Positive solution for the (p, q)-Laplacian systems by a new version of sub-super solution method

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Abstract
Purpose – In this paper, the authors give a new version of the sub-super solution method and prove the existence of positive solution for a (p, q)-Laplacian system under weak assumptions than usually made in such systems. In particular, nonlinearities need not be monotone or positive.
Design/methodology/approach – The authors prove that the sub-super solution method can be proved by the Shcauder fixed-point theorem and use the method to prove the existence of a positive solution in elliptic systems, which appear in some problems of population dynamics.
Findings – The results complement and generalize some results already published for similar problems.
Originality/value – The result is completely new and does not appear elsewhere and will be a reference for this line of research.

Keywords Nonlinear PDE system, p-Laplacian operator, Sub-super solution method
Paper type Research paper

1. Introduction
Consider the following (p_1, p_2)-Laplacian system,

$$\begin{cases} -\Delta_{p_1} u_1 & = \mu_1 F_1(x, u_1, u_2) \quad in \quad \Omega \\ -\Delta_{p_2} u_2 & = \mu_2 F_2(x, u_1, u_2) \quad in \quad \Omega \\ u_1 = u_2 & = 0 \quad on \quad \partial\Omega \end{cases} \tag{1}$$

Ω is an open bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For $i = 1, 2$, $\Delta_{p_i} = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator, $p_i > 1$, μ_i is a positive parameter and $F_i : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Many authors have been interested by the problem (1) in different ways [1–4]. The sub-super solution method, given in [5] by using a monotony argument, is the principal tool used to prove the existence of solution of the problem (1) in [1, 3, 4]. Recently, a new version of the

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method of the sub-super solution is given to prove the existence of solution for the $(p(x), q(x))$-Laplacian systems by using the Schaefer's fixed-point theorem [6].

Our main contribution in this article is, in first, to give a new version of the sub-super solution method based on Schauder's famous fixed-point theorem and, in second, use the method to prove the existence of a positive solution of problem (1) under the continuity assumptions on functions F and G. The functions F and G need not to be nondecreasing as in [1, 4].

Recall that the sub-super solution method is a topological method, which does not require strong regularity assumptions as the variational method.

The paper is organized as follows: in Section 2, we present some preliminary results and our main results. In Section 3, we study some general problems studied previously. We end our paper by studying some concrete examples.

2. Preliminaries and main results

We start by the definition of sub-super solution of the problem (1).

Definition 2.1. *We say that* $(\underline{u}_1, \bar{u}_1), (\underline{u}_2, \bar{u}_2) \in \left(W^{1,p_1}(\Omega) \cap L^\infty(\Omega)\right) \times \left(W^{1,p_2}(\Omega) \cap L^\infty(\Omega)\right)$ *is a pair of sub-super solution of the problem (1) if they satisfy*

H1. $\underline{u}_i \le \bar{u}_i$ *a.e in* Ω and $\underline{u}_i \le 0 \le \bar{u}_i$ *on* $\partial\Omega$ *for* $i = 1, 2$.

H2. $-\Delta_{p_1}\underline{u}_1 - F_1(x, \underline{u}_1, v) \le 0 \le -\Delta_{p_1}\bar{u}_1 - F_1(x, \bar{u}_1, v)$, $\forall v \in [\underline{u}_2, \bar{u}_2]$,

H3. $-\Delta_{p_2}\underline{u}_2 - F_2(x, u, \underline{u}_2) \le 0 \le -\Delta_{p_2}\bar{u}_2 - F_2(x, u, \bar{u}_2)$, $\forall u \in [\underline{u}_1, \bar{u}_1]$.

Inequalities in H1 *and* H2 *are in the weak sense. Where, for* $u \le v$ *a.e. in* Ω, $[u, v] = \{z : u(x) \le z(x) \le v(x), \text{ a.e.} \in \Omega\}$.

Theorem 2.1. *For* $i = 1, 2$, *assume that* F_i *is continuous in* $\bar{\Omega} \times \mathbb{R}^2$. *Then, if there exists a pair of sub-super solution of* (1) *in the sense of* Definition (2.1), *system* (1) *has a positive weak solution* $(u_1, u_2) \in [\underline{u}_1, \bar{u}_1] \times [\underline{u}_2, \bar{u}_2]$.

