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Uniformity on generalized topological spaces

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Abstract

Purpose – The present article deals with the initiation and study of a uniformity like notion, captioned μ-uniformity, in the context of a generalized topological space.

Design/methodology/approach – The existence of uniformity for a completely regular topological space is well-known, and the interrelation of this structure with a proximity is also well-studied. Using this idea, a structure on generalized topological space has been developed, to establish the same type of compatibility in the corresponding frameworks.

Findings – It is proved, among other things, that a μ-uniformity on a non-empty set X always induces a generalized topology on X, which is μ-completely regular too. In the last theorem of the paper, the authors develop a relation between μ-proximity and μ-uniformity by showing that every μ-uniformity generates a μ-proximity, both giving the same generalized topology on the underlying set.

Originality/value – It is an original work influenced by the previous works that have been done on generalized topological spaces. A kind of generalization has been done in this article, that has produced an intermediate structure to the already known generalized topological spaces.

Keywords Generalized topology, μ-uniformity, μ-completely regular, μ-proximity

Paper type Research paper

1. Introduction and prerequisites

It was Császár [1] who first initiated the idea of generalized topological space. This opened up a new direction which was pursued by many mathematicians toward generalizations of many topological concepts to this new arena. A generalized topology (GT, for short) μ on a set X is a collection of subsets of X such that $\phi \in \mu$ and arbitrary unions of members of μ belong to μ; and the ordered pair (X, μ) then stands for a generalized topological space (henceforth abbreviated as GTS). The sets in μ are called μ-open sets and their complements μ-closed sets. A GTS (X, μ) is called a strong GTS if $X \in \mu$. For any subset A of a GTS (X, μ), the μ-interior $i_\mu(A)$ and μ-closure $c_\mu(A)$ of A are defined in the usual way as:

$i_\mu(A) = \bigcup\{B \subseteq X : B \subseteq A \text{ and } B \in \mu\}$ and $c_\mu A = \bigcap\{B \subseteq X : A \subseteq B \text{ and } X \setminus B \in \mu\}$.

As is expected, μ-interior and μ-closure operators on a GTS (X, μ) obey the following basic properties:

(1) $i_\mu(A) \subseteq A$ and $A \subseteq c_\mu(A)$, for all $A \subseteq X$.

(2) $A \subseteq B \subseteq X \Rightarrow i_\mu(A) \subseteq i_\mu(B)$ and $c_\mu(A) \subseteq c_\mu(B)$.

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(3) $A(\subseteq X)$ is μ-open (μ-closed) if and only if $A = i_\mu(A)$ (resp. $A = c_\mu(A)$).

(4) $i_\mu(X \setminus A) = X \setminus c_\mu(A)$, for all $A \subseteq X$.

The notion of uniformity is well-known for a topological space. This article is intended to initiate the study of a uniformity-like structure, termed μ-uniformity, on a generalized topological space.

In what follows in Section 2, we define μ-uniformities on a nonempty set X axiomatically and show that such a μ-uniformity induces a generalized topology on X. Although a μ-uniformity is not necessarily a uniformity. In Section 3, we also prove that a μ-uniform space satisfies a sort of complete regularity condition. Finally in Section 4, we establish that for a μ-uniform space, there exists a μ-proximity relation [2] such that the same generalized topology originates from both the structures.

We now recall the definition of uniformity on a set and some well-known relevant results thereof; related details may be found in [3].

