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Qualitative properties and approximate solutions of thermostat fractional dynamics system via a nonsingular kernel operator

Received 23 June 2022
Revised 8 December 2022
Accepted 8 January 2023

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Abstract

Purpose – This paper aims to study qualitative properties and approximate solutions of a thermostat dynamics system with three-point boundary value conditions involving a nonsingular kernel operator which is called Atangana-Baleanu-Caputo (ABC) derivative for the first time. The results of the existence and uniqueness of the solution for such a system are investigated with minimum hypotheses by employing Banach and Schauder's fixed point theorems. Furthermore, Ulam-Hyers ($\mathcal{UH}$) stability, Ulam-Hyers-Rassias $\mathcal{UHR}$ stability and their generalizations are discussed by using some topics concerning the nonlinear functional analysis. An efficiency of Adomian decomposition method (ADM) is established in order to estimate approximate solutions of our problem and convergence theorem is proved. Finally, four examples are exhibited to illustrate the validity of the theoretical and numerical results.

Design/methodology/approach – This paper considered theoretical and numerical methodologies.

Findings – This paper contains the following findings: (1) Thermostat fractional dynamics system is studied under ABC operator. (2) Qualitative properties such as existence, uniqueness and Ulam–Hyers–Rassias stability are established by fixed point theorems and nonlinear analysis topics. (3) Approximate solution of the problem is investigated by Adomain decomposition method. (4) Convergence analysis of ADM is proved. (5) Examples are provided to illustrate theoretical and numerical results. (6) Numerical results are compared with exact solution in tables and figures.

Originality/value – The novelty and contributions of this paper is to use a nonsingular kernel operator for the first time in order to study the qualitative properties and approximate solution of a thermostat dynamics system.

Keywords Atangana–Baleanu–Caputo operator, Fractional boundary value problem, Thermostat dynamics system, Fixed point theorem, Adomian decomposition method

Paper type Research paper

JEL Classification — 34A08, 34B15, 65L10

The authors express their gratitude to the unknown referees for their helpful suggestions which improved the final version of this paper.

Arab Journal of Mathematical Sciences
Vol. 30 No. 2, 2024
pp. 197-217
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-06-2022-0147

1. Introduction

Fractional differential equations have been lately used as advantageous tools to learn about the modeling of many real phenomena. Comparing with integer derivatives, the most essential benefit of fractional derivatives is that it describes the quality of a heredity and memory of diverse materials and processes. For more important points about fractional calculus and its applications, we refer to these works [1–10, 49, 50], and the references given therein. Probably, sometimes the nonlocal fractional operators via a singular kernel cannot describe the complicated dynamics systems. Thus, the researchers used a new approach and another tool to provide different options for improving the description of real models of phenomena. For this regarding, there appeared new fractional operators with a nonsingular kernel [11–14]. Indeed, the most optimal emulative operator among a nonsingular kernel operator is that which depends on Mittag–Leffler function, which is called Atangana–Baleanu–Caputo (ABC) operator [12]. In view of this, many authors employed ABC derivative to study fractional differential equations and modeling of the infectious diseases, we refer to these works [15–20]. Particularly, Alnahdi *et al.* [21], studied the existence, uniqueness and continuous dependence of solutions of the nonlinear implicit fractional differential equation with nonlocal conditions involving the ABC fractional derivative. Furthermore, a lot of excellent materials on the mathematical models with different derivative operators applied to model real-life phenomena such as [22–28].

Adomian [29, 30], used ADM for estimating approximate solutions of integral equations, integro-differential equations, ordinary and partial differential equations, etc. Recently, the ADM algorithm received attention of researchers in fractional differential equations field; for more information see refs. [31–35] and the references therein.

On the other hand, the thermostat control is considered as the best physio-electrical type. A thermostat is a gauge device that regulates and measures the temperature of a particular physical model and takes a procedure related to its temperature, which is closed to a fit and preferred degree. This instrument is utilized in any controlling units and industrial systems, which includes building central heating, medical incubators, water heaters, refrigerators ovens, air conditioners and even vehicle engines, which increase or minimize the temperature.

In 2006, Infante and Webb [36], studied the following mathematical system for thermostat model:

$$\begin{cases} \mathfrak{u}''(s) + \mathfrak{h}(s, \mathfrak{u}(s)) = 0, s \in I = [0, 1], \\ \mathfrak{u}'(0) = 0, \qquad \eta\, \mathfrak{u}'(1) + \mathfrak{u}(\zeta) = 0, \end{cases} \tag{1.1}$$

where $\zeta \in I$ and $\eta > 0$. Furthermore, recently some authors extended equation (1.1) to fractional derivative of singular kernel such as Nieto and Pimentel [37], transferred the problem (1.1) to a Caputo fractional version. Baleanu *et al.* [38], formulated a hybrid fractional equation and inclusion forms for a thermostat dynamics system of fractional-order. Very recently, Etemad *et al.* [39], studied the qualitative properties of the solution for a new composition of the generalized thermostat dynamics model with multi-point by means of $\mu - \varphi$-contraction. For more research papers related to thermostat dynamics model, see these works [14, 40].

Motivated by the above ideas, the target of this paper is to investigate the existence, uniqueness, stability and approximate solutions of the following thermostat fractional differential equation involving ABC derivative:

$$^{ABC}\mathfrak{D}^{\sigma}\mathfrak{u}(s) + \mathfrak{h}(s, \mathfrak{u}(s)) = 0, s \in I = [0, 1], \tag{1.2}$$

with three-point boundary value conditions

$$\mathfrak{u}'(0) = \mathfrak{g}_1, \qquad \eta\, \mathfrak{u}'(1) + \mathfrak{u}(\zeta) = \mathfrak{g}_2, \tag{1.3}$$

where ${}^{ABC}\mathfrak{D}^{\sigma}$ denotes the σth ABC fractional derivative such that $\sigma \in (1, 2]$. The constants $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbb{R}, \mathfrak{h} : J \times \mathbb{R} \to \mathbb{R}$ be continuous, $\zeta \in I$ and the parameter $\eta > 0$. The novelty and contributions of this paper is to use a nonsingular kernel operator for the first time in order to study the qualitative properties and approximate solution of a thermostat dynamics system. Since singular kernel sometimes creates difficulty during numerical analysis. This is because of its local singular kernel. So, in order to overcome this difficulty, we use ABC operator of nonlocal nonsingular kernel type derivative.

Our manuscript is structured as follows: Several needful preliminaries are provided in Sec.2. The existence and uniqueness results are given in Sec.3. The $\mathcal{UH}$ stability and $\mathcal{UHR}$ stability results are investigated in Sec.4. An approximate solution and its convergence for our problem are established by ADM in Sec.5. Finally, four examples represent the validity of the main findings which are provided in Sec.6.

2. Preliminaries

Here, we will introduce several needful preliminaries for nonlinear analysis and fractional calculus [11, 12, 41–43]. In addition, we conclude an equivalent fractional integral equation corresponding to the thermostat fractional dynamics system (1.2)–(1.3).

We denote by $\mathcal{C}(I, \mathbb{R})$ the Banach space of all continuous functions equipped with usual norm $\|\mathfrak{u}\| = \sup\{|\mathfrak{u}(s)| : s \in I\}$.

Definition 2.1 *Consider $\sigma \in (0, 1]$ and $\mathfrak{h} \in \mathcal{H}^1(0, T)$. The σth left-sided ABC fractional derivative with the lower limit zero for a function $\mathfrak{h}$ is given by*

$$\left({}^{ABC}\mathfrak{D}^{\sigma}\mathfrak{h}\right)(s) = \frac{\phi(\sigma)}{1-\sigma}\int_0^s \mathbb{E}_{\sigma}\left(-\sigma\frac{(s-t)^{\sigma}}{1-\sigma}\right)\mathfrak{h}'(t)dt, \; s > 0, \tag{2.1}$$

and the associated σth left-sided AB fractional integral is given by

$$\left({}^{AB}\mathfrak{J}^{\sigma}\mathfrak{h}\right)(s) = \frac{1-\sigma}{\phi(\sigma)}\mathfrak{h}(s) + \frac{\sigma}{\phi(\sigma)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\mathfrak{h}(t)dt, \; s > 0, \tag{2.2}$$

where $\phi(\sigma)$ is the normalization function with $\phi(0) = \phi(1) = 1$, and $\mathbb{E}_{\sigma}$ is called the Mittag–Leffler function defined by

$$\mathbb{E}_{\sigma}(r) = \sum_{k=0}^{\infty}\frac{r^k}{\Gamma(\sigma k + 1)}, \tag{2.3}$$

here $\mathrm{Re}(\sigma) > 0, r \in \mathbb{C}$ and $\Gamma(.)$ is a well-known Gamma function.