Proof. Consider, for $i = 1, 2$, the truncation operators, $T_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ defined by

$$T_i(z_i)(x) = \begin{cases} \underline{u}_i(x) & \text{if} \quad z_i(x) \le \underline{u}_i(x) \\ z_i(x) & \text{if} \quad \underline{u}_i(x) \le z_i(x) \le \bar{u}_i(x) \\ \bar{u}_i(x) & \text{if} \quad z_i(x) \ge \bar{u}_i(x) \end{cases}$$

then $\forall z_i \in L^{p_i}(\Omega)$, $T_i(z_i) \in [\underline{u}_i, \bar{u}_i]$ and $\|T_i(z_i)\|_\infty \in [0, \|\bar{u}_i\|_\infty]$. By the continuity of F_i, there exists a positive constant C_i such that

$$|F_i(x, T_1(z_1)(x), T_2(z_2)(x))| \le C_i.$$

Let $\mathcal{F}_i : L^{p_i}(\Omega) \times L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ be the Nemytskii operator defined by

$$\mathcal{F}_i(u_1, u_2)(x) = F_i(x, T_1(u_1)(x), T_2(u_2)(x)).$$

Then, $\mathcal{F}_i$ is $L^{p_i}(\Omega)$-bounded. By the dominate convergence theorem and the continuity of F_i, we conclude the continuity of $\mathcal{F}_i$, and we have $\forall (u_1, u_2) \in L^{p_1}(\Omega) \times L^{p_2}(\Omega)$.

$$\|\mathcal{F}_i(u_1, u_2)\|_\infty \le C_i. \tag{2}$$

Now, fix $(z_1, z_2) \in L^{p_1}(\Omega) \times L^{p_2}(\Omega)$, there exists a unique pair $(w_1, w_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ solution of the problem,

$$\begin{cases} -\Delta_{p_1} w_1 = \mathcal{F}_1(T_1(z_1), T_2(z_2)) & \text{in} \quad \Omega \\ -\Delta_{p_2} w_2 = \mathcal{F}_2(T_1(z_1), T_2(z_2)) & \text{in} \quad \Omega \\ w_1 = w_2 = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3}$$

Therefore, we can define the operator $S: L^{p_1}(\Omega)\times L^{p_2}(\Omega)\to L^{p_1}(\Omega)\times L^{p_2}(\Omega)$ by $S(z_1, z_2) = (w_1, w_2)$, where (w_1, w_2) is the unique solution of problem (3).

S is a compact operator. Indeed, let $(z_{1,n}, z_{2,n})$ be a bounded sequence in $L^{p_1}(\Omega)\times L^{p_2}(\Omega)$ and $(w_{1,n}, w_{2,n}) = S(z_{1,n}, z_{2,n})$, then $\forall \varphi_i \in W_0^{1,p_i}(\Omega)$

$$\int_\Omega |\nabla w_{i,n}|^{p_i-2}\nabla w_{i,n}.\nabla\varphi_i = \int_\Omega \mathcal{F}_i(T_1(z_{1,n}), T_2(z_{2,n}))\varphi_i.$$

If the test function $\varphi_i = w_{i,n}$, by the Sobolev embedding theorem, there exists some constants K_i such tat

$$\|w_{i,n}\|^{p_i}_{W_0^{1,p_i}(\Omega)} = \int_\Omega |\nabla w_{i,n}|^{p_i} = \int_\Omega \mathcal{F}(T_1(z_{1,n}), T_2(z_{2,n}))w_{i,n} \le C_i\|w_{i,n}\|_{L^1(\Omega)} \le K_i\|w_{i,n}\|_{W_0^{1,p_i}(\Omega)}. \quad (4)$$

Then, $(w_{1,n}, w_{2,n})$ is bounded in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$. By the compact embedding, there exists a convergent sub-sequence of $(w_{1,n}, w_{2,n})$ in $L^{p_1}(\Omega)\times L^{p_2}(\Omega)$. So, S is compact.

