An analog of Titchmarsh's theorem for the Laguerre–Bessel transform

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Abstract

Purpose – Using a generalized translation operator, this study aims to obtain a generalization of Titchmarsh's theorem for the Laguerre–Bessel transform for functions satisfying the ψ-Laguerre–Bessel–Lipschitz condition in the space $L^2_\alpha(\mathbb{K})$, where $\mathbb{K} = [0, +\infty[\times [0, +\infty[$.
Design/methodology/approach – The author has employed the results developed by Titchmarsh, of reference number [1].
Findings – In this paper, an analogous of Titchmarsh's theorem is established for Laguerre–Bessel transform.
Originality/value – To the best of the authors' findings, at the time of submission of this paper, the results reported are new and interesting.

Keywords Laguerre–Bessel transform, Generalized translation operator, Lipschitz class, Titchmarsh theorem
Paper type Research paper

1. Introduction

The integral Fourier transform, as Fourier series, is widely used in various fields of calculus, computational mathematics, mathematical physics, etc.

Years ago, Titchmarsh established ([1], Theorem 84) that if f satisfies the Lipschitz condition $Lip(\delta; p)$ in the L^p norm $(1 < p \le 2)$ on the real line $\mathbb{R}$, that is

$$\left(\int_{\mathbb{R}} |f(x+h) - f(x)|^p\right)^{\frac{1}{p}} = o(h^\delta), \quad (0 < \delta \le 1) \quad h \to 0.$$

Then its Fourier transform $\mathcal{F}(f)$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{p}{p + \delta p - 1} < \beta \le \frac{p}{p-1}.$$

A second result ([1], Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy–Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform. Namely, we have:

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Theorem 1.1. If $f \in L^2(\mathbb{R})$. Then the following are equivalents:

(1) $\|f(t+h)-f(t)\|_{L^2(\mathbb{R})} = o(h^\delta), \quad (0<\delta<1)$ *as* $h \to 0$.

(2) $\int_{|\lambda|\geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = o(r^{-2\delta})$ *as* $r \to \infty$.

where $\mathcal{F}(f)$ stands for the Fourier transform of f.

Considerable attention has been devoted to discovering generalizations of new contexts for those theorems, see, e.g. ([2–7]). The aim of this paper is to give a generalization of these two theorems by using the harmonic analysis associated with the Laguerre–Bessel operators.

Throughout this paper, C denotes a positive constant which can differ from one line to another.

2. Preliminaries

Given $\alpha \geq 0$. The harmonic analysis on $\mathbb{K} = [0,+\infty[\times [0,+\infty[$ is generated by the following partial differential operators:

$$\begin{cases} \mathcal{D}_{1,\alpha} = \dfrac{\partial^2}{\partial t^2} + \dfrac{2\alpha}{t}\dfrac{\partial}{\partial t} \\ \mathcal{D}_{2,\alpha} = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\mathcal{D}_{1,\alpha}, \end{cases}$$

where $(x,t) \in \mathbb{K}$. For $(\lambda, m) \in [0,+\infty[\times \mathbb{N}$, the initial value problem:

$$\begin{cases} \mathcal{D}_{1,\alpha}u = -\lambda^2 u, \\ \mathcal{D}_{2,\alpha}u = -4\lambda\left(m + \dfrac{\alpha+1}{2}\right)u \\ u(0,0) = 1, \dfrac{\partial u}{\partial x}(0,0) = \dfrac{\partial u}{\partial t}(0,0) = 0, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\mathfrak{L}_m^\alpha(\lambda x^2), \quad (x,t) \in \mathbb{K}, \tag{1}$$

where $\mathfrak{L}_m^\alpha$ is the Laguerre function defined on $[0,+\infty[$, by

$$\mathfrak{L}_m^\alpha(x) = e^{-\frac{x}{2}}\frac{L_m^\alpha(x)}{L_m^\alpha(0)}, \tag{2}$$

L_m^α being the Laguerre polynomial of degree m and order α, given by

$$L_m^\alpha(x) = \sum_{k=0}^{m}(-1)^k\frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)}\frac{1}{k!(m-k)!}x^k, \tag{3}$$

and j_α is the normalized Bessel function given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{x}{2}\right)^{2k}. \tag{4}$$

Lemma 2.1. [8] *For all* $(\lambda, m) \in [0, +\infty[\times \mathbb{N}$, *the function* $\varphi_{\lambda,m}$ *is infinitely differentiable on* $\mathbb{R}^2$, *even with respect to each variable and we have*

$$sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{5}$$

Notation. We denote by:

(1) $| x,t | = | (x,t)|_{\mathbb{K}} = \left(x^4 + 4t^2\right)^{\frac{1}{4}}$ the homogeneous norm on $\mathbb{K}$.