Definition 2.2. *Consider $\mathfrak{h}^{(n)} \in \mathcal{H}^1(0, T)$ and $\sigma \in (n, n+1]$, $n = 0, 1, 2, \ldots$. Then, ABC fractional derivative satisfies*

$$\left({}^{ABC}\mathfrak{D}^{\sigma}\mathfrak{h}\right)(s) = \left({}^{ABC}\mathfrak{D}^{\sigma}\mathfrak{h}^{(n)}\right)(s),$$

and the associated fractional integral

$$\left({}^{AB}\mathfrak{J}^{\sigma}\mathfrak{h}\right)(s) = \left(I^n \, {}^{AB}\mathfrak{J}^{\vartheta}\mathfrak{h}\right)(s),$$

where $\vartheta = \sigma - n$ and I^n is an usual nth integral.

Lemma 2.1. *For $\sigma \in (n, n+1]$, $n = 0, 1, 2, \ldots$, the following relation holds:*

$$ {}^{AB}\mathfrak{I}^{\sigma}\left({}^{ABC}\mathfrak{D}^{\sigma}\mathfrak{h}\right)(s) = \mathfrak{h}(s) + c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n, $$

for an arbitrary constant c_j with $j = 0, 1, 2, \ldots, n$.

In the subsequent lemma, we derive an equivalent fractional integral equation corresponding to the system (1.2)–(1.3).

Lemma 2.2. *Let $\sigma \in (1, 2]$ and $\mathfrak{h} \in \mathcal{C}(I, \mathbb{R})$ with $\mathfrak{h}(0) = \mathfrak{h}(1) = 0$. Then, the system:*

$$ {}^{ABC}\mathfrak{D}^{\sigma}\mathfrak{u}(s) + \mathfrak{h}(s) = 0, \quad s \in I = [0, 1], \tag{2.4} $$

$$ \mathfrak{u}'(0) = \mathfrak{g}_1, \qquad \eta\, \mathfrak{u}'(1) + \mathfrak{u}(\zeta) = \mathfrak{g}_2, \tag{2.5} $$

has a solution given by:

$$ \begin{aligned} \mathfrak{u}(s) = \quad & \mathfrak{g}_2 + (s - \eta - \zeta)\mathfrak{g}_1 + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{\eta}{\Gamma(\sigma - 1)} \int_0^1 (1 - t)^{\sigma - 2}\mathfrak{h}(t)dt \\ & + \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^{\zeta} \mathfrak{h}(t)dt + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^{\zeta} (\zeta - t)^{\sigma - 1}\mathfrak{h}(t)dt \\ & - \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^{s} \mathfrak{h}(t)dt - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^{s} (s - t)^{\sigma - 1}\mathfrak{h}(t)dt. \end{aligned} \tag{2.6} $$

Proof. Consider $\mathfrak{h}$ satisfying the system (2.4)–(2.5). Then by applying σth AB fractional integral operator on both sides of (2.4) and using Lemma 2.1, we have

$$ \mathfrak{u}(s) = c_1 + c_2 s - \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^{s} \mathfrak{h}(t)dt - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^{s} (s - t)^{\sigma - 1}\mathfrak{h}(t)dt, \tag{2.7} $$

where $c_1, c_2 \in \mathbb{R}$. It follows that

$$ \mathfrak{u}'(s) = c_2 - \frac{2 - \sigma}{\phi(\sigma - 1)}\mathfrak{h}(s) - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma - 1)} \int_0^{s} (s - t)^{\sigma - 2}\mathfrak{h}(t)dt. \tag{2.8} $$

Now, by using the first nonlocal boundary value condition $\mathfrak{u}'(0) = \mathfrak{g}_1$, and the fact $\mathfrak{h}(0) = 0$, we get

$$ c_2 = \mathfrak{g}_1. \tag{2.9} $$

Next, by applying the second nonlocal boundary value condition $\eta\, \mathfrak{u}'(1) + \mathfrak{u}(\zeta) = \mathfrak{g}_2$, and by using $\mathfrak{h}(1) = 0,\ c_2 = \mathfrak{g}_1$, we obtain the following.

$$ \begin{aligned} & \eta\mathfrak{g}_1 - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{\eta}{\Gamma(\sigma - 1)} \int_0^1 (1 - t)^{\sigma - 2}\mathfrak{h}(t)dt \\ & + c_1 + \zeta\mathfrak{g}_1 - \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^{\zeta} \mathfrak{h}(t)dt - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^{\zeta} (\zeta - t)^{\sigma - 1}\mathfrak{h}(t)dt = \mathfrak{g}_2, \end{aligned} $$

which yields:

$$
\begin{aligned}
c_1 = \; & \mathfrak{g}_2 - (\eta + \zeta)\mathfrak{g}_1 + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{\eta}{\Gamma(\sigma - 1)} \int_0^1 (1 - t)^{\sigma - 2} \mathfrak{h}(t) dt \\
& + \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^\zeta \mathfrak{h}(t) dt + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta - t)^{\sigma - 1} \mathfrak{h}(t) dt.
\end{aligned}
$$

Substituting the values of c_1 and c_2 in (2.7), we have:

$$
\begin{aligned}
\mathfrak{u}(s) = \; & \mathfrak{g}_2 + (s - \eta - \zeta)\mathfrak{g}_1 + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{\eta}{\Gamma(\sigma - 1)} \int_0^1 (1 - t)^{\sigma - 2} \mathfrak{h}(t) dt \\
& + \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^\zeta \mathfrak{h}(t) dt + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta - t)^{\sigma - 1} \mathfrak{h}(t) dt \\
& - \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^s \mathfrak{h}(t) dt - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^s (s - t)^{\sigma - 1} \mathfrak{h}(t) dt.
\end{aligned}
$$

As an outcome of Lemma 2.2, we have the next lemma:

Lemma 2.3. *Consider* $\sigma \in (1, 2]$ *and* $\mathfrak{h} \in \mathcal{C}(I \times \mathbb{R}, \mathbb{R})$ *with* $\mathfrak{h}(0, \mathfrak{u}(0)) = \mathfrak{h}(1, \mathfrak{u}(1)) = 0$. *Then, the solution of the system* (1.2)–(1.3) *is given by*

$$
\begin{aligned}
\mathfrak{u}(s) = \; & \mathfrak{g}_2 + (s - \eta - \zeta)\mathfrak{g}_1 + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{\eta}{\Gamma(\sigma - 1)} \int_0^1 (1 - t)^{\sigma - 2} \mathfrak{h}(t, \mathfrak{u}(t)) dt \\
& + \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^\zeta \mathfrak{h}(t, \mathfrak{u}(t)) dt + \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta - t)^{\sigma - 1} \mathfrak{h}(t, \mathfrak{u}(t)) dt \\
& - \frac{2 - \sigma}{\phi(\sigma - 1)} \int_0^s \mathfrak{h}(t, \mathfrak{u}(t)) dt - \frac{\sigma - 1}{\phi(\sigma - 1)} \frac{1}{\Gamma(\sigma)} \int_0^s (s - t)^{\sigma - 1} \mathfrak{h}(t, \mathfrak{u}(t)) dt.
\end{aligned}
\tag{2.10}
$$

Now, we will state the Banach and Schauder's fixed point theorems, respectively.

Theorem 2.1. [43] *Consider* $\Pi : \mathcal{Y} \to \mathcal{Y}$ *as a contraction operator such that* $\mathcal{Y}$ *is a Banach space. Then, there is only one fixed point for* Π *in* $\mathcal{Y}$.

Theorem 2.2. [43] *Consider* $\mathcal{D}$ *as a closed, bounded and convex subset of a Banach space* $\mathcal{Y}$. *If* $\Pi : \mathcal{D} \to \mathcal{D}$ *is a continuous mapping such that* $\Pi\mathcal{D}$ *is relatively compact and* $\Pi\mathcal{D} \subset \mathcal{Y}$, *then there is at least one fixed point for* Π *in* $\mathcal{D}$.

3. Existence and uniqueness of solution

Firstly, we will discuss the existence and uniqueness of the solution for the system (1.2)–(1.3) by using Banach's fixed point theorem. In view of this, we are in need of the next hypothesis:

H1. $[(\mathcal{H}_1)]$ Let $\mathfrak{h} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with $\mathfrak{h}(0, \mathfrak{u}(0)) = \mathfrak{h}(1, \mathfrak{u}(1)) = 0$, and there is a constant $\ell_1 > 0$ such that

$$
|\mathfrak{h}(s, \mathfrak{u}_1) - \mathfrak{h}(s, \mathfrak{u}_2)| \leq \ell_1 \|\mathfrak{u}_1 - \mathfrak{u}_2\|,
$$

for all $s \in I = [0, 1]$ and $\mathfrak{u}_j \in \mathbb{R}$ $(j = 1, 2)$.