From (4), there exists $L_i > 0$ such that

$$\|w_{i,n}\|_{W_0^{1,p_i}(\Omega)} \le \bar{K}_i = K_i^{\frac{1}{p_i-1}} \Rightarrow \|w_{i,n}\|_{L^{p_i}(\Omega)} \le L_i. \quad (5)$$

Remark that in (2), (4), and (5), the constants are independent of the choice of (z_1, z_2). Then,

$$S(L^{p_1}(\Omega)\times L^{p_2}(\Omega)) \subset B_{L^{p_1}(\Omega)\times L^{p_2}(\Omega)}(0,\bar{L}).$$

for some $\bar{L} > 0$. By the Schauder fixed-point theorem, in $B_{L^{p_1}(\Omega)\times L^{p_2}(\Omega)}(0,\bar{L})$, there exists a unique $(u_1, u_2) \in L^{p_1}(\Omega)\times L^{p_2}(\Omega)$ such that $S(u_1, u_2) = (u_1, u_2)$.

$$\begin{cases} -\Delta_{p_1}u_1 = \mathcal{F}(T_1(u_1), T_2(u_2)) & in \quad \Omega \\ -\Delta_{p_2}u_2 = \mathcal{F}_2(T_1(u_1), T_2(u_2)) & in \quad \Omega \\ u_1 = u_2 = 0 & on \quad \partial\Omega \end{cases} \quad (6)$$

Finally, (u_1, u_2) is a solution of problem (1) if, and only if, $T_1(u_1) = u_1$ and $T_2(u_2) = u_2$, which means that $\underline{u}_1 \le u_1 \le \bar{u}_1$ and $\underline{u}_2 \le u_2 \le \bar{u}_2$. We need to prove that $(\underline{u}_1 - u_1)^+ = 0$, $(u_1 - \bar{u}_1)^+ = 0$, $(\underline{u}_2 - u_2)^+ = 0$ and $(u_2 - \bar{u}_2)^+ = 0$. Let us prove, for example, that $(\underline{u}_1 - u_1)^+ = 0$. The same argument works for the others cases.

Let $\Omega^+ = \{x \in \Omega,\ \underline{u}_1(x) > u_1(x)\}$. Since $(\underline{u}_1, \underline{u}_2)$ is a sub-solution, then for $\varphi = (\underline{u}_1 - u_1)^+$ and $v = T_2(u_2)$, we have,

$$\int_\Omega |\nabla\underline{u}_1|^{p_1-2}\nabla\underline{u}_1.\nabla(\underline{u}_1 - u_1)^+ \le \int_\Omega F_1(x, \underline{u}_1, T_2(u_2))(\underline{u}_1 - u_1)^+$$

and as (u_1, u_2) is a solution of (6),

$$\int_\Omega |\nabla u_1|^{p_1-2}\nabla u_1.\nabla(\underline{u}_1 - u_1)^+ = \int_\Omega F_1(x, T_1(u_1), T_2(u_2))(\underline{u}_1 - u_1)^+$$

Then,

$$\int_\Omega \left(|\nabla\underline{u}_1|^{p_1-2}\nabla\underline{u}_1 - |\nabla u_1|^{p_1-2}\nabla u_1\right).\nabla(\underline{u}_1 - u_1)^+ \le$$
$$\int_\Omega F_1(x, \underline{u}_1, T_2(u_2)) - F_1(x, T_1(u_1), T_2(u_2))(\underline{u}_1 - u_1)^+.$$

Remark that in Ω^+, $T_1(u_1) = \underline{u}_1$. So,

$$\int_{\Omega^+} \left(|\nabla \underline{u}_1|^{p_1-2}\nabla \underline{u}_1 - |\nabla u_1|^{p_1-2}\nabla u_1\right).\nabla(\underline{u}_1 - u_1)^+ = \int_{\Omega^+} \left(|\nabla \underline{u}_1|^{p_1-2}\nabla \underline{u}_1 - |\nabla u_1|^{p_1-2}\nabla u_1\right).(\nabla \underline{u}_1 - \nabla u_1) \leq 0.$$

Therefore, by the monotonicity of the p_1-Laplacian, $\underline{u}_1 - u_1 = 0$ in Ω^+. Then, $\underline{u}_1 \leq u_1$ in Ω. In the same way, we get $u_1 \leq \bar{u}_1$ in Ω. Then, $\underline{u}_1 \leq u_1 \leq \bar{u}_1$ and $\underline{u}_2 \leq u_2 \leq \bar{u}_2 \Rightarrow T_1(u_1) = u_1$ and $T_2(u_2) = u_2$. Finally, (u_1, u_2) is a solution of the problem (1). □