Definition 1.1. *Let X be a non-empty set:*

(1) *A non-void subset of $X \times X$ is called a binary relation on X.*

(2) *The identity relation on X is called the diagonal in $X \times X$ and is denoted by $\Delta(X)$ or simply by Δ. Thus $\Delta = \{(x, x) : x \in X\}$.*

(3) *The inverse of a relation U, denoted by U^{-1}, is defined by $U^{-1} = \{(y, x) : (x, y) \in U\}$.*

(4) *A relation U is said to be symmetric if $U = U^{-1}$.*

(5) *The composition of two relations U and V, denoted by $U \circ V$, is defined by $U \circ V = \{(x, y) : (x, z) \in U$ and $(z, y) \in V$, for some $z \in X\}$.*

Definition 1.2. *Let X be a non-empty set. A non-void family $\mathcal{U}$ of subsets of $X \times X$, is said to be a uniformity on X if the following conditions hold:*

(1) *$\Delta \subseteq U$, for every $U \in \mathcal{U}$.*

(2) *$U, V \in \mathcal{U} \Rightarrow U \cap V \in \mathcal{U}$.*

(3) *$U \in \mathcal{U}$ and $V \supseteq U \Rightarrow V \in \mathcal{U}$.*

(4) *$U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$.*

(5) *$U \in \mathcal{U} \Rightarrow$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.*

The pair $(X, \mathcal{U})$ is called a uniform space.

Definition 1.3. *Let U be a binary relation on X and A a non-void subset of X. Then we define, $U(A) = \{x \in X : (a, x) \in U$, for some $a \in A\}$. In particular, if $A = \{p\}$, for some $p \in X$, then $U(p) = U(\{p\}) = \{x \in X : (p, x) \in U\}$.*

Now we state some well-known results for a uniform space $(X, \mathcal{U})$.

Result 1.4. *Let $\mathcal{U}$ be a uniformity on a non-void set X. Let a family τ of subsets of X be defined as follows: A subset G of X belongs to τ if and only if to every element $p \in G$, there corresponds some $U_p \in \mathcal{U}$ such that $U_p(p) \subseteq G$. Then τ is a topology on X.*

Definition 1.5. *[4] If $(X, \mathcal{U})$ is a uniform space the topology $\tau(\mathcal{U})$ of the uniformity $\mathcal{U}$, or the uniform topology, is the family of all subsets G of X such that for each x in G there is U in $\mathcal{U}$ such that $U(x) \subseteq G$.*

Result 1.6. *A topological space (X, τ) is uniformizable if and only if it is completely regular.*

2. μ-uniformity

Before going into the details we first state two definitions which will be required later on.

Definition 2.1. *[5] Let X be a non-empty set and $\beta \subseteq \mathcal{P}(X)$. Then β is called a base for a generalized topology μ on X if $\mu = \{\cup\beta' : \beta' \subseteq \beta\}$.*

Definition 2.2. *[6] Let (X, μ) and (Y, ξ) be two generalized topological spaces. A function $f : (X, \mu) \to (Y, \xi)$ is said to be μ-continuous if for any $G \in \xi$, $f^{-1}(G) \in \mu$.*

In [7] the concept of generalized quasi uniformity was introduced, termed as g-quasi uniformity. In the same manner, we introduce the definition of μ-uniformity as follows.

Definition 2.3. *Let X be a non-empty set. A non-void family $\mathcal{U}_\mu$ of subsets of $X \times X$ is called a μ-uniformity on X if*

(1) $\Delta \subseteq U$ for every $U \in \mathcal{U}_\mu$,

(2) $U \in \mathcal{U}_\mu$ and $V \supseteq U \in \mathcal{U}_\mu \Rightarrow V \in \mathcal{U}_\mu$,

(3) $U \in \mathcal{U}_\mu \Rightarrow$ there exists a symmetric $V \in \mathcal{U}_\mu$ such that $V \circ V \subseteq U$.

The pair $(X, \mathcal{U}_\mu)$ is called a μ-uniform space.