Theorem 3.1. *Let $(\mathcal{H}_1)$ be fulfilled. If*

$$\Upsilon := \frac{\ell_1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)} + \frac{\ell_1(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\ell_1(\zeta^\sigma+1)}{\Gamma(\sigma+1)} < 1, \tag{3.1}$$

then, the system (1.2)–(1.3) has only one solution.

Proof. Define a mapping $\Omega : \mathcal{C}(I,\mathbb{R}) \to \mathcal{C}(I,\mathbb{R})$ as follows:

$$\begin{aligned}(\Omega\mathfrak{u})(s) = \; & \mathfrak{g}_2 + (s-\eta-\zeta)\mathfrak{g}_1 + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}\mathfrak{h}(t,\mathfrak{u}(t))dt \\ & + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^\zeta \mathfrak{h}(t,\mathfrak{u}(t))dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^\zeta (\zeta-t)^{\sigma-1}\mathfrak{h}(t,\mathfrak{u}(t))dt \\ & - \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s \mathfrak{h}(t,\mathfrak{u}(t))dt - \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\mathfrak{h}(t,\mathfrak{u}(t))dt.\end{aligned} \tag{3.2}$$

In order to show the system (1.2)–(1.3) has a unique solution, we will verify that a mapping Ω has a unique fixed point. Indeed, by utilizing $(\mathcal{H}_1)$, then for $\mathfrak{u},\mathfrak{v} \in \mathcal{C}(I,\mathbb{R})$ and $s \in I$, we have

$$\begin{aligned}& |(\Omega\mathfrak{u})(s) - (\Omega\mathfrak{v})(s)| \\ \le & \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}|\mathfrak{h}(t,\mathfrak{u}(t)) - \mathfrak{h}(t,\mathfrak{v}(t))|dt \\ & + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^\zeta |\mathfrak{h}(t,\mathfrak{u}(t)) - \mathfrak{h}(t,\mathfrak{v}(t))|dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^\zeta (\zeta-t)^{\sigma-1}|\mathfrak{h}(t,\mathfrak{u}(t)) - \mathfrak{h}(t,\mathfrak{v}(t))|dt \\ & + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s |\mathfrak{h}(t,\mathfrak{u}(t)) - \mathfrak{h}(t,\mathfrak{v}(t))|dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}|\mathfrak{h}(t,\mathfrak{u}(t)) - \mathfrak{h}(t,\mathfrak{v}(t))|dt \\ \le & \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}\ell_1|\mathfrak{u}(t) - \mathfrak{v}(t)|dt \\ & + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^\zeta \ell_1|\mathfrak{u}(t) - \mathfrak{v}(t)|dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^\zeta (\zeta-t)^{\sigma-1}\ell_1|\mathfrak{u}(t) - \mathfrak{v}(t)|dt \\ & + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s \ell_1|\mathfrak{u}(t) - \mathfrak{v}(t)|dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\ell_1|\mathfrak{u}(t) - \mathfrak{v}(t)|dt \\ \le & \frac{\ell_1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\|\mathfrak{u}-\mathfrak{v}\| \\ & + \frac{\ell_1\zeta(2-\sigma)}{\phi(\sigma-1)}\|\mathfrak{u}-\mathfrak{v}\| + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\ell_1\zeta^\sigma}{\Gamma(\sigma+1)}\|\mathfrak{u}-\mathfrak{v}\| \\ & + \frac{\ell_1(2-\sigma)}{\phi(\sigma-1)}\|\mathfrak{u}-\mathfrak{v}\| + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\ell_1}{\Gamma(\sigma+1)}\|\mathfrak{u}-\mathfrak{v}\| \\ \le & \left[\frac{\ell_1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)} + \frac{\ell_1(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\ell_1(\zeta^\sigma+1)}{\Gamma(\sigma+1)}\right]\|\mathfrak{u}-\mathfrak{v}\|.\end{aligned}$$

Hence,

$$\|\Omega\mathfrak{u} - \Omega\mathfrak{v}\| \le \Upsilon\|\mathfrak{u}-\mathfrak{v}\|.$$

Then, in view of the condition (3.1), the mapping Ω is contraction. Therefore, according to Theorem 2.1, there exists one fixed point for a mapping Ω, which represent a solution of the system (1.2)–(1.3).

Secondly, before stating and proving the second existence result by utilizing Schauder's fixed point theorem, we list the next hypothesis as follows.

H2. $[(\mathcal{H}_2)]$ Let $\mathfrak{h} : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function with $\mathfrak{h}(0,\mathfrak{u}(0)) = \mathfrak{h}(1,\mathfrak{u}(1)) = 0$, and there is a real number $\ell_2 > 0$ such that $|\mathfrak{h}(s,\mathfrak{u})| \le \ell_2(1+\|\mathfrak{u}\|)$, for all $s\in I=[0,1]$ and $\mathfrak{u}\in\mathbb{R}$.

Theorem 3.2. *Suppose that the hypothesis* $(\mathcal{H}_2)$ *holds. Then there will be at least one solution found for the system* (1.2)–(1.3), *provided that:*

$$\Psi = \frac{\ell_2}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)} + \frac{\ell_2(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\ell_2(\zeta^\sigma+1)}{\Gamma(\sigma+1)} < 1. \tag{3.3}$$

Proof. Consider an operator $\Omega : \mathcal{C}(I,\mathbb{R})\to\mathcal{C}(I,\mathbb{R})$ as defined in (3.2). Let the ball $\mathfrak{B}_\varrho = \{\mathfrak{u}\in\mathcal{C}(I,\mathbb{R}) : \|\mathfrak{u}\|\le\varrho\}$ with $\varrho \ge \frac{|\mathfrak{g}_2|+|(s-\eta-\zeta)|\ |\mathfrak{g}_1|+\Psi}{1-\Psi}$ and $\Psi < 1$.

Now, we prove that $(\Omega\mathfrak{B}_\varrho)\subset\mathfrak{B}_\varrho$. By using the hypothesis $(\mathcal{H}_2)$, we get

$$\begin{aligned}
&|(\Omega\mathfrak{u})(s)|\\
&\le|\mathfrak{g}_2| + |(s-\eta-\zeta)|\ |\mathfrak{g}_1|\\
&\quad+\frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1(1-t)^{\sigma-2}|\mathfrak{h}(t,\mathfrak{u}(t))|dt\\
&\quad+\frac{2-\sigma}{\phi(\sigma-1)}\int_0^\zeta|\mathfrak{h}(t,\mathfrak{u}(t))|dt+\frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^\zeta(\zeta-t)^{\sigma-1}|\mathfrak{h}(t,\mathfrak{u}(t))|dt\\
&\quad+\frac{2-\sigma}{\phi(\sigma-1)}\int_0^s|\mathfrak{h}(t,\mathfrak{u}(t))|dt+\frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s(s-t)^{\sigma-1}|\mathfrak{h}(t,\mathfrak{u}(t))|dt\\
&\le|\mathfrak{g}_2| + |(s-\eta-\zeta)|\ |\mathfrak{g}_1|\\
&\quad+\frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1(1-t)^{\sigma-2}\ell_2(1+|\mathfrak{u}(t)|)dt\\
&\quad+\frac{2-\sigma}{\phi(\sigma-1)}\int_0^\zeta\ell_2(1+|\mathfrak{u}(t)|)dt+\frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^\zeta(\zeta-t)^{\sigma-1}\ell_2(1+|\mathfrak{u}(t)|)dt\\
&\quad+\frac{2-\sigma}{\phi(\sigma-1)}\int_0^s\ell_2(1+|\mathfrak{u}(t)|)dt+\frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s(s-t)^{\sigma-1}\ell_2(1+|\mathfrak{u}(t)|)dt\\
&\le|\mathfrak{g}_2| + |(s-\eta-\zeta)|\ |\mathfrak{g}_1|\\
&\quad+\frac{1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\ell_2(1+\|\mathfrak{u}\|)\\
&\quad+\frac{\zeta(2-\sigma)}{\phi(\sigma-1)}\ell_2(1+\|\mathfrak{u}\|)+\frac{\sigma-1}{\phi(\sigma-1)}\frac{\zeta^\sigma}{\Gamma(\sigma+1)}\ell_2(1+\|\mathfrak{u}\|)\\
&\quad+\frac{(2-\sigma)}{\phi(\sigma-1)}\ell_2(1+\|\mathfrak{u}\|)+\frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma+1)}\ell_2(1+\|\mathfrak{u}\|)\\
&\le|\mathfrak{g}_2| + |(s-\eta-\zeta)|\ |\mathfrak{g}_1|\\
&\quad+\left[\frac{\ell_2}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}+\frac{\ell_2(\zeta+1)(2-\sigma)}{\phi(\sigma-1)}+\frac{\sigma-1}{\phi(\sigma-1)}\frac{\ell_2(\zeta^\sigma+1)}{\Gamma(\sigma+1)}\right](1+\|\mathfrak{u}\|).
\end{aligned}$$

For $\mathfrak{u} \in \mathfrak{B}_{\varrho}$, we have

$$|\Omega\mathfrak{u}| \le |\mathfrak{g}_2| + |(s-\eta-\zeta)|\ |\mathfrak{g}_1| + \Psi(1+\varrho) \le \varrho.$$

Hence, $(\Omega\mathfrak{B}_{\varrho}) \subset \mathfrak{B}_{\varrho}$.