3. Applications

Theorem 3.1. *Consider system (1) and assume that for i = 1, 2,*

A.1 $\exists\, C_i, \alpha_i, \beta_i > 0$, *such that* $|F_i(x, s_1, s_2)| \leq C_i\left(1 + |s_1|^{\alpha_i} + |s_2|^{\beta_i}\right)$,
A.2 $F_1(x, 0, 0) + F_2(x, 0, 0) > 0$ *a.e.* $x \in \Omega$.
Then,

(1) *If* $\max(\alpha_i, \beta_i) < p_i - 1$, *for* $i = 1, 2$, *then* $\forall \mu_i > 0$, *there exists a weak positive solution of problem (1).*

(2) *If* $\min(\alpha_i, \beta_i) \geq p_i - 1$, *for* $i = 1, 2$, *then, there exists positive numbers* $\bar{\mu}_i$ *such that* $\forall(\mu_1, \mu_2) \in]0, \bar{\mu}_1] \times]0, \bar{\mu}_2]$ *the problem (1) has at least a weak positive solution* (u_1, u_2) *such that* $\|u_i\|_\infty \leq \|e_i\|_\infty$. e_i *is the unique solution of problem (7).*

(3) *If* $\alpha_i + \beta_i < p_i - 1$ *and* $\alpha_j + \beta_j \geq p_j - 1$. $j = 2$ *if* $i = 1$ *and* $j = 1$ *if* $i = 2$. *Then there exists* $\bar{\mu}_j$ *such that problem (1) has at least a weak positive solution* $\forall \mu_i > 0$ *and* $0 < \mu_j \leq \bar{\mu}_j$.

Proof. By A.2, (0, 0) is a sub-solution, but not a solution, of problem (1). By Theorem 2.1, we need to find a super solution of problem. Let e_i be the unique positive solution of the Dirichlet boundary condition problem,

$$\begin{cases} -\Delta_{p_i} u = 1 & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{7}$$

(1) In the sub-linear case, as $\max(\alpha_i, \beta_i) < p_i - 1$, for $i = 1, 2$, there exists $K > 1$, large enough, such that

$$\begin{cases} -\Delta_{p_1}(Ke_1) = K^{p_1-1} & \geq \mu_1 C_1\left(1 + K^{\alpha_1}\|e_1\|_\infty^{\alpha_1} + K^{\beta_1}\|e_2\|_\infty^{\beta_1}\right) \\ & \geq \mu_1 C_1\left(1 + K^{\alpha_1}\|e_1\|_\infty^{\alpha_1} + \|v\|_\infty^{\beta_1}\right), \\ & \geq \mu_1 F_1(x, Ke_1, v), \quad \forall v \in [0, Ke_2] \\ -\Delta_{p_2}(Ke_2) = K^{p_2-1} & \geq \mu_2 C_2\left(1 + K^{\alpha_2}\|e_1\|_\infty^{\alpha_2} + K^{\beta_2}\|e_2\|_\infty^{\beta_2}\right) \\ & \geq \mu_2 C_2\left(1 + \|u\|_\infty^{\alpha_2} + K^{\beta_2}\|e_2\|_\infty^{\beta_2}\right), \\ & \geq \mu_2 F_2(x, u, Ke_2), \quad \forall u \in [0, Ke_1] \end{cases}$$

in the weak sense. So, $(\bar{u}_1, \bar{u}_2) = (Ke_1, Ke_2)$ is a super-solution of the problem (1). Therefore, there exists $(u_1, u_2) \in [0, Ke_1] \times [0, Ke_2]$ solution of system (1).