(2) $| \lambda, m | = | (\lambda, m)|_{[0,+\infty[\times\mathbb{N}} = 4\lambda\left(m + \frac{\alpha+1}{2}\right)$ the quasinorm on $[0, +\infty[\times \mathbb{N}$. Let us denote $\mathbb{B}_r$, the ball centered 0 and of radius r, defined by,

$$\mathbb{B}_r = \{(\lambda, m) \in [0, +\infty[\times \mathbb{N}; | \lambda, m| < r\} \quad and \quad \mathbb{B}_r^c = ([0, +\infty[\times \mathbb{N}) \backslash \mathbb{B}_r.$$

(3) $L_\alpha^p(\mathbb{K}), p \in [1, +\infty]$, the spaces of measurable functions on $\mathbb{K}$ such that

$$\begin{cases} \|f\|_{p,\alpha} = \left[\displaystyle\int_{\mathbb{K}} \left|f(x,t)\right|^p dm_\alpha(x,t)\right]^{\frac{1}{p}} < +\infty, & if \quad p \in [1, +\infty[\\ \|f\|_{\infty,\alpha} = \text{es ssup}_{(x,t)\in\mathbb{K}} |f(x,t)| < +\infty, \end{cases}$$

where dm_α is the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}t^{2\alpha}}{\pi\Gamma(\alpha+1)} dxdt.$$

(4) $L_{\gamma_\alpha}^p([0, +\infty[\times \mathbb{N}), p \in [1, +\infty]$, the spaces of measurable functions on $[0, +\infty[\times \mathbb{N}$ such that

$$\begin{cases} \|g\|_{\gamma_{\alpha,p}} = \left[\displaystyle\int_{[0,+\infty[\times\mathbb{N}} \left|g(\lambda,m)\right|^p d\gamma_\alpha(\lambda,m)\right]^{\frac{1}{p}} < +\infty, \text{ if } p \in [1, +\infty[\\ \|g\|_{\gamma_{\alpha,\infty}} = \text{ess sup}_{(\lambda,m)\in[0,+\infty[\times\mathbb{N}} |g(\lambda,m)| < +\infty, \end{cases}$$

where $d\gamma_\alpha$ is the positive measure defined on $[0, +\infty[\times \mathbb{N}$ by

$$\int_{[0,+\infty[\times\mathbb{N}} g(\lambda,m) d\gamma_\alpha(\lambda,m) = \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^{\infty} L_m^\alpha(0) \int_0^{+\infty} g(\lambda,m)\lambda^{3\alpha+1} d\lambda.$$

Definition 2.2.

(1) *The translation operators* $T_{(x,t)}^{(\alpha)}, (x,t) \in \mathbb{K}$ *are defined for a continuous function f on* $\mathbb{K}$, *by*

$$T_{(x,t)}^{(\alpha)} f(y,s) = \begin{cases} \displaystyle\frac{1}{4\pi} \sum_{i,j=0}^{1} \int_0^\pi f\left(\Delta_\theta(x,y), Y + (-1)^i t + (-1)^j s\right) d\theta, \text{ if } \alpha = 0 \\ b_\alpha \displaystyle\int_{[0,\pi]^3} f(\Delta_\theta(x,y), \Delta_\theta(x,y)\xi) \quad d\mu_\alpha(\xi,\psi,\theta), \text{ if } \alpha > 0. \end{cases}$$

where $\Delta_\theta(x,y) = \sqrt{x^2 + y^2 + 2xy\cos\theta}$, $b_\alpha = \frac{(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\pi^{\frac{3}{4}}\Gamma(\alpha)}$, $Y = xy\sin\theta$ *and*

$$d\mu_\alpha(\xi,\psi,\theta) = (\sin\xi)^{2\alpha-1}(\sin\psi)^{2\alpha-1}(\sin\theta)^{2\alpha}d\xi d\psi d\theta.$$