Next, we show that a mapping Ω be continuous. Let $\{\mathfrak{u}_n\}$ is a sequence convergence to $\mathfrak{u}$ in $\mathfrak{B}_{\varrho}$ as $n \to \infty$. Then for all $s \in I$, we obtain

$$\begin{aligned}
&|(\Omega\mathfrak{u}_n)(s) - (\Omega\mathfrak{u})(s)| \\
&\le \frac{\sigma-1}{\phi(\sigma-1)} \frac{\eta}{\Gamma(\sigma-1)} \int_0^1 (1-t)^{\sigma-2} |\mathfrak{h}(t,\mathfrak{u}_n(t)) - \mathfrak{h}(t,\mathfrak{u}(t))| dt \\
&\quad + \frac{2-\sigma}{\phi(\sigma-1)} \int_0^{\zeta} |\mathfrak{h}(t,\mathfrak{u}_n(t)) - \mathfrak{h}(t,\mathfrak{u}(t))| dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^{\zeta} (\zeta-t)^{\sigma-1} |\mathfrak{h}(t,\mathfrak{u}_n(t)) - \mathfrak{h}(t,\mathfrak{u}(t))| dt \\
&\quad + \frac{2-\sigma}{\phi(\sigma-1)} \int_0^{s} |\mathfrak{h}(t,\mathfrak{u}_n(t)) - \mathfrak{h}(t,\mathfrak{u}(t))| dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^{s} (s-t)^{\sigma-1} |\mathfrak{h}(t,\mathfrak{u}_n(t)) - \mathfrak{h}(t,\mathfrak{u}(t))| dt \\
&\le \frac{\|\mathfrak{h}(.,\mathfrak{u}_n(.)) - \mathfrak{h}(.,\mathfrak{u}(.))\|}{\phi(\sigma-1)} \frac{\eta}{\Gamma(\sigma-1)} + \frac{(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} \|\mathfrak{h}(.,\mathfrak{u}_n(.)) - \mathfrak{h}(.,\mathfrak{u}(.))\| \\
&\quad + \frac{(\zeta^{\sigma}+1)(\sigma-1)}{\phi(\sigma-1)} \frac{\|\mathfrak{h}(.,\mathfrak{u}_n(.)) - \mathfrak{h}(.,\mathfrak{u}(.))\|}{\Gamma(\sigma+1)}.
\end{aligned}$$

According to the continuity of the function $\mathfrak{h}$, we find:

$$\|\Omega\mathfrak{u}_n - \Omega\mathfrak{u}\| \to 0 \quad as \quad n \to \infty.$$

So, Ω is continuous on $\mathfrak{B}_{\varrho}$.

Subsequently, we show that $\Omega(\mathfrak{B}_{\varrho})$ is relatively compact. Since we have $(\Omega\mathfrak{B}_{\varrho}) \subset \mathfrak{B}_{\varrho}$, hence $(\Omega\mathfrak{B}_{\varrho})$ is an uniformly bounded.

To establish that a mapping Ω be equicontinuous operator in $\mathfrak{B}_{\varrho}$, let $\mathfrak{u} \in \mathfrak{B}_{\varrho}$ and $s_1, s_2 \in I$ with $s_1 < s_2$. Then, by using $(\mathcal{H}_2)$, we have

$$\begin{aligned}
&|(\Omega\mathfrak{u})(s_1) - (\Omega\mathfrak{u})(s_2)| \\
&\le |(s_1-s_2)|\ |\mathfrak{g}_1| + \frac{(2-\sigma)}{\phi(\sigma-1)} \left| \int_0^{s_1} \mathfrak{h}(t,\mathfrak{u}(t)) dt - \int_0^{s_2} \mathfrak{h}(t,\mathfrak{u}(t)) dt \right| \\
&\quad + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \left| \int_0^{s_1} (s_1-t)^{\sigma-1} \mathfrak{h}(t,\mathfrak{u}(t)) dt - \int_0^{s_2} (s_2-t)^{\sigma-1} \mathfrak{h}(t,\mathfrak{u}(t)) dt \right| \\
&\le |(s_1-s_2)|\ |\mathfrak{g}_1| + \frac{(2-\sigma)}{\phi(\sigma-1)} \ell_2(1+\|\mathfrak{u}\|)|(s_1-s_2)| \\
&\quad + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma+1)} \ell_2(1+\|\mathfrak{u}\|)\left|(s_1^{\sigma}-s_2^{\sigma})\right|.
\end{aligned}$$

Clearly, as $s_2 \to s_1$, then $|(\Omega\mathfrak{u})(s_1) - (\Omega\mathfrak{u})(s_2)| \to 0$. Since $\mathfrak{u}$ is an arbitrary in $\mathfrak{B}_{\varrho}$, therefore Ω be an equicontinuous mapping. In view of well-known Arzela–Ascoli Theorem, it follows that $(\Omega\mathfrak{B}_{\varrho})$ be relatively compact, and consequently Ω is completely continuous. As an outcome of Theorem 2.2, we deduce that the system (1.2)–(1.3) admits at least one solution. The proof is finished.

4. Stability of solution

The $\mathcal{UH}$ stability concept is initiated by the authors Ulam and Hyers [44, 45], and it has a significant effect in the fractional differential equations field [17, 46, 47]. Throughout this section, we will discuss $\mathcal{UH}$ stability, $\mathcal{UHR}$ stability and their generalizations for the solution of the system (1.2)–(1.3).

Let $\rho > 0$ and $\beta, \mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$. Then, the following identities hold:

$$\left| {}^{ABC}\mathfrak{D}^{\sigma}\widetilde{\mathfrak{u}}(s) + \mathfrak{h}(s, \widetilde{\mathfrak{u}}(s)) \right| \leq \rho, s \in I, \tag{4.1}$$

$$\left| {}^{ABC}\mathfrak{D}^{\sigma}\tilde{\mathfrak{u}}(s) + \mathfrak{h}(s, \tilde{\mathfrak{u}}(s)) \right| \leq \rho\beta(s), s \in I, \tag{4.2}$$

$$\left| {}^{ABC}\mathfrak{D}^{\sigma}\tilde{\mathfrak{u}}(s) + \mathfrak{h}(s, \tilde{\mathfrak{u}}(s)) \right| \leq \beta(s), s \in I. \tag{4.3}$$

Definition 4.1. *The system (1.2)–(1.3) is $\mathcal{UH}$ stable, if $\Xi_{\mathfrak{h}} > 0$ be a real number such that for every $\widetilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ verify the identity (4.1), $\forall \rho > 0$, there is only one solution $\mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$ of the system (1.2)–(1.3) such that*

$$|\widetilde{\mathfrak{u}}(s) - \mathfrak{u}(s)| \leq \Xi_{\mathfrak{h}}\rho, s \in I.$$

Definition 4.2. *The system (1.2)–(1.3) is generalized $\mathcal{UH}$ is stable, if $B_{\mathfrak{h}} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ be a function with $B_{\mathfrak{h}}(0) = 0$ such that for every $\tilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ verify the identity (4.1), $\forall \rho > 0$, there is only one solution $\mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$ of the system (1.2)–(1.3) such that*

$$|\tilde{\mathfrak{u}}(s) - \mathfrak{u}(s)| \leq B_{\mathfrak{h}}(\rho), s \in I.$$

Definition 4.3. *The system (1.2)–(1.3) is $\mathcal{UHR}$ stable with respect to $\beta \in \mathcal{C}(I, \mathbb{R})$, if $\Xi_{\mathfrak{h},\beta} > 0$ be a real number such that for all $\tilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ satisfy the identity (4.2), $\forall \rho > 0$, there is only one solution $\mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$ of the system (1.2)–(1.3), such that*

$$|\tilde{\mathfrak{u}}(s) - \mathfrak{u}(s)| \leq \Xi_{\mathfrak{h},\beta}\rho\beta(s), s \in I.$$

Definition 4.4. *The system (1.2)–(1.3) is generalized $\mathcal{UHR}$ is stable with respect to $\beta \in \mathcal{C}(I, \mathbb{R})$, if $\Xi_{\mathfrak{h},\beta} > 0$ be a real number such that for every $\tilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ satisfy the identity (4.3), and there is only one solution $\mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$ of the system (1.2)–(1.3), such that*

$$|\tilde{\mathfrak{u}}(s) - \mathfrak{u}(s)| \leq \Xi_{\mathfrak{h},\beta}\beta(s), s \in I.$$

Remark 4.1. *Let $\tilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ be a function verifying the identity (4.1), if and only if there exists a function $\alpha_1 \in \mathcal{C}(I, \mathbb{R})$ such that*

(1) $|\alpha_1(s)| \leq \rho, \ \forall s \in I$;

(2) $-\ {}^{ABC}\mathfrak{D}^{\sigma}\tilde{\mathfrak{u}}(s) = \mathfrak{h}(s, \tilde{\mathfrak{u}}(s)) + \alpha_1(s), s \in I$.