Result 2.4. *Let $(X, \mathcal{U}_\mu)$ be a μ-uniform space, then for any $U \in \mathcal{U}_\mu$, $U \subseteq U \circ U$.*

Proof. Let $(x, y) \in U$. Then as $(y, y) \in U$[from (i)], we have $(x, y) \in U \circ U$, hence $U \subseteq U \circ U$. □

Proposition 2.5. *Let $(X, \mathcal{U}_\mu)$ be a μ-uniform space, then for any $U \in \mathcal{U}_\mu$, $U^{-1} \in \mathcal{U}_\mu$.*

Proof. Let $U \in \mathcal{U}_\mu$. Then by axiom (iii), there exists a symmetric $V \in \mathcal{U}_\mu$ such that $V \circ V \subseteq U$. Again by Result 2.4, $V \subseteq V \circ V$ which implies $V \subseteq U$ and so $V^{-1} \subseteq U^{-1}$, i.e. $V \subseteq U^{-1}$ [since V is symmetric]. So by axiom (ii), $U^{-1} \in \mathcal{U}_\mu$. □

Result 2.6. *Every uniform space $(X, \mathcal{U})$ is a μ-uniform space.*

Proof. Axioms (i) and (ii) of Definition 2.3 are obvious from the definition of uniformity given in Definition 1.2. Now for axiom (iii) of Definition 2.3, consider $U \in \mathcal{U}$, then by axiom (v) of Definition 1.2 there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$; we set $W = V \cap V^{-1}$. By axioms (ii) and (iv) of Definition 1.2, we see that $W \in \mathcal{U}$, and it is also clear that W is symmetric and $W \circ W \subseteq U$. Hence, $(X, \mathcal{U})$ is a μ-uniform space. □

Note 2.7. *The converse of the above stated result is false i.e. a μ-uniformity on a set X need not be a uniformity on X. In fact, consider $X = \{a, b, c\}$ and $A = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$, $B = \{(a, a), (b, b), (c, c), (c, b), (b, c)\}$. We set $\mathcal{U}_\mu = \{U \subseteq X \times X : A \subseteq U \text{ or } B \subseteq U\}$. It is clear that $\mathcal{U}_\mu$ is a μ-uniformity on X. But $A \cap B = \{(a, a), (b, b), (c, c)\} \notin \mathcal{U}_\mu$, which does not satisfy (ii) of Definition 1.2, and hence it is not a uniformity.*

Definition 2.8. *[7] Let X be a nonempty set. A nonempty family $\mathcal{U}$ of subsets of $X \times X$ is called a generalized quasi uniformity (or g-quasi uniformity) on X if the following hold:*

(1) $\Delta \subseteq U, \forall U \in \mathcal{U}$.

(2) $U \in \mathcal{U}$ *and* $U \subseteq V \Rightarrow V \in \mathcal{U}$.

(3) $U \in \mathcal{U} \Rightarrow \exists V \in \mathcal{U}$ *such that* $V \circ V \subseteq U$.

Remark 2.9. *It is a straightforward to observe that every μ-uniform space is also a g-quasi uniform space as defined in [7]. But the converse is not true.*
Consider the set $X = \{a, b, c\}$ and the subset U of $X \times X$ given by $U = \{(a, a), (b, b), (c, c), (a, b)\}$. Set $\mathcal{U}_\mu = \{V \subseteq X \times X : U \subseteq V\}$. It is clear that $\mathcal{U}_\mu$ is a g-quasi uniformity on X. Now $U \in \mathcal{U}_\mu$ but there does not exist any symmetric $A \subseteq X \times X$ in $\mathcal{U}_\mu$ such that $A \circ A \subseteq U$. Hence $(X, \mathcal{U}_\mu)$ is not a μ-uniform space.
So the family of all μ-uniform spaces is coarser than the family of all g-quasi uniform spaces but finer than the collection of all uniform spaces.

Theorem 2.10. Let $\mathcal{U}_\mu$ be a μ-uniformity on a non-empty set X. Let a family τ_μ of subsets of X be defined by:
A subset $G \in \tau_\mu$ if and only if for every $p \in G$, there exists some $U_p \in \mathcal{U}_\mu$ such that $U_p(p) \subseteq G$. Then τ_μ is a strong generalized topology on X.