(2) In the super-linear case, for $i = 1, 2$, $\min(\alpha_i, \beta_i) \geq p_i - 1$. Put $\bar{\mu}_i = \frac{1}{C_i\left(1+\|e_1\|_\infty^{\alpha_i}+\|e_2\|_\infty^{\beta_i}\right)}$ and let $0 < \mu_i \leq \bar{\mu}_i$. Then,

$$\begin{cases} -\Delta_{p_1} e_1 = 1 & = \bar{\mu}_1 C_1\left(1 + \|e_1\|_\infty^{\alpha_1} + \|e_2\|_\infty^{\beta_1}\right) \\ & \geq \mu_1 C_1\left(1 + \|e_1\|_\infty^{\alpha_1} + \|e_2\|_\infty^{\beta_1}\right) \\ & \geq \mu_1 C_1\left(1 + \|e_1\|_\infty^{\alpha_1} + v^{\beta_1}\right) \\ & \geq \mu_1 F_1(x, e_1, v), \ \forall v \in [0, e_2] \\ -\Delta_{p_2} e_2 = 1 & = \bar{\mu}_2 C_2\left(1 + \|e_1\|_\infty^{\alpha_2} + \|e_2\|_\infty^{\beta_2}\right) \\ & \geq \mu_2 C_2\left(1 + \|e_1\|_\infty^{\alpha_2} + \|e_2\|_\infty^{\beta_2}\right) \\ & \geq \mu_2 C_2\left(1 + u^{\alpha_2} + \|e_2\|_\infty^{\beta_2}\right) \\ & \geq \mu_2 F_2(x, u, e_2), \ \forall u \in [0, e_1]. \end{cases}$$

(e_1, e_2) is a super-solution, and we deduce that the problem (1) admits a weak positive solution (u_1, u_2), $\forall (\mu_1, \mu_2) \in]0, \bar{\mu}_1] \times]0, \bar{\mu}_2]$ such that $0 < \|u_i\|_\infty \leq \|e_i\|_\infty$.

(3) Consider the sub-super linear case, $0 < \alpha_1 + \beta_1 < p_1 - 1$ and $\alpha_2 + \beta_2 \geq p_2 - 1$ and put $\bar{\mu}_2 = \frac{1}{C_2\left(1+\|e_1\|_\infty^{\alpha_2}+\|e_2\|_\infty^{\beta_2}\right)}$. Let $\mu_1 > 0$ and $0 < \mu_2 \leq \bar{\mu}_2$. Then, there exists K, large enough, such that

$$\begin{cases} -\Delta_{p_1}(Ke_1) & = K^{p_1 - 1} \geq \mu_1 F_1(x, Ke_1, v), \ \forall v \in [0, Ke_2] \\ -\Delta_{p_2} e_2 & = 1 \qquad \geq \mu_2 F_2(x, u, e_2), \qquad \forall u \in [0, e_1]. \end{cases}$$

So, (Ke_1, e_2) is as super solution of problem (P).

By the same argument, if $\alpha_1 + \beta_1 \geq p_1 - 1$ and $0 < \alpha_2 + \beta_2 < p_2 - 1$. Put $\bar{\mu}_1 = \frac{1}{C_1\left(1+\|e_1\|_\infty^{\alpha_1}+\|e_2\|_\infty^{\beta_1}\right)}$. For all $0 < \mu_1 \leq \bar{\mu}_1$ and $\mu_2 > 0$, there exists K, large enough, such that

$$\begin{cases} -\Delta_{p_1} e_1 & = 1 \qquad \geq \mu_1 F_1(x, e_1, v), \qquad \forall v \in [0, e_2] \\ -\Delta_{p_2}(Ke_2) & = K^{p_2 - 1} \geq \mu_2 F_2(x, u, e_2), \qquad \forall u \in [0, Ke_1]. \end{cases}$$

So, (e_1, Ke_2) is as super solution; hence, the problem (1) admits a weak positive solution (u_1, u_2). The proof of Theorem 3.1 is complete.

□

Remark 3.1. *The second point of Theorem 3.1 is of great importance because no restriction was made on the growth of nonlinearities but only on the parameter μ_i which must not be large.*

The hypothesis A.2 plays an essential role in the way that we need only to find a super-solution. In what follows, we provide an example without the hypothesis A.2 of Theorem (3.1). We will see that the assumptions on nonlinearities become more restrictive.

Proposition 3.1. *For $i = 1, 2$, assume that $\exists\ 0 < c_i \leq C_i$, $0 < \alpha_i, \beta_i < p_i - 1$, $\forall (s_1, s_2) \in \mathbb{R}^+ \times \mathbb{R}^+$,*

$$c_i s_i^{\alpha_i} \leq F_i(x, s_1, s_2) \leq C_i\left(s_1^{\alpha_i} + s_2^{\beta_1}\right).$$

Then, $\forall \mu_i > 0$, there exists a weak positive solution of the problem (1).