Remark 4.2. *Let $\tilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ be a function satisfying the identity (4.2), if and only if there exists a function $\alpha_2 \in \mathcal{C}(I, \mathbb{R})$ such that*

(1) $|\alpha_2(s)| \leq \rho\beta(s), \ \forall s \in I$;

(2) $-\ {}^{ABC}\mathfrak{D}^{\sigma}\tilde{\mathfrak{u}}(s) = \mathfrak{h}(s, \tilde{\mathfrak{u}}(s)) + \alpha_2(s)$, $s \in I$.

Remark 4.3. *There exists a real number* $\xi_\beta > 0$ *and nondecreasing function* $\beta(s) \in \mathcal{C}(I, \mathbb{R})$ *such that* ${}^{AB}\mathfrak{J}^\sigma|\beta(s)| \le \xi_\beta \beta(s), \forall s \in I.$

Now, we introduce the main results related to the $\mathcal{UH}$ and $\mathcal{UHR}$ stable of the solution for the system (1.2)–(1.3).

Theorem 4.1. *If the hypothesis* $(\mathcal{H}_1)$ *holds with* $\mathfrak{h}(1, \mathfrak{u}(1)) = 0$, *subject to*

$$\mathcal{K} = \frac{\zeta(2-\sigma)}{\phi(\sigma-1)}\ell_1 + \frac{\sigma-1}{\phi(\sigma-1)} \frac{\zeta^\sigma}{\Gamma(\sigma+1)}\ell_1 < 1.$$

Then, the unique solution of the system (1.2)–(1.3) is $\mathcal{UH}$ *stable and consequently generalized* $\mathcal{UH}$ *stable.*

Proof. Consider $\rho > 0$ and let $\tilde{\mathfrak{u}} \in \mathcal{C}(I, \mathbb{R})$ verifies the identity (4.1). Then, by remark 4.1, we have:

$$\begin{cases} -{}^{ABC}\mathfrak{D}^\sigma \tilde{\mathfrak{u}}(s) = \mathfrak{h}(s, \tilde{\mathfrak{u}}(s)) + \alpha_1(s), s \in I, \\ \tilde{\mathfrak{u}}'(0) = \mathfrak{g}_1, \qquad \eta\, \tilde{\mathfrak{u}}'(1) + \tilde{\mathfrak{u}}(\zeta) = \mathfrak{g}_2. \end{cases} \tag{4.4}$$

According to Lemma 2.3, we get

$$\begin{aligned} \tilde{\mathfrak{u}}(s) &= \Sigma_{\tilde{\mathfrak{u}}} + \frac{\sigma-1}{\phi(\sigma-1)} \frac{\eta}{\Gamma(\sigma-1)} \int_0^1 (1-t)^{\sigma-2}\alpha_1(t)dt \\ &+ \frac{2-\sigma}{\phi(\sigma-1)} \int_0^\zeta \alpha_1(t)dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta-t)^{\sigma-1}\alpha_1(t)dt \\ &- \frac{2-\sigma}{\phi(\sigma-1)} \int_0^s \Big[\mathfrak{h}(t, \tilde{\mathfrak{u}}(t)) + \alpha_1(t)\Big]dt - \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^s (s-t)^{\sigma-1}\Big[\mathfrak{h}(t, \tilde{\mathfrak{u}}(t)) + \alpha_1(t)\Big]dt, \end{aligned}$$

where

$$\begin{aligned} \Sigma_{\tilde{\mathfrak{u}}} &= \mathfrak{g}_2 + (s-\eta-\zeta)\mathfrak{g}_1 + \frac{\sigma-1}{\phi(\sigma-1)} \frac{\eta}{\Gamma(\sigma-1)} \int_0^1 (1-t)^{\sigma-2}\mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt \\ &+ \frac{2-\sigma}{\phi(\sigma-1)} \int_0^\zeta \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta-t)^{\sigma-1}\mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt, \end{aligned}$$

which it follows that,

$$\begin{aligned} &\left|\tilde{\mathfrak{u}}(s) - \Sigma_{\tilde{\mathfrak{u}}} + \frac{2-\sigma}{\phi(\sigma-1)} \int_0^s \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^s (s-t)^{\sigma-1}\mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt\right| \\ &\le \frac{\sigma-1}{\phi(\sigma-1)} \frac{\eta}{\Gamma(\sigma-1)} \int_0^1 (1-t)^{\sigma-2}|\alpha_1(t)|dt \\ &+ \frac{2-\sigma}{\phi(\sigma-1)} \int_0^\zeta |\alpha_1(t)|dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta-t)^{\sigma-1}|\alpha_1(t)|dt \\ &+ \frac{2-\sigma}{\phi(\sigma-1)} \int_0^s |\alpha_1(t)|dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^s (s-t)^{\sigma-1}|\alpha_1(t)|dt \\ &\le \rho\left(\frac{1}{\phi(\sigma-1)} \frac{\eta}{\Gamma(\sigma-1)} + \frac{(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} + \frac{\sigma-1}{\phi(\sigma-1)} \frac{(\zeta^\sigma+1)}{\Gamma(\sigma+1)}\right). \end{aligned} \tag{4.5}$$

Now, let $\mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$ be a solution of the following problem:

$$\begin{cases} -{}^{ABC}\mathfrak{D}^{\sigma}\mathfrak{u}(s) = \mathfrak{h}(s, \mathfrak{u}(s)) + \alpha_1(s), s \in I, \\ \mathfrak{u}'(0) = \tilde{\mathfrak{u}}'(0), \qquad \mathfrak{u}(\zeta) = \tilde{\mathfrak{u}}(\zeta). \end{cases} \tag{4.6}$$

Since $\mathfrak{u}(\zeta) = \tilde{\mathfrak{u}}(\zeta), \forall \zeta \in I$, it follows that $\mathfrak{u}(1) = \tilde{\mathfrak{u}}(1)$.
Next, in view of Lemma 2.3 the equivalent fractional integral equation of (4.6) is given by

$$\mathfrak{u}(s) = \Sigma_{\mathfrak{u}} - \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s \mathfrak{h}(t, \mathfrak{u}(t))dt - \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\mathfrak{h}(t, \mathfrak{u}(t))dt. \tag{4.7}$$

Obviously, $\Sigma_{\mathfrak{u}} = \Sigma_{\tilde{\mathfrak{u}}}$, as follows:

$$\begin{aligned}
&|\Sigma_{\mathfrak{u}} - \Sigma_{\tilde{\mathfrak{u}}}| \\
&\leq \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}|\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))|dt \\
&\quad + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta} |\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))|dt \\
&\quad + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta} (\zeta-t)^{\sigma-1}|\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))|dt \\
&\leq \frac{\eta(2-\sigma)}{\phi(\sigma-1)}\mathfrak{h}(1, \mathfrak{u}(1)) + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}|\mathfrak{u}(t) - \tilde{\mathfrak{u}}(t)|dt \\
&\quad + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta} |\mathfrak{u}(t) - \tilde{\mathfrak{u}}(t)|dt \\
&\quad + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta} (\zeta-t)^{\sigma-1}|\mathfrak{u}(t) - \tilde{\mathfrak{u}}(t)|dt \\
&\leq \eta\, {}^{AB}\mathfrak{I}^{\sigma-1}|\mathfrak{u}(1) - \tilde{\mathfrak{u}}(1)| + {}^{AB}\mathfrak{I}^{\sigma}|\mathfrak{u}(\zeta) - \tilde{\mathfrak{u}}(\zeta)| = 0.
\end{aligned}$$