(2) *The convolution product of two continuous functions f, g on $\mathbb{K}$, with compact support is defined by*

$$(f * g)(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s) g(y,s) dm_\alpha(y,s), \quad (x,t) \in \mathbb{K}.$$

We have the following properties:

(1) If $f \in L^p_\alpha(\mathbb{K}), g \in L^q_\alpha(\mathbb{K})$ such that $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, then the function $f * g \in L^r_\alpha(\mathbb{K})$, and

$$\|f * g\|_{r,\alpha} \leq \|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(2) For all $(\lambda, m) \in [0, +\infty[\times \mathbb{N}$, the kernel $\varphi_{\lambda,m}$ verifies the following product formula

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = T^{(\alpha)}_{(x,t)}\varphi_{\lambda,m}(y,s), (x,t), (y,s) \in \mathbb{K}.$$

(3) For $f \in L^p_\alpha(\mathbb{K}), p \in [1, +\infty]$,we have $T^{(\alpha)}_{(x,t)} f \in L^p_\alpha(\mathbb{K})$ and

$$\|T^{(\alpha)}_{(x,t)} f\|_{p,\alpha} \leq \|f\|_{p,\alpha}.$$

The Fourier–Laguerre–Bessel transform of a function in $L^1_\alpha(\mathbb{K})$ is given by

$$\mathcal{F}_{LB} f(\lambda,m) = \int_{\mathbb{K}} f(x,t)\varphi_{\lambda,m}(x,t) dm_\alpha(x,t), (\lambda,m) \in [0,+\infty[\times \mathbb{N}.$$

From Ref. [8], it is well known that Fourier–Laguerre–Bessel transform can be inverted to

$$\mathcal{F}^{-1}_{LB} f(x,t) = \int_{[0,+\infty[\times \mathbb{N}} f(\lambda,m)\varphi_{\lambda,m}(x,t) d\gamma_\alpha(\lambda,m), (x,t) \in \mathbb{K}.$$

It is well-known (see Refs. [8–11]) that the Fourier–Laguerre–Bessel transform $\mathcal{F}_{LB}$ satisfies the following properties.

Theorem 2.3. (Inversion formula). If $f \in L^1_\alpha(\mathbb{K})$ such that $\mathcal{F}_{LB}(f) \in L^1_{\gamma_\alpha}([0, +\infty[\times \mathbb{N})$, then for all $(x,t) \in \mathbb{K}$ we have

$$f(x,t) = \int_{[0,+\infty[\times \mathbb{N}} \mathcal{F}_{LB} f(\lambda,m)\varphi_{\lambda,m}(x,t) d\gamma_\alpha(\lambda,m).$$

Theorem 2.4. (Plancherel Theorem for $\mathcal{F}_{LB}$). The generalized Fourier transform $\mathcal{F}_{LB}$ extends to an isometric isomorphism from $L^2_\alpha(\mathbb{K})$. Onto $L^2_{\gamma_\alpha}([0, +\infty[\times \mathbb{N})$.

Proposition 2.5. For $f \in L^1_\alpha(\mathbb{K})$, $(x,t) \in \mathbb{K}$ and $(\lambda, m) \in [0, +\infty[\times \mathbb{N}$, we have

$$\mathcal{F}_{LB}\left(T^{(\alpha)}_{(x,t)} f\right)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}_{LB}(f)(\lambda,m). \tag{6}$$

Remark 1. From (6) (see Ref. [14]), *we get*

$$\mathcal{F}_{LB}\left(T^{(\alpha)}_{(x,t)}f-f\right)(\lambda,m)=\left(\varphi_{\lambda,m}(x,t)-1\right)\mathcal{F}_{LB}(f)(\lambda,m). \tag{7}$$

3. Main results

In order to give the main results, we begin with auxiliary results interesting in themselves.

Lemma 3.1. Let $\eta>0$.