Proof. According to the proof of Theorem 3.1, there exists K, large enough, such that (Ke_1, Ke_2) is a super-solution of the problem (1). Then, we need only to find a sub-solution. Let $\underline{u}_i = \varepsilon\varphi_i$, φ_i is the principal eigenfunction (positive) associated with the principal eigenvalue of p_i-Laplacian operator such that $\|\varphi_i\|_\infty = 1$,

$$\begin{cases} -\Delta_{p_i}\varphi_i & = \lambda_{1,p_i}\varphi_i^{p_i-1} \quad in \quad \Omega \\ \varphi_i & = 0 \quad on \quad \partial\Omega \end{cases} \tag{8}.$$

$(\underline{u}_1, \underline{u}_2)$ is as sub-solution of the problem (1). We need to have

$$\begin{cases} -\Delta_{p_1}\underline{u}_1 = \lambda_{1,p_1}\varepsilon^{p_1-1}\varphi_{1,p_1}^{p_1-1} & \leq \mu_1 c_1 \varepsilon^{\alpha_1}\varphi_{1,p_1}^{\alpha_1} \leq \mu_1 F_1(x, \underline{u}_1, v), \quad \forall v \in [\underline{u}_2, \bar{u}_2] \\ -\Delta_{p_2}\underline{u}_2 = \lambda_{1,p_2}\varepsilon^{p_2-1}\varphi_{1,p_2}^{p_2-1} & \leq \mu_2 c_2 \varepsilon^{\beta_2}\varphi_{1,p_2}^{\beta_2} \leq \mu_2 F_2(x, u, \underline{u}_2), \quad \forall u \in [\underline{u}_1, \bar{u}_1] \end{cases}$$

As $\|\varphi_i\|_\infty = 1$, it is enough to have $\lambda_{1,p_i}\varepsilon^{p_i-1} \leq \mu_i c_i \varepsilon^{\delta_i}$ ($\delta_1 = \alpha_1$ and $\delta_2 = \beta_2$). This is possible for ε, small enough, $\varepsilon < 1$ and for all $\mu_i > 0$. To end the proof, we need to verify that $\varepsilon\varphi_i = \underline{u}_i \leq \bar{u}_i = Ke_i$ for a small ε and K large. By the maximum principle, we have

$$-\Delta_{p_i}(\varepsilon\varphi_i) = \varepsilon^{p_i-1}\lambda_{1,p_i}\varphi_i \leq \varepsilon^{p_i-1}\lambda_{1,p_i} \leq K^{p_i-1} = -\Delta_{p_i}(Ke_i) \Rightarrow \varepsilon\varphi_i \leq Ke_i.$$

□

4. Examples

In this section, we use Theorem 3.1 and solve some elliptic (p_1, p_2)-Laplacian systems studied in some published articles see [7].

Consider the following (p_1, p_2)-Laplacian system

$$(P) \quad \begin{cases} -\Delta_{p_1}u_1 & = \mu_1\left(a_1(x) + b_1(x)u_1^{\alpha_1}u_2^{\beta_1}\right) \quad in \quad \Omega \\ -\Delta_{p_2}u_2 & = \mu_2\left(a_2(x) + b_2(x)u_1^{\alpha_2}u_2^{\beta_2}\right) \quad in \quad \Omega \\ u_1 & = u_2 = 0 \quad on \quad \partial\Omega \end{cases}$$

4.1 The case where (0, 0) is a sub-solution

Assume, for $i = 1, 2$, that $a_i(x)$ and $b_i(x)$ are continuous and nonnegative in $\bar{\Omega}$, a_1 or a_2 not identically null. Then, (0, 0) is a sub-solution of problem (P). Taking into account Theorem 3.1 and its proof, we get the following propositions,

Proposition 4.1. *Problem (P) has a positive weak solution provided that for i = 1, 2,*

(1) $\mu_i > 0$ *if* $0 < \alpha_i + \beta_i < p_i - 1$ ***(The sub-linear case)***,

(2) $0 < \mu_i \leq \bar{\mu}_i = \frac{1}{C_i\left(1 + \|e_1\|_\infty^{\alpha_i}\|e_2\|_\infty^{\beta_i}\right)}$ *if* $\alpha_i + \beta_i \geq p_i - 1$. $C_i = max(\|a_i\|_\infty, \|b_i\|_\infty)$ ***(The super linear case)*** *and*

(3) $\mu_i > 0$ *and* $0 < \mu_j \leq \bar{\mu}_j$ *if* $\alpha_i + \beta_i < p_i - 1$ *and* $\alpha_j + \beta_j \geq p_j - 1$. *j = 2 if i = 1 and j = 1 if i = 2* ***(The sub-super linear case)***.

