 2, \ldots, k-1)$ be sequences such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0.

Let $\{h_n\}_{n=0}^{\infty}$ be an arbitrary sequence in D and set:

$$\begin{cases} \varepsilon_n = \rho\left(h_{n+1} - (1-\alpha_n)h_n - \alpha_n z_n^1\right) \\ \delta_n = \rho(h_{n+1} - f) - \rho\left((1-\alpha_n)h_n + \alpha_n z_n^1 - f\right) \\ \gamma_n = \sup\limits_{\lambda \in (0,1]} \lambda\rho\left(\dfrac{h_{n+1} - (1-\alpha_n)h_n - \alpha_n z_n^1}{\lambda}\right) \\ s_n^i = \left(1-\beta_n^i\right)h_n + \beta_n^i z_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ s_n^{k-1} = \left(1-\beta_n^{k-1}\right)h_n + \beta_n^{k-1} w_n, \quad n = 0, 1, 2, \ldots \end{cases} \tag{3.8}$$

where $w_n \in P_\rho^T(h_n)$ and $z_n^i \in P_\rho^T(s_n^i), i = 1, 2, \ldots, k-1$.

Let:

$$r_{n,m} = (1-\alpha_n)\rho\left(\frac{h_n - f}{m}\right) + \frac{\alpha_n}{m}\rho\left(z_n^1 - f\right). \tag{3.9}$$

By the convexity of ρ, we have:

$$\begin{aligned} \rho(h_{n+1} - f) &= \delta_n + \rho\left((1-\alpha_n)h_n + \alpha_n z_n^1 - f\right) \\ &= \delta_n + \rho\left((1-\alpha_n)(h_n - f) + \alpha_n\left(z_n^1 - f\right)\right) \\ &\le \delta_n + r_{n,1}. \end{aligned} \tag{3.10}$$

If $m > 1$, we have:

$$\begin{aligned} \rho\left(\frac{h_{n+1} - f}{m}\right) &= \rho\left(\frac{\alpha_n}{m}\,\frac{h_{n+1} - (1-\alpha_n)h_n - \alpha_n z_n^1}{\alpha_n} + (1-\alpha_n)\frac{h_n - f}{m} + \frac{\alpha_n}{m}\left(z_n^1 - f\right)\right) \\ &\le \frac{\alpha_n}{m}\rho\left(\frac{h_{n+1} - (1-\alpha_n)h_n - \alpha_n z_n^1}{\alpha_n}\right) + r_{n,m} \\ &\le \frac{\gamma_n}{m} + r_{n,m}. \end{aligned} \tag{3.11}$$

and if in addition $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$,

$$\begin{aligned} \rho\left(\frac{h_{n+1} - f}{m}\right) &= \rho\left(\frac{h_{n+1} - (1-\alpha_n)h_n - \alpha_n z_n^1}{m} + \frac{1-\alpha_n}{m}(h_n - f) + \frac{\alpha_n}{m}\left(z_n^1 - f\right)\right) \\ &\le \frac{1}{m}\varepsilon_n + r_{n,m}. \end{aligned} \tag{3.12}$$

Since $z_n^1 \in P_\rho^T(s_n^1)$, then $\rho(z_n^1 - f) = dist_\rho(z_n^1, P_\rho^T(f)) \leq H_\rho(P_\rho^T(s_n^1), P_\rho^T(f))$ hence:

$$r_{n,m} \leq (1-\alpha_n)\rho\left(\frac{h_n - f}{m}\right) + \frac{\alpha_n}{m} H_\rho\left(P_\rho^T(s_n^1), P_\rho^T(f)\right). \tag{3.13}$$

Using (3.7) and (3.8), and noting that $\varphi(0) = 0$, we get the following:

$$r_{n,m} \leq (1-\alpha_n)\rho\left(\frac{h_n - f}{m}\right) + \delta\alpha_n\rho\left(\frac{s_n^1 - f}{m}\right) \tag{3.14}$$