Proof. Clearly $\phi \in \tau_\mu$. For each $p \in X$, $U(p) \subseteq X$, for any $U \in \mathcal{U}_\mu$ so $X \in \tau_\mu$.
Let $G_\alpha \in \tau_\mu$, where $\alpha \in \Lambda$, an index set. Let $G = \bigcup_{\alpha \in \Lambda} G_\alpha$ and $p \in G$. Then $p \in G_\beta$ for some $\beta \in \Lambda$, so there exists $U_p \in \mathcal{U}_\mu$ such that $U_p(p) \subseteq G_\beta \subseteq G$. Hence, $G \in \tau_\mu$.
So, τ_μ is a strong generalized topology on X. □

Definition 2.11. *The generalized topology τ_μ obtained in the previous theorem from the μ-uniformity $\mathcal{U}_\mu$ on X is called the generalized topology on X induced by $\mathcal{U}_\mu$ and will be denoted by $\tau(\mathcal{U}_\mu)$.*
Henceforth, the GTS $(X, \tau(\mathcal{U}_\mu))$ will be called a μ-uniform space.

3. μ-uniformity and μ-complete regularity

Definition 3.1. *[2] A GTS (X, μ) is said to be μ-completely regular if for any μ-closed set A in X and for $x \notin A$, there exists a μ-continuous function $f : (X, \mu) \to (\mathbb{R}, \nu)$ such that $f(x) = 0$ and $f(A) = \{1\}$, where ν is the generalized topology on the set $\mathbb{R}$ of reals generated by the base $\beta = \{(-\infty, t) : t \in \mathbb{R}\} \cup \{(t, \infty) : t \in \mathbb{R}\}$.*

Theorem 3.2. A μ-uniformizable GTS (X, μ) is μ-completely regular.

Proof. Given that the GTS (X, μ) is μ-uniformizable, i.e. there exists a μ-uniformity $\mathcal{U}_\mu$ on X such that $\mu = \tau(\mathcal{U}_\mu)$. Let F be μ-closed and $p \notin F$. Thus $X \setminus F = W$(say) is μ-open and $p \in W$, so there exists $U \in \mathcal{U}_\mu$ such that $U(p) \subseteq W$.
Now we shall show by induction that for every $n \in \mathbb{N} \cup \{0\}$, we can construct a symmetric member $U_n \in \mathcal{U}_\mu$ such that $U_n \subseteq U$ and $U_n \circ U_n \subseteq U_{n-1} \subseteq U$, when n is positive with $U = U_0$.
In fact, let $U = U_0$; then there exists a symmetric $U_1 \in \mathcal{U}_\mu$ such that $U_1 \circ U_1 \subseteq U_0$, where $U_1 = U_1 \circ \Delta \subseteq U_1 \circ U_1 \subseteq U_0$. Let U_{n-1} have been constructed in this way, then there exists a symmetric $U_n \in \mathcal{U}_\mu$ such that $U_n \circ U_n \subseteq U_{n-1}$ and similarly $U_n = U_n \circ \Delta \subseteq U_n \circ U_n \subseteq U_{n-1} \subseteq U$.
So, we get a decreasing sequence $\{U_n : n \geq 0\}$ with each member being a subset of U.
Next for every diadic rational [A diadic rational number r is of the form $r = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \cdots + \frac{1}{2^{n_m}} = \frac{p}{2^{n_m}}$, where p is some positive integer] $r \in (0, 1]$, we define $V_r = U_{n_1} \circ U_{n_2} \circ \ldots \circ U_{n_m}$, where $r = \Sigma_{i=1}^{m} 2^{-n_i}$ with $0 \leq n_1 < n_2 < \ldots < n_m$; since every diadic rational number has unique expression, V_r is well-defined. We define $V_0 = \Delta$, though it may not be in $\mathcal{U}_\mu$ and also note that $V_1 = U_0$. Then it can be shown that (Lemma 3.3 below)

$$V_{k2^{-n}} \subseteq V_{k2^{-n}} \circ U_n \subseteq V_{(k+1)2^{-n}} \qquad \ldots(\bigstar)$$