(1) *The behavior in 0 of the kernel $\varphi_{\lambda,m}$ could be expressed as follows:*

$$\varphi_{\lambda,m}(x,t)=1-\frac{(\lambda t)^2}{4\left(\alpha+\frac{1}{2}\right)}-\frac{|\lambda,m|\,x^2}{4(\alpha+1)}+\kappa_{\alpha,m}\lambda^2x^4+o\left(|\lambda,m|^2|x,t|^4\right), \tag{8}$$

where $\kappa_{\alpha,m}=\frac{m^2}{2(\alpha+1)(\alpha+2)}+\frac{m}{2(\alpha+2)}+\frac{1}{8}$.

(2) *There exists a constant C, such that if $|\lambda,m|x^2<\eta$, then*

$$|\varphi_{\lambda,m}(x,t)-1|\geq C|\lambda,m|x^2. \tag{9}$$

(3) *There exist $C>0$ such that for all $(x,t)\in\mathbb{K}$,*

$$|\lambda,m|\,|x,t|^2<\eta \quad\Rightarrow\quad \left|\varphi_{\lambda,m}(x,t)-1\right|^2\leq C|\lambda,m|^2|x,t|^4 \tag{10}$$

(4) *There exist $C>0$ and $A>0$ such that for all $|x,t|^2|\lambda,m|>A$ and $(x,t)\in\mathbb{K}$,*

$$|\varphi_{\lambda,m}(x,t)-1|\,.\geq C. \tag{11}$$

Proof.

(1) From the relation (2) and (3), we have

$$\mathfrak{L}^{\alpha}_m(x)=1-\frac{m+\frac{\alpha+1}{2}}{\alpha+1}x+\left(\frac{m^2}{2(\alpha+1)(\alpha+2)}+\frac{m}{2(\alpha+2)}+\frac{1}{8}\right)x^2+o\left(x^2\right). \tag{12}$$

Then (i) could be deduced easily using the relation (1),(12) and the behavior in 0 of the normalized Bessel function which states

$$j_\alpha(u)=1-\frac{1}{4(\alpha+1)}u^2+o\left(u^2\right).$$

(2) Using relations (8), we obtain

$$\lim_{|\lambda,m|x^2\to0}\left(\frac{|\varphi_{\lambda,m}(x,t)-1|}{|\lambda,m|x^2}\right)=\frac{1}{4(\alpha+1)}>0,$$

which proves the wanted result.

(3) Using relation (8).

(4) From ([6], Lemma 4.3), we have

$$\lim_{|\lambda,m|\to+\infty}\psi_{\lambda,m}(x,t)=0,$$

where $\psi_{\lambda,m}(x,t)=e^{i\lambda t}\mathfrak{L}_m^\alpha(\lambda x^2)$ the Laguerre kernel, and from Ref. [12], we have the asymptotic formula for the normalized Bessel function j_α when $x\to+\infty$:

$$j_\alpha(x)=\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{2}{x}\right)^{\alpha+\frac{1}{2}}\cos\left(x-(2\alpha+1)\frac{\pi}{4}\right)+o\left(\frac{1}{x^{\frac{3}{2}}}\right).$$

Hence as

$$\varphi_{\lambda,m}(x,t)=j_{\alpha-\frac{1}{2}}(\lambda t)\frac{1}{e^{i\lambda t}}\psi_{\lambda,m}(x,t),$$

then $\lim_{|\lambda,m|\to+\infty}\varphi_{\lambda,m}(x,t)=0$, we get $\lim_{|\lambda,m|\to+\infty}|\varphi_{\lambda,m}(x,t)-1|=1$, which completes the proof. □

Lemma 3.2. (Hausdorff-Young inequality) Let $1<p\leq 2$. *If* $f\in L^p_\alpha(\mathbb{K})$, *then* $\mathcal{F}_{LB}f\in L^q_{\gamma_\alpha}([0,+\infty[\times\mathbb{N})$ *and we have*

$$\|\mathcal{F}_{LB}f\|_{\gamma_{\alpha,q}}\leq C\|f\|_{p,\alpha},$$

where the numbers p and q above are conjugate exponents:

$$\frac{1}{p}+\frac{1}{q}=1.$$

Proof. By applying the Riesz–Thorin interpolation theorem to the elementary estimate [13] and Plancherel theorem, we obtain the desired inequality. □