Using the convexity of ρ in (3.8), and the fact that $z_n^2 \in P_\rho^T(s_n^2)$, we have

$$\begin{aligned}\rho\left(\frac{s_n^1 - f}{m}\right) &\leq (1-\beta_n^1)\rho\left(\frac{h_n - f}{m}\right) + \beta_n^1\rho\left(\frac{z_n^2 - f}{m}\right) \\ &\leq (1-\beta_n^1)\rho\left(\frac{h_n - f}{m}\right) + \frac{\beta_n^1}{m} dist_\rho\left(z_n^2, P_\rho^T(f)\right) \\ &\leq (1-\beta_n^1)\rho\left(\frac{h_n - f}{m}\right) + \frac{\beta_n^1}{m} H_\rho\left(P_\rho^T(s_n^2), P_\rho^T(f)\right).\end{aligned} \tag{3.15}$$

Using (3.7) and noting that $\varphi(0) = 0$, then we get the following:

$$\rho\left(\frac{s_n^1 - f}{m}\right) \leq (1-\beta_n^1)\rho\left(\frac{h_n - f}{m}\right) + \delta\beta_n^1\rho\left(\frac{s_n^2 - f}{m}\right). \tag{3.16}$$

Substituting (3.16) in (3.15), then in (3.14), we obtain

$$\begin{aligned} r_{n,m} &\leq (1-\alpha_n)\rho\left(\frac{h_n - f}{m}\right) + \delta\alpha_n\rho\left(\frac{s_n^1 - f}{m}\right) \\ &\leq (1-\alpha_n)\rho\left(\frac{h_n - f}{m}\right) + \delta\alpha_n(1-\beta_n^1)\rho\left(\frac{h_n - f}{m}\right) + \delta^2\alpha_n\beta_n^1\rho\left(\frac{s_n^2 - f}{m}\right) \\ &\leq [1-(1-\delta)\alpha_n - \alpha_n\beta_n^1\delta]\rho\left(\frac{h_n - f}{m}\right) + \delta^2\alpha_n\beta_n^1\rho\left(\frac{s_n^2 - f}{m}\right).\end{aligned} \tag{3.17}$$

Similarly, successive applications of (1.8) and (3.3) give:

$$\begin{cases} \rho\left(\frac{s_n^2 - f}{m}\right) \leq (1-\beta_n^2)\rho\left(\frac{h_n - f}{m}\right) + \delta\beta_n^2\rho\left(\frac{s_n^3 - f}{m}\right) \\ \rho\left(\frac{s_n^3 - f}{m}\right) \leq (1-\beta_n^3)\rho\left(\frac{h_n - f}{m}\right) + \delta\beta_n^3\rho\left(\frac{s_n^4 - f}{m}\right) \\ \rho\left(\frac{s_n^{k-2} - f}{m}\right) \leq (1-\beta_n^{k-2})\rho\left(\frac{h_n - f}{m}\right) + \delta\beta_n^{k-2}\rho\left(\frac{s_n^{k-1} - f}{m}\right) \\ \rho\left(\frac{s_n^{k-1} - f}{m}\right) \leq (1-\beta_n^{k-1})\rho\left(\frac{h_n - f}{m}\right) + \delta\beta_n^{k-1}\rho\left(\frac{h_n - f}{m}\right) \end{cases} \tag{3.18}$$

Substituting (3.18) in (3.17), and simplifying, we obtain

$$r_{n,m} \leq [1-(1-\delta)\alpha_n]\rho\left(\frac{h_n - f}{m}\right). \tag{3.19}$$