which holds for every non-negative n and all $k = 0, 1, \ldots, 2^n - 1$. Also for two diadic rational numbers r, s with $0 \le r \le s \le 1$, there exists positive integer n such that $r = i \cdot 2^{-n}$ and $s = j \cdot 2^{-n}$, where i, j are positive integers satisfying $0 \le i \le j \le 2^n$.
Hence, we have $V_r = V_{i\cdot 2^{-n}} \subseteq V_{(i+1)2^{-n}} \subseteq \cdots \subseteq V_{j\cdot 2^{-n}} = V_s$. Thus if $0 \le r \le s \le 1$ and r, s are diadic rationals then $V_r \subseteq V_s$.

Next, we define a function $g: X \to [0, 1]$ by taking

$$g(x) = \begin{cases} \sup\{r : x \notin V_r(p)\}, & \text{for } x \neq p \\ 0, & \text{for } x = p \end{cases}.$$

Since $V_0 = \Delta$, $V_0(p) = \{p\}$. For each $x(\neq p) \in X, x \notin V_0(p) \Rightarrow 0 \in \{r : x \notin V_r(p)\} \Rightarrow \{r : x \notin V_r(p)\} \neq \phi$. Also, $r \le 1 \Rightarrow \{r : x \notin V_r(p)\}$ is bounded above and so its supremum exists.
Now for any point $q \in F$, i.e. $q \in X \setminus W$, we have $q \notin V_1(p)$, as $U(p) \subseteq W$ and $V_1 = U_0 \subseteq U$. Again, $q \notin V_1(p) \Rightarrow 1 \in \{r : q \notin V_r(p), r \le 1\} \Rightarrow g(q) = 1$.
Finally, we shall show that g is μ-continuous in (X, μ). For this it is enough to show that $g^{-1}([0, t))$ and $g^{-1}((t, 1])$ are μ-open [since $[0, t), (t, 1]$ are the basic μ-open sets of $[0, 1]$ where $t \in (0, 1)$, when it is considered as a subspace of the GTS $(\mathbb{R}, \nu)$ defined previously]. Let $x \in g^{-1}([0, t))$, then $g(x) \in [0, t)$; let us take $g(x) = s$ then $s < t \le 1$. We set $r = t - s > 0$, now there exists $n \in \mathbb{N}$ such that $2^n > \frac{2}{r}$. We show that $U_n(x) \subseteq g^{-1}([0, t))$, consequently $g^{-1}([0, t)) \in \tau(\mathcal{U}_\mu) = \mu$.
Now let k be the uniquely determined positive integer satisfying $k - 1 \le s \cdot 2^n < k$ i.e. $(k-1)2^{-n} \le s < k \cdot 2^{-n}$, then $g(x) = s < k \cdot 2^{-n}$. Now, $x \notin V_{k2^{-n}}(p) \Rightarrow k \cdot 2^{-n} \in \{r : x \notin V_r(p)\} \Rightarrow s = \sup\{r : x \notin V_r(p)\} \ge k \cdot 2^{-n}$, which is a contradiction. So $x \in V_{k2^{-n}}(p) \Rightarrow (p, x) \in V_{k2^{-n}}$. Also for $y \in U_n(x)$ we get $(x, y) \in U_n$. Hence, $(p, y) \in V_{k2^{-n}} \circ U_n \subseteq V_{(k+1)2^{-n}}$, by (a), and so $y \in V_{(k+1)2^{-n}}(p)$, and hence $g(y) \le (k+1)2^{-n}$. Therefore, $g(y) - s \le (k+1)2^{-n} - (k-1)2^{-n} = \frac{2}{2^n} < r = t - s$ i.e. $g(y) < t \Rightarrow y \in g^{-1}([0, t))$. Hence, $U_n(x) \subseteq g^{-1}([0, t))$, so $g^{-1}([0, t)) \in \tau(\mathcal{U}_\mu) = \mu$.
Next, for $g^{-1}((t, 1])$, let $x \in g^{-1}((t, 1])$, then $g(x) = s > t \ge 0$. Let $r = s - t > 0$ and $n \in \mathbb{N}$ so that $2^n > \frac{2}{r}$. We shall show that $U_n(x) \subseteq g^{-1}((t, 1])$. Let k be the uniquely determined positive integer satisfying $(k-1)2^{-n} \le t < k \cdot 2^{-n}$. If possible, let $y \in U_n(x)$ and $y \notin g^{-1}((t, 1])$. Then $g(y) \le t < k \cdot 2^{-n}$ and so $y \in V_{k2^{-n}}(p)$ (in fact otherwise, $y \notin V_{k2^{-n}}(p) \Rightarrow g(y) \ge k \cdot 2^{-n}$). Therefore $(p, y) \in V_{k2^{-n}}$ and since $y \in U_n(x)$, $(x, y) \in U_n$ and hence, as U_n is symmetric, $(y, x) \in U_n$. Thus $(p, x) \in V_{k2^{-n}} \circ U_n \subseteq V_{(k+1)2^{-n}}$ [by (★)]. So, $x \in V_{(k+1)2^{-n}}(p)$. Consequently, $g(x) \le (k+1)2^{-n}$. Now $g(x) - t \le (k+1)2^{-n} - (k-1)2^{-n} = \frac{2}{2^n} < r \Rightarrow s - t < r$, a contradiction to the equality.
Hence $U_n(x) \subseteq g^{-1}((t, 1])$, so $g^{-1}((t, 1]) \in \tau(\mathcal{U}_\mu) = \mu$. Hence, g is μ-continuous and so (X, μ) is μ-completely regular. □