Proposition 3.3. Let f be a function in $L^p_\alpha(\mathbb{K})$, *such that* $\left\|T^{(\alpha)}_{(x,t)}f-f\right\|_{p,\alpha}=O(x^\gamma)$ *for* $1<p\leq 2$ *and* $0<\gamma\leq 1$. *Then* $\mathcal{F}_{LB}f$ *belongs to* $L^\beta_{\gamma_\alpha}([0,+\infty[\times\mathbb{N})$, *where*

$$\frac{(\alpha+2)p}{(\alpha+2)(p-1)+\frac{\gamma p}{2}}<\beta\leq\frac{p}{p-1}.$$

Proof. By proceeding similarly to theorem (Theorem 3.1 [6]). For fixed $(x,t)\in\mathbb{K}$, we have using relations (7) and Lemma 3.2

$$\int_{[0,+\infty[\times\mathbb{N}}|\varphi_{\lambda,m}(x,t)-1|^q|\mathcal{F}_{LB}f(\lambda,m)|^q d\gamma_\alpha(\lambda,m)=O(x^{\gamma q}).$$

Using relations (9), we get

$$\begin{aligned}\int_{\mathbb{B}_{\frac{\eta}{x^2}}}|\lambda,m|^q|\mathcal{F}_{LB}f(\lambda,m)|^q d\gamma_\alpha &\leq x^{-2q}\int_{\mathbb{B}_{\frac{\eta}{x^2}}}|\varphi_{\lambda,m}(x,t)-1|^q|\mathcal{F}_{LB}f(\lambda,m)|^q d\gamma_\alpha\\ &\leq Cx^{(\gamma-2)q}.\end{aligned}$$

Now, let $\beta \le q$. From Hölder inequality, one gets

$$\int_{\mathbb{B}_X} |\lambda, m|^\beta |\mathcal{F}_{LB} f(\lambda, m)|^\beta d\gamma_\alpha \le \left(\int_{\mathbb{B}_X} \Big|\lambda, m|^q |\mathcal{F}_{LB} f(\lambda, m)|^q d\gamma_\alpha\right)^{\frac{\beta}{q}} \left(\int_{\mathbb{B}_X} 1 d\gamma_\alpha\right)^{1-\frac{\beta}{q}}.$$

Therefore

$$\int_{\mathbb{B}_X} |\lambda, m|^\beta |\mathcal{F}_{LB} f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = O\Big(X^{\frac{(2-\gamma)q}{2}\frac{\beta}{q} + (\alpha+2)\left(1-\frac{\beta}{q}\right)}\Big). \quad (13)$$

Recall that $\mathbb{B}_1^c = ([0, +\infty[\times \mathbb{N}) \backslash \mathbb{B}_1$. To get the theorem, it is enough to prove that $\int_{\mathbb{B}_1^c \cap \mathbb{B}_X} |\mathcal{F}_{LB} f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m)$ is bounded when $X \to +\infty$. Therefore, we can write

$$\int_{\mathbb{B}_1^c \cap \mathbb{B}_X} |\mathcal{F}_{LB} f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^{+\infty} L_m^\alpha(0) I,$$

where I depend on m and X and has the expression

$$I = \int_{\frac{1}{4m+2\alpha+2}}^{\frac{X}{4m+2\alpha+2}} |\mathcal{F}_{LB} f(\lambda, m)|^\beta \lambda^{3\alpha+1} \mathrm{d}\lambda.$$

Then

$$I = \int_{\frac{1}{4m+2\alpha+2}}^{\frac{X}{4m+2\alpha+2}} (4m + 2\alpha + 2)|\lambda, m|^{-\beta} \Phi_m^{'}(|\lambda, m|) \mathrm{d}\lambda,$$

where

$$\Phi_m(X) = \int_{\frac{1}{xm+2\alpha+2}}^{\frac{X}{2m+2\alpha+2}} |\lambda, m|^\beta |\mathcal{F}_{LB} f(\lambda, m)|^\beta \lambda^{3\alpha+1} \mathrm{d}\lambda$$

Making a change of variables and an integration by parts, we get

$$I = \Phi_m(X) X^{-\beta} + \beta \int_1^X t^{-\beta-1} \Phi_m(t) \mathrm{d}t.$$