Hence we have the equations:

$$\rho(h_{n+1} - f) \leq \delta_n + [1-(1-\delta)\alpha_n]\rho(h_n - f) \tag{3.20}$$

and if $m > 1$,

$$\rho\left(\frac{h_{n+1} - f}{m}\right) \leq \frac{\gamma_n}{m} + [1-(1-\delta)\alpha_n]\rho\left(\frac{h_n - f}{m}\right). \tag{3.21}$$

and if in addition $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$,

$$\rho(h_{n+1} - f) \leq \frac{1}{m}\varepsilon_n + [1-(1-\delta)\alpha_n]\rho(h_n - f). \tag{3.22}$$

(1) If $m = 1$, then from (3.20) and Lemma 1.2, $\lim_{n\to\infty}\delta_n = 0 \Rightarrow h_n \to f$. Thus, the fixed-point iteration (1.9) is relatively ρ-stable.

(2) Suppose now that $m > 1$ and that $\lim_{n\to\infty}\gamma_n = 0$.

Then by (3.21) and Lemma 1.2, $\rho\left(\frac{h_n - f}{m}\right) \to 0$ and $m\rho\left(\frac{h_n - f}{m}\right) \to 0$. Thus, the fixed-point iteration (1.9) is weakly ρ-stable.

(3) Suppose that $m > 1$ and that $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$. If $\lim_{n\to\infty}\varepsilon_n = 0$, then by (3.22) and Lemma 1.2, $h_n \to f$. Thus, the fixed-point iteration (1.9) is ρ-stable. ∎

Theorem 3.1 leads to the following corollary:

Corollary 3.1. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is an m-strong ρ-quasi-contractive-like mapping, satisfying contractive-like condition (3.7), where $m \in \mathbb{N}$. Suppose that $F_\rho(T) \neq \emptyset/$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be the explicit Noor (1.10), the explicit Ishikawa (1.11) or the explicit Mann (1.12) iterative sequence, where the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset [0,1)$ are such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0. Then $\{f_n\}$ is

(1) relatively ρ-stable with respect to T if $m = 1$;

(2) weakly ρ-stable with respect to T if $m > 1$.

(3) ρ-stable with respect to T if $m > 1$ and $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$ where $f \in F_\rho(T)$ (in this case, P_ρ^T is a ρ-quasi-contractive-like map).

3.2 Stability results for explicit multistep-SP iterative sequences in modular function spaces

Theorem 3.2. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is an m-strong

ρ-quasi-contractive-like mapping, satisfying contractive-like condition (3.7), where $m \in \mathbb{N}$. Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the explicit multistep iterative sequence (1.13), where the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset [0,1)$ $(i = 1, 2, \ldots, k-1)$ are such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0. Then, (1.13) is:

(1) relatively ρ-stable with respect to T if $m = 1$;

(2) weakly ρ-stable with respect to T if $m > 1$.

(3) ρ-stable with respect to T if $m > 1$ and $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$ where $f \in F_\rho(T)$ (in this case, P_ρ^T is a ρ-quasi-contractive-like map).

Proof. The method of proof is similar to that of Theorem 3.1. ■

Theorem 3.2 leads to the following corollary:

Corollary 3.2. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is an m-strong ρ-quasi-contractive-like mapping, satisfying contractive-like condition (3.7), where $m \in \mathbb{N}$. Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the explicit SP iterative sequence (1.14), with the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset [0, 1)$ such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0. Then (1.14) is:

(1) relatively ρ-stable with respect to T if $m = 1$;

(2) weakly ρ-stable with respect to T if $m > 1$;

(3) ρ-stable with respect to T if $m > 1$ and $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$ where $f \in F_\rho(T)$ (in this case, P_ρ^T is a ρ-quasi-contractive-like map).

3.3 Stability results for implicit multistep iterative sequences in modular function spaces

Theorem 3.3. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping satisfying property (I), such that P_ρ^T is an m-strong ρ-quasi-contractive-like mapping, satisfying contractive-like condition (3.7), where $m \in \mathbb{N}$. Suppose that $F_\rho(T) \neq \emptyset/$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the implicit multistep iterative sequence (1.15), where the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset [0, 1)$ $(i = 1, 2, \ldots, k-1)$ are such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0. Then, (1.15) is:

(1) relatively ρ-stable with respect to T if $m = 1$;

(2) weakly ρ-stable with respect to T if $m > 1$.