Lemma 3.3. *Following the same notations as in Theorem 3.2, the inclusion relation $V_{k\cdot 2^{-n}} \subseteq V_{k\cdot 2^{-n}} \circ U_n \subseteq V_{(k+1)2^{-n}}$ holds for every non-negative integer n and for $k = 0, 1, 2, \ldots, 2^n - 1$.*

Proof. This relation holds for $n = 0$, since for $n = 0$, $k = 0$ and $V_0 = \Delta$ so that $V_0 \circ U_0 = U_0 = V_1$. Let $n > 0$ and we assume that the inclusions hold for $n - 1$. We shall prove the inclusions for n. Since $V_{k\cdot 2^{-n}} = V_{k\cdot 2^{-n}} \circ \Delta \subseteq V_{k\cdot 2^{-n}} \circ U_n$ is always true, it remains only to prove $V_{k\cdot 2^{-n}} \circ U_n \subseteq V_{(k+1)2^{-n}}$, for $k = 0, 1, 2, \ldots, 2^n - 1$.
If k is an even integer, say $k = 2m$, we have $k \cdot 2^{-n} = (2m) \cdot 2^{-n} = m \cdot 2^{-(n-1)}$, i.e. $(k+1) \cdot 2^{-n} = m \cdot 2^{-(n-1)} + 2^{-n} = (2m+1) \cdot 2^{-n}$.
It then follows from the definition of the sets V_r, given in Theorem 3.2, that $V_{(k+1)\cdot 2^{-n}} = V_{m\cdot 2^{-(n-1)}} \circ U_n = V_{k\cdot 2^{-n}} \circ U_n$, thus the inclusion is proved in this case.
If k is an odd integer, say $k = 2m + 1$, then $k \cdot 2^{-n} = (2m+1) \cdot 2^{-n} = m \cdot 2^{-(n-1)} + 2^{-n}$ and $(k+1) \cdot 2^{-n} = (2m+2) \cdot 2^{-n} = (m+1) \cdot 2^{-(n-1)}$. By our induction hypothesis, we get $V_{m\cdot 2^{-(n-1)}} \circ U_{n-1} \subseteq V_{(m+1)\cdot 2^{-(n-1)}}$. ...(*)