Consequently

$$\int_{\mathbb{B}_1^c \cap \mathbb{B}_X} |\mathcal{F}_{LB} f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = X^{-\beta} \psi(X) + \beta \int_1^X t^{-\beta-1} \psi(t) \mathrm{d}t,$$

where

$$\begin{aligned} \psi(X) &= \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^{+\infty} L_m^\alpha(0) \Phi_m(X), \\ &= \int_{\mathbb{B}_1^c \cap \mathbb{B}_X} |\lambda, m|^\beta |\mathcal{F}_{LB} f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m). \end{aligned}$$

From relation (13), we have

$$\int_{\mathbb{B}_1^c \cap \mathbb{B}_X} |\mathcal{F}_{LB} f(\lambda, m)|^{\beta} d\gamma_{\alpha}(\lambda, m) = O\Big(X^{-\beta+\frac{2-\gamma}{2}\beta+(\alpha+2)\left(1-\frac{\beta}{q}\right)}\Big) + O\Big(\int_1^X t^{-\beta-1} t^{\frac{2-\gamma}{2}\beta+(\alpha+2)\left(1-\frac{\beta}{q}\right)} \mathrm{d}t\Big).$$

This is bounded as $X \to +\infty$ if $-\beta\left(\frac{\gamma}{2}+\frac{\alpha+2}{q}\right)+(\alpha+2)<0$ that gives $\beta > \frac{(\alpha+2)p}{(\alpha+2)(p-1)+\frac{\gamma p}{2}}$. □

Next we define the ψ-Laguerre–Bessel–Lipschitz class:

Definition 3.4. A function f is said to be in ψ-Laguerre–Bessel–Lipschitz class and is denoted by $Lip_{\alpha}(\psi, 2)$, if f belongs to $L_{\alpha}^2(\mathbb{K})$ and verifies, for all $(x,t) \in \mathbb{K}$

$$\left\| T_{(x,t)}^{(\alpha)} f - f \right\|_{2,\alpha} = O(\psi(|x,t|)) as |x,t| \to 0, \tag{14}$$

where

(1) $\psi(t)$ is a continuous increasing function on $[0; \infty[$.

(2) $\psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0; \infty[$.

(3) $\int_0^{\frac{1}{h}} s\psi(s^{-1}) ds = o\left(\frac{1}{h^2}\psi(h)\right) \; as \;\; h \to 0.$

Example 3.5. Let $\psi(t) = t^{\gamma}$, where $0 < \gamma < 1$. In this case, the relation (14) is a generalization of Lipschitz condition $\|f(x+h) - f(x)\| = O(h^{\gamma})$, and the ψ-Laguerre–Bessel–Lipschitz class $Lip_{\alpha}(\psi, 2)$, are called the Laguerre–Bessel–Lipschitz class $Lip_{\alpha}(\gamma, 2)$.

Now, we are able to generalise the equivalence theorem.

Theorem 3.6. Let $f \in L_{\alpha}^2(\mathbb{K})$*, the following two conditions are equivalent:*

(1) $f \in Lip_{\alpha}(\psi, 2)$.

(2)

$$\int_{\mathbb{B}_r^c} |\mathcal{F}_{LB} f(\lambda, m)|^2 d\gamma_{\alpha}(\lambda, m) = o\left(\psi\left(r^{-1}\right)\right) \;\; as \;\; r \to +\infty.$$

Proof. ($i \Rightarrow ii$): Let $f \in L_{\alpha}^2(\mathbb{K})$ from (i), we have

$$\left\| T_{(x,t)}^{(\alpha)} f - f \right\|_{2,\alpha}^2 = \int_{[0,+\infty[\times \mathbb{N}} |\varphi_{\lambda,m}(x,t) - 1|^2 |\mathcal{F}_{LB} f(\lambda, m)|^2 d\gamma_{\alpha}.$$

Therefore, using relation (11), we have

$$\int_{\mathbb{B}^c_{\frac{A}{|x,t|^2}}} |\mathcal{F}_{LB} f(\lambda, m)|^2 d\gamma_{\alpha} \leq C \int_{\mathbb{B}^c_{\frac{A}{|x,t|^2}}} |\varphi_{\lambda,m}(x,t) - 1|^2 |\mathcal{F}_{LB} f(\lambda, m)|^2 d\gamma_{\alpha} \leq C \left\| T_{(x,t)}^{(\alpha)} f - f \right\|_{2,\alpha}^2 = O\left(\psi\left(|x,t|^2\right)\right).$$

Consequently, (ii) holds.