(3) ρ-stable with respect to T if $m > 1$ and $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$ where $f \in F_\rho(T)$ (in this case, P_ρ^T is a ρ-quasi-contractive-like map).

Proof.

Let $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset [0, 1)$ be sequences such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0. Suppose $f \in F_\rho(T)$. Let $\{h_n\}_{n=0}^{\infty}$ is an arbitrary sequence and set:

$$\begin{cases} \varepsilon_n = \rho\big(h_{n+1} - (1-\alpha_n)h_n^1 - \alpha_n z_{n+1}\big), \\ \delta_n = \rho(h_{n+1} - f) - \rho\big((1-\alpha_n)h_n^1 + \alpha_n z_{n+1} - f\big), \\ \gamma_n = \sup\limits_{\lambda\in(0,1]} \lambda\rho\left(\dfrac{h_{n+1} - (1-\alpha_n)h_n^1 - \alpha_n z_{n+1}}{\lambda}\right) \\ h_n^i = \big(1-\beta_n^i\big)h_n^{i+1} + \beta_n^i z_n^i, \quad i = 1, 2, \ldots, k-2 \\ h_n^{k-1} = \big(1-\beta_n^{k-1}\big)h_n + \beta_n^{k-1} z_n^{k-1}, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{3.23}$$

where $z_{n+1} \in P_\rho^T(h_{n+1})$, $z_n^i \in P_\rho^T(h_n^i)$, $i = 1, 2, \ldots, k-1$.

Let:

$$r_{n,m} = (1-\alpha_n)\rho\left(\frac{h_n^1 - f}{m}\right) + \frac{\alpha_n}{m}\rho(z_{n+1} - f). \tag{3.24}$$

By the convexity of ρ, we have:

$$\begin{aligned} \rho(h_{n+1} - f) &= \delta_n + \rho\big((1-\alpha_n)h_n^1 + \alpha_n z_{n+1} - f\big) \\ &= \delta_n + \rho\big((1-\alpha_n)\big(h_n^1 - f\big) + \alpha_n(z_{n+1} - f)\big) \\ &\le \delta_n + r_{n,1}. \end{aligned} \tag{3.25}$$

If $m > 1$, we have:

$$\begin{aligned} \rho\left(\frac{h_{n+1} - f}{m}\right) &= \rho\left(\frac{\alpha_n}{m}\frac{h_{n+1} - (1-\alpha_n)h_n^1 - \alpha_n z_{n+1}}{\alpha_n} + (1-\alpha_n)\frac{h_n^1 - f}{m} + \frac{\alpha_n}{m}(z_{n+1} - f)\right) \\ &\le \frac{\alpha_n}{m}\rho\left(\frac{h_{n+1} - (1-\alpha_n)h_n^1 - \alpha_n z_{n+1}}{\alpha_n}\right) + r_{n,m} \\ &\le \frac{\gamma_n}{m} + r_{n,m}. \end{aligned} \tag{3.26}$$

and if in addition $\forall g \in D\ \rho(g - f) = m\rho\left(\frac{g-f}{m}\right)$,

$$\begin{aligned} \rho\left(\frac{h_{n+1} - f}{m}\right) &= \rho\left(\frac{h_{n+1} - (1-\alpha_n)h_n^1 - \alpha_n z_{n+1}}{m} + \frac{1-\alpha_n}{m}\big(h_n^1 - f\big) + \frac{\alpha_n}{m}(z_{n+1} - f)\right) \\ &\le \frac{1}{m}\varepsilon_n + r_{n,m}. \end{aligned} \tag{3.27}$$