Since $U_n \circ U_n \subseteq U_{n-1}$, it implies that $V_{k\cdot 2^{-n}} \circ U_n = V_{m\cdot 2^{-(n-1)}+2^{-n}} \circ U_n = V_{m\cdot 2^{-(n-1)}} \circ U_n \circ U_n \subseteq V_{m\cdot 2^{-(n-1)}} \circ U_{n-1}$ and by using (*) we get $V_{k\cdot 2^{-n}} \circ U_n \subseteq V_{(m+1)\cdot 2^{-(n-1)}} = V_{(k+1)\cdot 2^{-n}}$. Thus, the inclusion also holds for odd integers. □

Remark 3.4. *It is still an open problem whether a μ-completely regular GT is μ-uniformizable.*

4. μ-uniformity and μ-proximity

In a uniform space $(X, \mathcal{U})$, there is a result that a uniformity always induces a proximity on X which generates the same topology as is induced by $\mathcal{U}$ on X. In the following theorem, we also have a similar result for a GTS. First we state the definition of μ-proximity.

Definition 4.1. *[2] A binary relation δ_μ on the power set $\mathcal{P}(X)$ of a set X is called a μ-proximity on X if δ_μ satisfies the following axioms:*

(1) $A\delta_\mu B$ *iff* $B\delta_\mu A, \ \forall A, B \in \mathcal{P}(X)$

(2) *If* $A\delta_\mu B$, $A \subseteq C$ *and* $B \subseteq D$, *then* $C\delta_\mu D$

(3) $\{x\}\delta_\mu\{x\}, \forall x \in X$

(4) $A \not\delta_\mu B \Rightarrow \exists\, E(\subseteq X)$ *such that* $A \not\delta_\mu E$ *and* $(X \setminus E) \not\delta_\mu B$.

Now δ_μ generates a generalized topology on X which is given below:

Proposition 4.2. *[2] Let a subset A of a μ-proximity space (X, δ_μ) be defined to be δ_μ-closed iff $(\{x\}\delta_\mu A \Rightarrow x \in A)$. Then the collection of complements of all δ_μ-closed sets so defined, yields a generalized topology $\mu = \tau(\delta_\mu)$ on X.*

Proposition 4.3. *[2] Let (X, δ_μ) be a μ-proximity space and $\mu = \tau(\delta_\mu)$. Then the μ-closure $c_\mu(A)$ of a set A in (X, μ) is given by $c_\mu(A) = \{x : \{x\}\delta_\mu A\}$.*

Lemma 4.4. *Let $(X, \mathcal{U}_\mu)$ be a μ-uniform space. Then for $A, B \subseteq X$, $U(A) \cap U(B) \neq \phi$, for all $U \in \mathcal{U}_\mu$ if and only if $U(A) \cap B \neq \phi$ for all $U \in \mathcal{U}_\mu$.*

Proof. Let $U(A) \cap B \neq \phi$. Since $B \subseteq U(B)$ (as $\Delta \subseteq U$), we get $U(A) \cap U(B) \neq \phi$ for all $U \in \mathcal{U}_\mu$.Conversely, let $U(A) \cap U(B) \neq \phi$ for all $U \in \mathcal{U}_\mu$ and if possible let there exist $V \in \mathcal{U}_\mu$ such that $V(A) \cap B = \phi$. Now there exists a symmetric $W \in \mathcal{U}_\mu$ such that $W \circ W \subseteq V$. By the given condition, $W(A) \cap W(B) \neq \phi$ and let $p \in W(A) \cap W(B)$, i.e. $(a, p) \in W$ and $(b, p) \in W$ for some $a \in A$, $b \in B$. Since W is symmetric, we get $(a, b) \in W \circ W \subseteq V$ which implies $b \in V(a) \subseteq V(A)$. Thus $V(A) \cap B \neq \phi$, a contradiction. □

Theorem 4.5. For a μ-uniform space $(X, \mathcal{U}_\mu)$, the relation δ_μ defined on $\mathcal{P}(X)$ by

$A\delta_\mu B$ if and only if for every $U \in \mathcal{U}_\mu$, $U(A) \cap U(B) \neq \phi$
is a μ-proximity structure on X such that $\tau(\mathcal{U}_\mu) = \tau(\delta_\mu)$.