Proof. We have to look for a super solution of our problem.

(1) Since $0 < \alpha_i + \beta_i < p_i - 1$, for $\forall \mu_i > 0$, there exists K such that for $i = 1, 2$, $K^{p_i-1} \geq \mu_i C_i \left(1 + K^{\alpha_i+\beta_i} \|e_1\|_\infty^{\alpha_i} \|e_2\|_\infty^{\beta_i}\right)$. Then, we have

$$\begin{cases} -\Delta_{p_1}(Ke_1) & = K^{p_1-1} \\ & \geq \mu_1 C_1 \left[1 + (K\|e_1\|_\infty)^{\alpha_1} (K\|e_2\|_\infty)^{\beta_1}\right] \\ & \geq \mu_1 \left[a_1(x) + b_1(x)(Ke_1)^{\alpha_1} (Ke_2)^{\beta_1}\right] \\ & \geq \mu_1 \left[a_1(x) + b_1(x)(Ke_1)^{\alpha_1} v^{\beta_1}\right], \quad \forall v \in [0, Ke_2]. \end{cases}$$

In the same way, we show that $-\Delta_{p_2}(Ke_2) \geq \mu_2 \left[a_2(x) + b_2(x) u^{\alpha_2} (Ke_2)^{\beta_2}\right]$, $\forall u \in [0, Ke_1]$. So, (Ke_1, Ke_2) is a super solution of problem *(P),* and then the problem has a weak positive solution.

(2) Consider the super linear case, for $i = 1, 2$, $\alpha_i + \beta_i \geq p_i - 1$. Let $0 < \mu_i \leq \bar{\mu}_i$, (e_1, e_2) is a super solution of problem (*P*). Indeed, we have

$$\begin{cases} -\Delta_{p_1} e_1 = 1 & = \bar{\mu}_1 C_1 \left[1 + \|e_1\|_\infty^{\alpha_1} \|e_2\|_\infty^{\beta_1}\right] \\ & \geq \mu_1 C_1 \left[1 + \|e_1\|_\infty^{\alpha_1} \|e_2\|_\infty^{\beta_1}\right] \\ & \geq \mu_1 \left[a_1(x) + b_1(x) e_1^{\alpha_1} e_2^{\beta_1}\right] \\ & \geq \mu_1 \left[a_1(x) + b_1(x) e_1^{\alpha_1} v^{\beta_1}\right], \quad \forall v \in [0, e_2]. \end{cases}$$

In the same way, we obtain $-\Delta_{p_2} e_2 \geq \mu_2 \left[a_2(x) + b_2(x) u^{\alpha_2} e_2^{\beta_2}\right]$, $\forall u \in [0, e_1]$. We conclude that problem (*P*) has a positive weak solution in $[0, e_1] \times [0, e_2]$.

(3) Finally, the third point, the sub-super linear case, follows from the two previous ones.

□

4.2 The case where (0, 0) is a trivial solution

We examine only the case where $p_1 = p_2 = p$. (0, 0) is a trivial solution of (*P*) if a_i, $i = 1, 2$, are identically null. Our goal is to find a positive solution of problem (*P*). To do this, we need to find a pair of positive sub-super solution. Nevertheless, we have to add some more assumptions. Assume that, for $i = 1, 2$, b_i is positive continuous in $\bar{\Omega}$. So, $c_i \leq \|b_i\|_\infty \leq C_i$ for some positive constants c_i and C_i. The problem becomes

$$(P_1) \quad \begin{cases} -\Delta_p u_1 & = \mu_1 b_1(x) u_1^{\alpha_1} u_2^{\beta_1} \quad \text{in } \Omega \\ -\Delta_p u_2 & = \mu_2 b_2(x) u_1^{\alpha_2} u_2^{\beta_2} \quad \text{in } \Omega \\ u_1 & = u_2 = 0 \quad \text{on } \partial\Omega \end{cases}$$

Proposition 4.2. *Assume that, for* $i = 1, 2$, $0 < \alpha_i + \beta_i < p - 1$. *Then, the problem* (P_1) *has a positive weak solution* $\forall \mu_i > 0$.