Since $z_{n+1} \in P_\rho^T(h_{n+1})$, from (3.24) and (3.7) we have that:

$$\begin{aligned} r_{n,m} &= (1-\alpha_n)\rho\left(\frac{h_n^1 - f}{m}\right) + \frac{\alpha_n}{m}\rho(z_{n+1} - f) \\ &= (1-\alpha_n)\rho\left(\frac{h_n^1 - f}{m}\right) + \frac{\alpha_n}{m} dist_\rho\left(z_{n+1}, P_\rho^T(f)\right) \\ &\leq (1-\alpha_n)\rho\left(\frac{h_n^1 - f}{m}\right) + \frac{\alpha_n}{m} H_\rho\left(P_\rho^T(h_{n+1}), P_\rho^T(f)\right) \\ &\leq (1-\alpha_n)\rho\left(\frac{h_n^1 - f}{m}\right) + \delta\alpha_n\rho\left(\frac{h_{n+1} - f}{m}\right). \end{aligned} \tag{3.28}$$

Using the convexity of ρ in (3.23), and the fact that $z_n^1 \in P_\rho^T(h_n^1)$, we have

$$\begin{aligned} \rho\left(\frac{h_n^1 - f}{m}\right) &\leq \left(1-\beta_n^1\right)\rho\left(\frac{h_n^2 - f}{m}\right) + \beta_n^1\rho\left(\frac{z_n^1 - f}{m}\right) \\ &\leq \left(1-\beta_n^1\right)\rho\left(\frac{h_n^2 - f}{m}\right) + \frac{\beta_n^1}{m} dist_\rho\left(z_n^1, P_\rho^T(f)\right) \\ &\leq \left(1-\beta_n^1\right)\rho\left(\frac{h_n^2 - f}{m}\right) + \frac{\beta_n^1}{m} H_\rho\left(P_\rho^T(h_n^1), P_\rho^T(f)\right) \\ &\leq \left(1-\beta_n^1\right)\rho\left(\frac{h_n^2 - f}{m}\right) + \delta\beta_n^1\rho\left(\frac{h_n^1 - f}{m}\right). \end{aligned}$$

Thus:

$$\rho\left(\frac{h_n^1 - f}{m}\right) \leq \left[\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right]\rho\left(\frac{h_n^2 - f}{m}\right). \tag{3.29}$$

Similarly, we have the following:

$$\rho\left(\frac{h_n^2 - f}{m}\right) \leq \left[\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right]\rho\left(\frac{h_n^3 - f}{m}\right) \tag{3.30}$$

$$\vdots$$

$$\rho\left(\frac{h_n^{k-2} - f}{m}\right) \leq \left[\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right]\rho\left(\frac{h_n^{k-1} - f}{m}\right) \tag{3.31}$$

$$\rho\left(\frac{h_n^{k-1}-f}{m}\right) \leq \left[\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right]\rho\left(\frac{h_n-f}{m}\right) \tag{3.32}$$

Substituting (3.29) – (3.32), and simplifying, we obtain

$$\begin{aligned} r_{n,m} &\leq (1-\alpha_n)\left[\prod_{i=1}^{m}\frac{1-\beta_n^{i-1}}{1-\delta\beta_n^{i-1}}\right]\rho\left(\frac{h_n-f}{m}\right)+\delta\alpha_n\rho\left(\frac{h_{n+1}-f}{m}\right) \\ &\leq (1-\alpha_n)\rho\left(\frac{h_n-f}{m}\right)+\delta\alpha_n\rho\left(\frac{h_{n+1}-f}{m}\right). \end{aligned} \tag{3.33}$$

Hence, substituting (3.33) in (3.25)–(3.27), we have the equations:

$$\rho(h_{n+1}-f) \leq \delta_n + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho(h_n-f) \tag{3.34}$$

and if $m > 1$,

$$\rho\left(\frac{h_{n+1}-f}{m}\right) \leq \frac{\gamma_n}{m} + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho\left(\frac{h_n-f}{m}\right). \tag{3.35}$$

and if in addition $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$,

$$\rho\left(\frac{h_{n+1}-f}{m}\right) \leq \frac{\varepsilon_n}{m} + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho\left(\frac{h_n-f}{m}\right). \tag{3.36}$$

(1) If $m = 1$, then from (3.34) and Lemma 1.2, $\lim_{n\to\infty}\delta_n = 0 \Rightarrow h_n \to f$. Thus the fixed-point iteration (1.15) is relatively ρ-stable.