Now, by using the hypothesis $(\mathcal{H}_1)$ and (4.5), we have

$$\begin{aligned}
&|\tilde{\mathfrak{u}}(s) - \mathfrak{u}(s)| \\
&\leq \left|\tilde{\mathfrak{u}}(s) - \Sigma_{\tilde{\mathfrak{u}}} + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\mathfrak{h}(t, \tilde{\mathfrak{u}}(t))dt\right| \\
&\quad + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta} |\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))|dt \\
&\quad + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta} (\zeta-t)^{\sigma-1}|\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))|dt \\
&\leq \rho\left(\frac{1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)} + \frac{(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} + \frac{\sigma-1}{\phi(\sigma-1)}\frac{(\zeta^{\sigma}+1)}{\Gamma(\sigma+1)}\right) \\
&\quad + \frac{\zeta(2-\sigma)}{\phi(\sigma-1)}\ell_1\|\mathfrak{u} - \tilde{\mathfrak{u}}\| + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\zeta^{\sigma}}{\Gamma(\sigma+1)}\ell_1\|\mathfrak{u} - \tilde{\mathfrak{u}}\|.
\end{aligned}$$

Therefore,

$$\|\mathfrak{u}-\tilde{\mathfrak{u}}\| \leq \frac{\rho\mathcal{H}}{1-\mathcal{K}} = \Xi_{\mathfrak{h}}\rho,$$

such that

$$\mathcal{H} = \left(\frac{1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)} + \frac{(\zeta+1)(2-\sigma)}{\phi(\sigma-1)} + \frac{\sigma-1}{\phi(\sigma-1)}\frac{(\zeta^{\sigma}+1)}{\Gamma(\sigma+1)}\right),$$

$$\mathcal{K} = \frac{\zeta(2-\sigma)}{\phi(\sigma-1)}\ell_1 + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\zeta^{\sigma}}{\Gamma(\sigma+1)}\ell_1,$$

and $\Xi_{\mathfrak{h}} := \frac{\mathcal{H}}{1-\mathcal{K}}$. This satisfies that the system (1.2)–(1.3) is $\mathcal{UH}$ stable. Furthermore, if $\|\mathfrak{u}-\tilde{\mathfrak{u}}\| \leq B_{\mathfrak{h}}(\rho)$ so that $B_{\mathfrak{h}}(0)=0$; hence, the solution for the system (1.2)–(1.3) is generalized and $\mathcal{UH}$ stable.

Theorem 4.2. *If the hypothesis* $(\mathcal{H}_1)$ *holds with* $\mathfrak{h}(1,\mathfrak{u}(1))=0$ *subject to*

$$\mathcal{K} = \frac{\zeta(2-\sigma)}{\phi(\sigma-1)}\ell_1 + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\zeta^{\sigma}}{\Gamma(\sigma+1)}\ell_1 < 1.$$

Then, the unique solution of the system (1.2)–(1.3) *is* $\mathcal{UHR}$ *stable and consequently generalized* $\mathcal{UHR}$.

Proof. Let $\rho > 0$ and assume that $\tilde{\mathfrak{u}} \in \mathcal{C}(I,\mathbb{R})$verifies the identity (4.2). By remark 4.2, we have:

$$\begin{cases} -{}^{ABC}\mathfrak{D}^{\sigma}\tilde{\mathfrak{u}}(s) = \mathfrak{h}(s,\tilde{\mathfrak{u}}(s)) + \alpha_2(s), s\in I, \\ \tilde{\mathfrak{u}}'(0) = \mathfrak{g}_1, \qquad \eta\,\tilde{\mathfrak{u}}'(1) + \tilde{\mathfrak{u}}(\zeta) = \mathfrak{g}_2. \end{cases} \tag{4.8}$$

As an outcome of Lemma 2.3, we find that

$$\begin{aligned}\tilde{\mathfrak{u}}(s) &= \Sigma_{\tilde{\mathfrak{u}}} + \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}\alpha_2(t)dt \\ &+ \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta}\alpha_2(t)dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta}(\zeta-t)^{\sigma-1}\alpha_2(t)dt \\ &- \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s\left[\mathfrak{h}(t,\tilde{\mathfrak{u}}(t)) + \alpha_2(t)\right]dt - \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\left[\mathfrak{h}(t,\tilde{\mathfrak{u}}(t)) + \alpha_2(t)\right]dt.\end{aligned}$$

Hence, due to Remark 4.3, we obtain

$$\begin{aligned}&\left|\tilde{\mathfrak{u}}(s) - \Sigma_{\tilde{\mathfrak{u}}} + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^s \mathfrak{h}(t,\tilde{\mathfrak{u}}(t))dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^s (s-t)^{\sigma-1}\mathfrak{h}(t,\tilde{\mathfrak{u}}(t))dt\right| \\ &\leq \eta\,{}^{AB}\mathfrak{J}^{\sigma-1}|\alpha_2(1)| + {}^{AB}\mathfrak{J}^{\sigma}|\alpha_2(\zeta)| + {}^{AB}\mathfrak{J}^{\sigma}|\alpha_2(s)| \\ &\leq (\eta+2)\rho\;\xi_{\beta}\beta(s).\end{aligned} \tag{4.9}$$

Now, let $\mathfrak{u} \in \mathcal{C}(I, \mathbb{R})$ be a solution of (4.6). Therefore, by the hypotheses $(\mathcal{H}_1)$ and (4.9), for any $s \in I$, we have:

$$
\begin{aligned}
&|\tilde{\mathfrak{u}}(s) - \mathfrak{u}(s)| \\
&\leq \left| \tilde{\mathfrak{u}}(s) - \Sigma_{\tilde{\mathfrak{u}}} + \frac{2-\sigma}{\phi(\sigma-1)} \int_0^s \mathfrak{h}(t, \tilde{\mathfrak{u}}(t)) dt + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^s (s-t)^{\sigma-1} \mathfrak{h}(t, \tilde{\mathfrak{u}}(t)) dt \right| \\
&\quad + \frac{2-\sigma}{\phi(\sigma-1)} \int_0^\zeta |\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))| dt \\
&\quad + \frac{\sigma-1}{\phi(\sigma-1)} \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta - t)^{\sigma-1} |\mathfrak{h}(t, \mathfrak{u}(t)) - \mathfrak{h}(t, \tilde{\mathfrak{u}}(t))| dt \\
&\leq (\eta+2)\rho \, \xi_\beta \beta(s) + \mathcal{K} \|\mathfrak{u} - \tilde{\mathfrak{u}}\|.
\end{aligned}
$$

Thus,

$$
\|\mathfrak{u} - \tilde{\mathfrak{u}}\| \leq \frac{(\eta+2)\rho \, \xi_\beta \beta(s)}{1-\mathcal{K}} = \Xi_{\mathfrak{h},\beta} \rho \beta(s),
$$

such that $\Xi_{\mathfrak{h},\beta} := \frac{(\eta+2)\xi_\beta}{1-\mathcal{K}}$. This establish that the system (1.2)–(1.3) is $\mathcal{UHR}$ stable. In addition, if $\rho = 1$, then the solution of the system (1.2)–(1.3) is the generalized $\mathcal{UHR}$ stable.

5. Approximate solutions

In this section, we will introduce approximate solutions of the system (1.2)–(1.3) by using ADM. In the light of Lemma 2.3, we have proved that the solutions of system (1.2)–(1.3) and Eq. (2.10) are equivalent. Therefore, we can express decomposition of the solution of Eq. (3.2) as follows.

$$
(\Omega \mathfrak{u})(s) = \mathfrak{G}(s) + \mathfrak{N}(\mathfrak{h}(s, \mathfrak{u}(s))), \tag{5.1}
$$

where $\mathfrak{G}$ is a known function and $\mathfrak{N}$ is the nonlinear terms. Thus, we formulate Eq. (2.10) in the following decomposed format:

$$
\mathfrak{u}(s) = \mathfrak{G}(s) + \mathfrak{N}(\mathfrak{h}(s, \mathfrak{u}(s))). \tag{5.2}
$$

Suppose that the solution of (5.2) is given in a series version as next:

$$
\mathfrak{u}(u) = \sum_{n=0}^{\infty} \mathfrak{u}_n(s). \tag{5.3}
$$

So, yields that

$$
\sum_{n=0}^{\infty} \mathfrak{u}_n(s) = \mathfrak{G}(s) + \mathfrak{N}(\mathfrak{h}(s, \mathfrak{u}(s))). \tag{5.4}
$$

Now, we can be decompose the nonlinear term $\mathfrak{N}(\mathfrak{h}(s, \mathfrak{u}(s))$ by Adomian polynomials as follows:

$$
\mathfrak{h}(s, \mathfrak{u}(s)) = \sum_{n=0}^{\infty} \mathfrak{A}_n(s), \tag{5.5}
$$

where $\mathfrak{A}_n(s)$ is obtained by

$$\mathfrak{A}_n(s) = \frac{1}{n!}\frac{\partial^n}{\partial s^n}\left[\mathfrak{N}\left(\sum_{k=0}^{\infty}\mathfrak{u}_k s^k\right)\right]_{u=0}, \quad n = 0, 1, \ldots.$$

Therefore, we rewrite Eq. (5.4) as following format.

$$\sum_{n=0}^{\infty}\mathfrak{u}_n(s) = \mathfrak{G}(s) + \mathfrak{N}\left(\sum_{n=0}^{\infty}\mathfrak{A}_n(s)\right),$$

which admits the iterative technique as next:

$$\begin{cases} \mathfrak{u}_0(s) = \mathfrak{G}(s), \\ \mathfrak{u}_1(s) = \mathfrak{N}(\mathfrak{A}_0(s)), \\ \mathfrak{u}_2(s) = \mathfrak{N}(\mathfrak{A}_1(s)), \\ \mathfrak{u}_3(s) = \mathfrak{N}(\mathfrak{A}_2(s)), \\ \vdots \\ \mathfrak{u}_n(s) = \mathfrak{N}(\mathfrak{A}_{n-1}(s)), n \geq 1, \\ \vdots \end{cases} \tag{5.6}$$

For numerical targets, the n-terms approximation solution of Eq. (2.10) is represented by:

$$\mathfrak{Y}_n(u) = \sum_{i=0}^{n}\mathfrak{u}_i(s). \tag{5.7}$$

Now, we will prove the convergence theorem of ADM algorithm for the system (1.2)–(1.3).