Proof. To show that δ_μ is a μ-proximity on X we proceed in the following manner:

(1) For $A, B \subseteq X$, clearly $A\delta_\mu B$ iff $B\delta_\mu A$.

(2) Let $A\delta_\mu B$ with $A \subseteq C$ and $B \subseteq D$, so for any $U \in \mathcal{U}_\mu$, $U(A) \cap U(B) \neq \phi$. Now $U(A) \subseteq U(C)$ and $U(B) \subseteq U(D)$, therefore $U(C) \cap U(D) \neq \phi$. Hence $C\delta_\mu D$.

(3) For all $x \in X$, $x \in U(x) \cap U(x)$, for all $U \in \mathcal{U}_\mu$ which implies $U(x) \cap U(x) \neq \phi$ for all $U \in \mathcal{U}_\mu$ and so $\{x\}\delta_\mu\{x\}$.

(4) Let $A, B \in \mathcal{P}(X)$ such that $A\not\delta_\mu B$. Then for some $U \in \mathcal{U}_\mu$, $U(A) \cap U(B) = \phi$; we set $C = U(A)$ and $D = U(B)$. It is clear that $A \subseteq C$. We show that $A\not\delta_\mu(X \setminus C)$. In fact, $A\delta_\mu(X \setminus C) \Rightarrow$ for

every $V \in \mathcal{U}_\mu$, $V(A) \cap V(X \setminus U(A)) \neq \phi$. Let W be a symmetric member of $\mathcal{U}_\mu$ such that $W \circ W \subseteq U$, then $W(A) \cap W(X \setminus U(A)) \neq \phi$ and so there exists $p \in W(A) \cap W(X \setminus U(A))$. Therefore, there exists $a \in A$, $b \in X \setminus U(A)$ such that $(a, p) \in W$ and $(b, p) \in W$, now W being symmetric, $(a, b) \in W \circ W \subseteq U$ which implies $b \in U(a) \subseteq U(A)$, a contradiction to the fact that $b \in X \setminus U(A)$. Thus $A \not\delta_\mu (X \setminus C)$. Similarly, $B \subseteq D$ and $B \not\delta_\mu (X \setminus D)$, also as $C \cap D = U(A) \cap U(B) = \phi$, $B \not\delta_\mu C$. In fact, if $B \delta_\mu C$ then as $C \subseteq (X \setminus D)$ that implies $B \delta_\mu (X \setminus D)$ [using (ii) in this proof shown above], a contradiction. Thus, we see that axiom (iv) of μ-proximity is satisfied. Finally, we show that $\tau(\mathcal{U}_\mu) = \tau(\delta_\mu)$. Let $A \subseteq X$ and $x \in X$. Then $x \in c_{\tau(\mathcal{U}_\mu)} A \Leftrightarrow U(x) \cap A \neq \phi$, for all $U \in \mathcal{U}_\mu \Leftrightarrow U(x) \cap U(A) \neq \phi$, for all $U \in \mathcal{U}_\mu$ [by Lemma 4.4] $\Leftrightarrow \{x\} \delta_\mu A \Leftrightarrow x \in c_{\tau(\delta_\mu)} A$ [by Proposition 4.3]. Thus, $\tau(\mathcal{U}_\mu) = \tau(\delta_\mu)$. □

Remark 4.6. *It is still an open problem whether a μ-proximity structure δ_μ on a set X induces a μ-uniformity $\mathcal{U}_\mu$ on X such that $\tau(\mathcal{U}_\mu) = \tau(\delta_\mu)$.*

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