(2) Suppose now that $m > 1$ and that $\lim_{n\to\infty}\gamma_n = 0$.
Then by (3.35) and Lemma 1.2, $\rho\left(\frac{h_n-f}{m}\right) \to 0$. Thus $m\rho\left(\frac{h_n-f}{m}\right) \to 0$. Thus, the fixed-point iteration (1.15) is weakly ρ-stable.

(3) Suppose that $m > 1$ and that $\forall g \in D\, \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$. If $\lim_{n\to\infty}\varepsilon_n = 0$, then by (3.36) and Lemma 1.2, $h_n \to f$. Thus, the fixed-point iteration (1.15) is ρ-stable. ■

Theorem 3.3 leads to the following corollary:

Corollary 3.3. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping satisfying property (I), such that P_ρ^T is an m-strong ρ-quasi-contractive-like mapping, satisfying contractive-like condition (3.7), where $m \in \mathbb{N}$. Suppose that $F_\rho(T) \neq \varnothing$. Let $f_0 \in D$ and $\{f_n\} \subset D$ be defined by the implicit Noor (1.16), implicit Ishikawa (1.17), implicit Mann (1.18) iterative sequence respectively, where the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset [0,1)$ are such that $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0. Then, (1.16)–(1.18) are:

(1) relatively ρ-stable with respect to T if $m = 1$;

(2) weakly ρ-stable with respect to T if $m > 1$.

(3) ρ-stable with respect to T if $m > 1$ and $\forall g \in D\ \rho(g-f) = m\rho\left(\frac{g-f}{m}\right)$ where $f \in F_\rho(T)$ (in this case, P_ρ^T is a ρ-quasi-contractive-like map).

3.3.1 Numerical example. Let $M[0,1]$ be the collection of all real-valued measurable functions on $[0,1]$ and $\rho : M[0,\ 1] \to \mathbb{R}$ a convex function modular defined by $\rho(f) = \int_0^1 |f|$ $\forall f \in M[0,\ 1]$. Let $D = \{f \in L_\rho :\ 0 \le f(x) \le 2\ \forall x \in [0,\ 1]\}$ be a subset of the modular function space $L_\rho = M[0,\ 1]$ defined by ρ. D is nonempty, closed and convex.

Define map $T : D \to P_\rho(D)$ by $Tf = \{\delta f\}$, where $\delta = 0.9$. T satisfies property (*I*), has a unique fixed point $f = 0$ (since $0 \in T(0)$), and P_ρ^T is a ρ-contraction, with $P_\rho^T(f) = \{Tf\}\ \forall f \in D$. In fact, P_ρ^T is an m-strong ρ-strong contraction for all $m \in \mathbb{N}$, since $\rho(g) = m\rho\left(\frac{g}{m}\right)$.

We present the results of convergence to $f = 0$ of a multistep iterative sequence (1.9), an explicit multistep-SP iterative sequence (1.13) and an implicit multistep iterative sequence (1.15) using MATLAB. The parameters used are the following: $f_0(x) = 0.5x + 0.95\ \forall x \in [0,\ 1]$, $\alpha_n = \frac{1}{4} + \frac{1}{n+2}$, $\beta_n^i = \frac{1}{n+2}$ for $i = 1,\ 2,\ \ldots,\ k-1$, where $k = 11$ and $n = 1, 2, \ldots, 100$ (see Tables 1 and 2).