($ii \Rightarrow i$): Denote $r = \frac{\eta}{|x,t|^2}$, by Plancherel theorem, we get

$$\left\| T^{(\alpha)}_{(x,t)} f - f \right\|^2_{2,\alpha} = I_1 + I_2,$$

where

$$I_1 = \int_{\mathbb{B}_r} \left|\varphi_{\lambda,m}(x,t) - 1\right|^2 |\mathcal{F}_{LB} f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)$$

and

$$I_2 = \int_{\mathbb{B}_r^c} \left|\varphi_{\lambda,m}(x,t) - 1\right|^2 |\mathcal{F}_{LB} f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).$$

Using relation (5), we find that

$$I_2 \leq 4 \int_{\mathbb{B}_r^c} |\mathcal{F}_{LB} f(\lambda,m)|^2 d\gamma_\alpha = o\left(\psi\left(r^{-1}\right)\right) = o\left(\psi\left(\left|x,t\right|^2\right)\right).$$

Denote

$$g(X) = \int_X^\infty |\mathcal{F}_{LB} f(\lambda,m)|^2 \lambda^{3\alpha+1} \mathrm{d}\lambda,$$

then $g'(\lambda) = -|\mathcal{F}_{LB} f(\lambda,m)|^2 \lambda^{3\alpha+1}$. Using relation (10), which gives

$$I_1 \leq C|x,t|^4 \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^\infty L_m^\alpha(0) \left(\int_0^{\frac{\eta}{4\kappa_m \left|x,t\right|^2}} (4\kappa_m)^2 \lambda^2 (-g'(\lambda)) \mathrm{d}\lambda \right),$$

by integration by parts, we have

$$I_1 \leq C|x,t|^4 \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^\infty L_m^\alpha(0) \left(-\frac{\eta^2}{\left|x,t\right|^4} g\left(\frac{\eta}{4\kappa_m \left|x,t\right|^2} \right) + (4\kappa_m)^2 \int_0^{\frac{\eta}{4\kappa_m \left|x,t\right|^2}} 2\lambda g(\lambda) \mathrm{d}\lambda \right).$$

Remark that

$$\frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^\infty L_m^\alpha(0) g\left(\frac{R}{4\kappa_m}\right) = \int_{\mathbb{B}_R^c} |\mathcal{F}_{LB} f(\lambda,m)|^2 d\gamma_\alpha = o\left(\psi\left(R^{-1}\right)\right).$$

Making a change of variable, one gets

$$\begin{aligned} I_1 &\leq o\left(\psi\left(\left|x,t\right|^2\right)\right) + C|x,t|^4 \int_0^{\frac{\eta}{|x,t|^2}} u \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{m=0}^\infty L_m^\alpha(0) g\left(\frac{u}{4\kappa_m}\right) \mathrm{d}u \\ &= o\left(\psi\left(\left|x,t\right|^2\right)\right) + |x,t|^4 o\left(\int_0^{\frac{\eta}{|x,t|^2}} u\psi\left(u^{-1}\right) du \right) \\ &= o\left(\psi\left(\left|x,t\right|^2\right)\right). \end{aligned}$$

Then

$$\left\| T^{(\alpha)}_{(x,t)}f - f \right\|^2_{2,\alpha} = o\left(\psi\left(\left|x,t\right|^2\right)\right) = o\left(\psi(|x,t|)^2\right) \quad \text{as } |x,t| \to 0.$$

□

We conclude this work by the following immediate consequence. It is analogous of Titchmarsh theorem ([1], Theorem 85) is established for Laguerre–Bessel transform.

Corollary 3.7. Let $\psi(t) = t^\gamma$, *where* $0 < \gamma < 1$. *The following two conditions are equivalent:*

(1) *f is in Laguerre–Bessel–Lipschitz class* $Lip_\alpha(\gamma, 2)$.

(2) $\int_{\mathbb{B}^c_r} |\mathcal{F}_{LB}f(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) = O(r^{-\gamma})$ *as* $r \to +\infty$.

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