Theorem 5.1. *Let* $(\mathcal{H}_1)$ *and condition* (3.1) *hold. Assume that* $\mathfrak{u}(s) = \sum_{i=0}^{\infty}\mathfrak{u}_i(s)$ *be a series solution of Eq.* (2.10) *which obtained by ADM is convergent, then it converges to the exact solution of Eq.* (2.10)*, whenever* $\|\mathfrak{u}_1\| < \infty$.

Proof. For $n \geq m$, consider $\mathfrak{Y}_n, \mathfrak{Y}_m$ be an arbitrary partial sums, then we have

$$\begin{aligned} |\mathfrak{Y}_n(s) - \mathfrak{Y}_m(u)| &= \left|\sum_{i=0}^{n}\mathfrak{u}_i(s) - \sum_{i=0}^{m}\mathfrak{u}_i(s)\right| = \left|\sum_{i=m+1}^{n}\mathfrak{u}_i(s)\right| \\ &\leq \sum_{i=m+1}^{n}|\mathfrak{N}(\mathfrak{A}_{i-1}(s))| \\ &\leq \sum_{i=m+1}^{n}\left|\frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1(1-t)^{\sigma-2}\mathfrak{A}_{i-1}(t)dt\right. \\ &\quad + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta}\mathfrak{A}_{i-1}(t)dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta}(\zeta-t)^{\sigma-1}\mathfrak{A}_{i-1}(t)dt \\ &\quad \left. - \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{s}\mathfrak{A}_{i-1}(t)dt - \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{s}(s-t)^{\sigma-1}\mathfrak{A}_{i-1}(t)dt\right| \\ &\leq \left|\frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1(1-t)^{\sigma-2}\sum_{i=m}^{n-1}\mathfrak{A}_i(t)dt\right. \\ &\quad + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta}\sum_{i=m}^{n-1}\mathfrak{A}_i(t)dt + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta}(\zeta-t)^{\sigma-1}\sum_{i=m}^{n-1}\mathfrak{A}_i(t)dt \\ &\quad \left. - \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{s}\sum_{i=m}^{n-1}\mathfrak{A}_i(t)dt - \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{s}(s-t)^{\sigma-1}\sum_{i=m}^{n-1}\mathfrak{A}_i(t)dt\right|. \end{aligned}$$

From (5.3), we get

$$\mathfrak{h}(s, \mathfrak{Y}_{n-1}) - \mathfrak{h}(s, \mathfrak{Y}_{m-1}) = \sum_{i=m}^{n-1} \mathfrak{A}_n. \tag{5.8}$$

Thus, by using $(\mathcal{H}_1)$ and taking supremum, we find:

$$\begin{aligned}
\|\mathfrak{Y}_n - \mathfrak{Y}_m\| \quad & \leq \sup_{t\in I}\Big| \frac{\sigma-1}{\phi(\sigma-1)}\frac{\eta}{\Gamma(\sigma-1)}\int_0^1 (1-t)^{\sigma-2}(\mathfrak{h}(t,\mathfrak{Y}_{n-1}) - \mathfrak{h}(t,\mathfrak{Y}_{m-1}))dt \\
& + \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{\zeta}(\mathfrak{h}(t,\mathfrak{Y}_{n-1}) - \mathfrak{h}(t,\mathfrak{Y}_{m-1}))dt \\
& + \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{\zeta}(\zeta-t)^{\sigma-1}(\mathfrak{h}(t,\mathfrak{Y}_{n-1}) - \mathfrak{h}(t,\mathfrak{Y}_{m-1}))dt \\
& - \frac{2-\sigma}{\phi(\sigma-1)}\int_0^{s}(\mathfrak{h}(t,\mathfrak{Y}_{n-1}) - \mathfrak{h}(t,\mathfrak{Y}_{m-1}))dt \\
& - \frac{\sigma-1}{\phi(\sigma-1)}\frac{1}{\Gamma(\sigma)}\int_0^{s}(s-t)^{\sigma-1}(\mathfrak{h}(t,\mathfrak{Y}_{n-1}) - \mathfrak{h}(t,\mathfrak{Y}_{m-1}))dt\Big| \\
& \leq \Upsilon\|\mathfrak{Y}_{n-1} - \mathfrak{Y}_{m-1}\| \\
& \leq \Upsilon^2\|\mathfrak{Y}_{n-2} - \mathfrak{Y}_{m-2}\| \leq \cdots \leq \Upsilon^m\|\mathfrak{Y}_1 - \mathfrak{Y}_0\| \leq \Upsilon^m\|\mathfrak{u}_1\|.
\end{aligned}$$

Since $0 < \Upsilon < 1$ and $\|\mathfrak{u}_1\| < \infty$, then the right-side of above inequality tends to be zero whenever $m \to \infty$. Therefore, $\|\mathfrak{Y}_n - \mathfrak{Y}_m\| \to 0$. So, we deduce that $\mathfrak{Y}_n$ is a Cauchy sequence in the Banach space $\mathcal{C}(I, \mathbb{R})$, hence the series convergence and the proof is finished.

6. Examples

Herein, we examine the validity of the main results by illustrating the following examples:

Example 6.1. *Consider the following system:*

$$\begin{cases} {}^{ABC}\mathfrak{D}^{\frac{3}{2}}\mathfrak{u}(s) + \mathfrak{h}(s, \mathfrak{u}(s)) = 0, s \in [0, 1], 1 < \sigma \leq 2, \\ \mathfrak{u}'(0) = 0, \qquad \frac{3}{4}\mathfrak{u}'(1) + \mathfrak{u}(\frac{2}{3}) = \frac{1}{4}, \end{cases} \tag{6.1}$$

where $\sigma = \frac{3}{2}$ $\eta = \frac{3}{4}$ $\zeta = \frac{2}{3}$ $\mathfrak{g}_1 = 0$, $\mathfrak{g}_2 = \frac{1}{4}$ *and* $I = [0, 1]$. *Define the function* $\mathfrak{h} : I \times \mathbb{R} \to \mathbb{R}$ *by*

$$\mathfrak{h}(s, \mathfrak{u}(s)) = \sin(\pi s)\left[\frac{s+1}{9(1+\mathfrak{u}(s))}\right].$$

Clearly, $\mathfrak{h}(0, \mathfrak{u}(0)) = \mathfrak{h}(1, \mathfrak{u}(1)) = 0$. Now, we are going to check that the hypothesis $(\mathcal{H}_1)$ holds. For any $\mathfrak{u}, \mathfrak{v} \in \mathcal{C}(I, \mathbb{R})$, we have

$$|\mathfrak{h}(s, \mathfrak{u}(s)) - \mathfrak{h}(s, \mathfrak{v}(s))| \leq \frac{2}{9}|u - v|,$$

thus, $\ell_1 = \frac{2}{9}$. *Therefore, by applying the condition (3.1), and choosing* $\phi(\sigma - 1) = 1$, *we get*

$$\Upsilon := 0.408298 < 1. \tag{6.2}$$

Hence, all hypotheses of Theorem 3.1 are fulfilled. So, the system (6.1) has only one solution. On the other hand, since $\mathcal{K} = 0.119571 < 1$, with $\Xi_{\mathfrak{h}} := \frac{\mathcal{H}}{1-\mathcal{K}} = 2.0869 > 0$. Thus, in view of Theorem 4.1, we conclude that the system (6.1) is $\mathcal{UH}$ and generalized $\mathcal{UH}$ stable. Similarly, the conditions of the $\mathcal{UHR}$ and the generalized $\mathcal{UHR}$ stability can be smoothly establish by choosing an increasing function $\beta(s) = s$.