N	Explicit multistep $f_n(x)$	Explicit multistep-SP $f_n(x)$	Implicit multistep $f_n(x)$
0	0.5000x + 0.9500	0.5000x + 0.9500	0.5000x + 0.9500
1	0.4583x + 0.8708	0.3470x + 0.6593	0.3904x + 0.7418
⋮	⋮	⋮	⋮
16	0.2461x + 0.4676	0.0443x + 0.0842	0.0695x + 0.1320
17	0.2383x + 0.4527	*0.0410x + 0.0779*	0.0648x + 0.1230
⋮	⋮	⋮	⋮
24	0.1917x + 0.3642	0.0252x + 0.0480	0.0417x + 0.0792
25	0.1860x + 0.3534	0.0237x + 0.0450	*0.0394x + 0.0748*
⋮	⋮	⋮	⋮
60	0.0690x + 0.1311	0.0043x + 0.0081	0.0080x + 0.0152
61	0.0671x + 0.1276	*0.0041x + 0.0078*	0.0077x + 0.0146
⋮	⋮	⋮	⋮
77	0.0435x + 0.0827	0.0022x + 0.0042	0.0042x + 0.0080
78	0.0424x + 0.0805	0.0021x + 0.0040	*0.0041x + 0.0077*
79	*0.0412x + 0.0783*	0.0020x + 0.0039	0.0039x + 0.0075
⋮	⋮	⋮	⋮
101	0.0229x + 0.0435	0.0009x + 0.0017	0.0018x + 0.0035

Table 1. Convergence

N	Explicit multistep	Explicit multistep-SP	Implicit multistep
0	1.2	1.2	1.2
⋮	⋮	⋮	⋮
16	0.5906	0.1064	0.1667
17	0.5719	*0.0984*	0.1554
⋮	⋮	⋮	⋮
24	0.4601	0.0606	0.1001
25	0.4464	0.0569	*0.0945*
⋮	⋮	⋮	⋮
60	0.1656	0.0103	0.0192
61	0.1611	*0.0099*	0.0184
⋮	⋮	⋮	⋮
77	0.1044	0.0053	0.0101
78	0.1016	0.0051	*0.0098*
79	*0.0990*	0.0049	0.0094
⋮	⋮	⋮	⋮
101	0.0550	0.0022	0.0044

Table 2. Approximates $\rho(f_n - f)$

For this example, the explicit multistep-SP sequence seems to converge to the fixed point $f = 0$ slightly faster than the implicit multistep sequence, with approximates $\rho(f_n - f)$ under 10^{-2} at $n = 17$ and $n = 25$ respectively, while the explicit multistep sequence is considerably slower, with $\rho(f_n - f) < 10^{-2}$ only from $n = 79$.

4. Conclusion

In Theorems 2.1–2.3, the fixed points of multivalued maps T with a ρ-contractive-like associate map P_ρ^T in modular spaces are successfully approximated, with supporting proofs and a numerical example, via the explicit multistep (1.9), the explicit multistep-SP (1.13) and the implicit multistep (1.15) iterative sequences. These sequences involve more steps ($k \geq 1$) than the iterations considered in [6, 7].

In an attempt to prove the stability of these iterations, a new approach is used to match the convexity structure of ρ: the concepts of relative ρ-stability (3.3) and weak ρ-stability (3.4) are introduced for the first time in literature, as well as the notions of m-strong ρ-quasi-contraction types (3.5)–(3.7), where $m \in \mathbb{N}$, which coincide with quasi-contraction types when ρ is nonnegative homogeneous. Theorems 3.1–3.3 then state conditions under which schemes (1.9), (1.13) and (1.15) are ρ-stable, relatively ρ-stable and weakly ρ-stable, when P_ρ^T is an m-strong ρ-quasi-contractive-like mapping. The proofs of this theorem are fundamentally different from those of parallel results in metric spaces as they elegantly cut out the use of triangle inequality.

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