Proof. According to Theorem 2.1, we need to find a pair of sub-super solution of the problem (P_1). Assume that for $i = 1, 2, 0 < \alpha_i + \beta_i < p - 1$, then $\forall \mu_i > 0$, we can choose $0 < \varepsilon < 1$, such that $\lambda_{1,p}\varepsilon^{p-1-(\alpha_i+\beta_i)} \leq \mu_i c_i$ for $i = 1, 2$. Fix such ε and choose $K \geq \max(\lambda_{1,p}^{\frac{1}{p-1}}\varepsilon, 1)$ such that $K^{p-1-(\alpha_i+\beta_i)} \geq \mu_i C_i \|e_1\|_\infty^{\alpha_i+\beta_i}$ for $i = 1, 2$. Then, $(\varepsilon\varphi_1, \varepsilon\varphi_1)$ and (Ke_1, Ke_1) is a pair of sub-super linear solution of the problem (P_1) $(\|\varphi_1\|_\infty = 1)$. Indeed, we have, by the maximum principle, $\varepsilon\varphi_1 \leq Ke_1$ because

$$-\Delta_p(\varepsilon\varphi_1) = \lambda_{1,p}\varepsilon^{p-1}\varphi_1^{p-1} \leq \lambda_{1,p}\varepsilon^{p-1} \leq K^{p-1} = -\Delta_p(Ke_1).$$

$\forall (u, v) \in [\varepsilon\varphi_1, Ke_1] \times [\varepsilon\varphi_1, Ke_1]$, we have

$$\begin{cases} -\Delta_p(\varepsilon\varphi_1) = \lambda_{1,p}\varepsilon^{p-1}\varphi_1^{p-1} & \leq \mu_1 c_1 \varepsilon^{\alpha_1+\beta_1}\varphi_1^{\alpha_1+\beta_1} \\ & \leq \mu_1 c_1 (\varepsilon\varphi_1)^{\alpha_1}(\varepsilon\varphi_1)^{\beta_1} \\ & \leq \mu_1 b_1(x)(\varepsilon\varphi_1)^{\alpha_1} v^{\beta_1}. \\ -\Delta_p(\varepsilon\varphi_1) = \lambda_{1,p}\varepsilon^{p-1}\varphi_1^{p-1} & \leq \mu_2 c_2 \varepsilon^{\alpha_2+\beta_2}\varphi_1^{\alpha_2+\beta_2} \\ & \leq \mu_2 c_2 (\varepsilon\varphi_1)^{\alpha_2}(\varepsilon\varphi_1)^{\beta_2} \\ & \leq \mu_2 b_2(x) u^{\alpha_2}(\varepsilon\varphi_1)^{\beta_2}. \end{cases}$$

and

$$\begin{cases} -\Delta_p(Ke_1) = K^{p-1} & \geq \mu_1 C_1 K^{\alpha_1+\beta_1}\|e_1\|_\infty^{\alpha_1+\beta_1} \\ & \geq \mu_1 C_1 (Ke_1)^{\alpha_1}(Ke_1)^{\beta_1} \\ & \geq \mu_1 b_1(x)(Ke_1)^{\alpha_1} v^{\beta_1}. \\ -\Delta_p(Ke_1) = K^{p-1} & \geq \mu_2 C_2 K^{\alpha_2+\beta_2}\|e_1\|_\infty^{\alpha_2+\beta_2} \\ & \geq \mu_2 C_2 (Ke_1)^{\alpha_2}(Ke_1)^{\beta_2} \\ & \geq \mu_2 b_2(x) u^{\alpha_2}(Ke_1)^{\beta_2}. \end{cases}$$

The problem (P_1) has a weak positive solution in the set $[\varepsilon\varphi_1, Ke_1] \times [\varepsilon\varphi_1, Ke_1]$. The proof is complete. □

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