Example 6.2. *Consider the following system:*

$$\begin{cases} {}^{ABC}\mathfrak{D}^{\frac{5}{4}}\mathfrak{u}(s) + \mathfrak{h}(s,\mathfrak{u}(s)) = 0, s \in [0,1], 1 < \sigma \leq 2, \\ \mathfrak{u}'(0) = 1, \qquad \frac{1}{2}\mathfrak{u}'(1) + \mathfrak{u}(\frac{1}{3}) = \frac{1}{2}, \end{cases} \tag{6.3}$$

where $\sigma = \frac{5}{4}$, $\eta = \frac{1}{2}$, $\zeta = \frac{1}{3}$, $\mathfrak{g}_1 = 1$, $\mathfrak{g}_2 = \frac{1}{2}$ and $I = [0,1]$. Define the function $\mathfrak{h} : I \times \mathbb{R} \to \mathbb{R}$ as

$$\mathfrak{h}(s,\mathfrak{u}(s)) = (s^2 - s)\left[\frac{1 + \sin^{-1}|\mathfrak{u}(s)|}{5 + s}\right].$$

Obviously, $\mathfrak{h}(0,\mathfrak{u}(0)) = \mathfrak{h}(1,\mathfrak{u}(1)) = 0$. Now, we will check the hypothesis $(\mathcal{H}_2)$, for any $\mathfrak{u},\mathfrak{v} \in \mathcal{C}(I,\mathbb{R})$, we have

$$|\mathfrak{h}(s,\mathfrak{u}(s))| \leq \frac{1}{5}(1 + |\mathfrak{u}(s)|),$$

so, $\ell_2 = \frac{1}{5}$. Moreover, set $\phi(\sigma - 1) = 1$, then the condition (3.3) holds, i.e.

$$\Psi := 0.282889 < 1. \tag{6.4}$$

Therefore, all hypotheses of Theorem 3.2 are fulfilled. Thus, the system (6.3) has at least one solution.

Example 6.3. *Consider the following system:*

$$\begin{cases} {}^{ABC}\mathfrak{D}^{1.9}\mathfrak{u}(s) + \left(s^2 - s\right)\left(s^2 - \mathfrak{u}^2(s)\right) = 0, s \in [0,1], 1 < \sigma \leq 2, \\ \mathfrak{u}'(0) = 1, \qquad \frac{1}{2}\mathfrak{u}'(1) + \mathfrak{u}(\frac{1}{2}) = 1, \end{cases} \tag{6.5}$$

where $\sigma = 1.9$, $\eta = \frac{1}{2}$, $\zeta = \frac{1}{2}$, $\mathfrak{g}_1 = 1$, $\mathfrak{g}_2 = 1$ and it has the exact solution $\mathfrak{u}(s) = s$. Table 1 and Figure 1, show an efficiency of ADM algorithm which estimates rapid convergence approximate solution with the exact solution of problem (6.5).

Example 6.4. *Consider the following system:*

$$\begin{cases} {}^{ABC}\mathfrak{D}^{1.8}\mathfrak{u}(s) + (s - \sqrt{s}\,)\left((1+s)^2 - \mathfrak{u}^2(s)\right) = 0, s \in [0,1], 1 < \sigma \leq 2, \\ \mathfrak{u}'(0) = 1, \qquad \frac{3}{4}\mathfrak{u}'(1) + \mathfrak{u}(\frac{1}{4}) = 2, \end{cases} \tag{6.6}$$

where $\sigma = 1.8$, $\eta = \frac{3}{4}$, $\zeta = \frac{1}{4}$, $\mathfrak{g}_1 = 1$, $\mathfrak{g}_2 = 2$ and it has exact solution $\mathfrak{u}(s) = 1 + s$. Table 2 and Figure 2, show a good agreement of approximate solution obtained by ADM with the exact solution of problem (6.6).

s	Exact sol $u(s) = s$	ADM-solution	Absolute error -ADM
0	0	−0.000159	0.000159
0.1	0.1	0.099841	0.000159
0.2	0.2	0.199844	0.000156
0.3	0.3	0.299850	0.000150
0.4	0.4	0.399861	0.000139
0.5	0.5	0.499878	0.000122
0.6	0.6	0.599899	0.000101
0.7	0.7	0.699924	0.000076
0.8	0.8	0.799951	0.000049
0.9	0.9	0.899977	0.000023
1	1	1	1.725×10^{-6}

Table 1. Numerical results of exact solution and ADM-solution at $\sigma = 1.9$ of Example 6.3

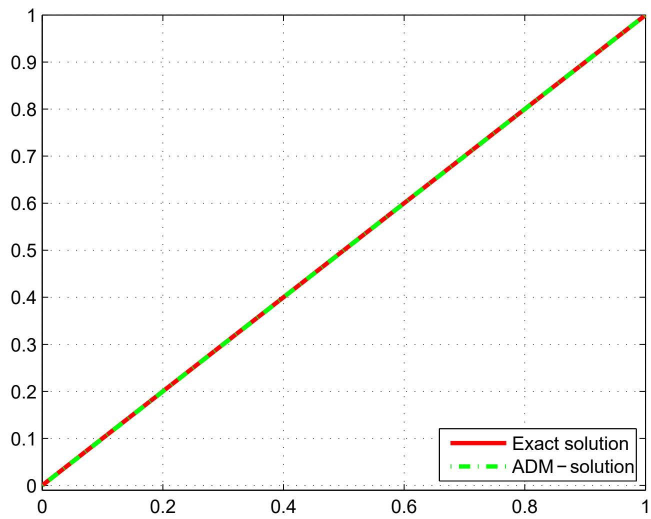

Figure 1. Exact solution compared with ADM-solution at $\sigma = 1.9$ of Example 6.3

s	Exact sol $u(s) = 1 + s$	ADM-solution	Absolute error -ADM
0	1	0.989180	0.010820
0.1	1.1	1.090730	0.009268
0.2	1.2	1.193520	0.006484
0.3	1.3	1.296940	0.003065
0.4	1.4	1.400580	0.000584
0.5	1.5	1.504100	0.004104
0.6	1.6	1.607210	0.007213
0.7	1.7	1.709750	0.009746
0.8	1.8	1.811680	0.011679
0.9	1.9	1.913130	0.013129
1	2	2.014320	0.014316

Table 2. Numerical results of exact solution and ADM-solution at $\sigma = 1.8$ of Example 6.4

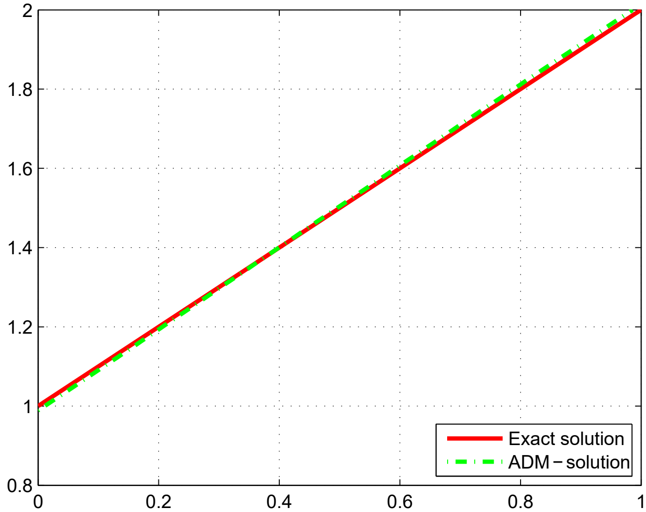

Figure 2.
Exact solution compared with ADM-solution at $\sigma = 1.8$ of Example 6.4

7. Conclusion

In the fractional calculus field, there appeared many derivative and integral definitions involving an arbitrary order. It is important to focus our attention to study the real phenomena by utilizing those definitions. In particular, a thermostat dynamics system is one of the beneficial topics in life. In this paper, we introduced the system (1.2)–(1.3) in framework of a nonsingular kernel operator (*ABC*) for the first time. Moreover, Schauder and Banach fixed point theorems were applied for discussion the existence and uniqueness of solution of the system (1.2)–(1.3) with minimum hypotheses. In addition, the $\mathcal{UH}$ and $\mathcal{UHR}$ stability of the solution for the system (1.2)–(1.3) were proved. Approximate solutions of problem (1.2)–(1.3) were established by ADM algorithm and convergence theorem of series solution was investigated. In addition, the efficiency of ADM algorithm which estimates that rapid convergence approximate solution was proved by compared ADM solution with exact solution. Finally, the validity of the main outcomes was described by four examples.

As a future direction, the studied problem would be interesting if it was studied under nonlocal boundary conditions via generalized ABC fractional operators, which is introduced by Fernandez and Baleanu [48].

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