Some fixed-point theorems for a general class of mappings in modular G-metric spaces

Godwin Amechi Okeke and Daniel Francis
Functional Analysis and Optimization Research Group Laboratory (FANORG), Department of Mathematics, School of Physical Sciences, Federal University of Technology, Owerri, Nigeria

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Abstract

Purpose – This paper aims to prove some fixed-point theorems for a general class of mappings in modular G-metric spaces. The results of this paper generalize and extend several known results to modular G-metric spaces, including the results of Mutlu *et al.* [1]. Furthermore, the authors produce an example to demonstrate the applicability of the results.

Design/methodology/approach – The results of this paper are theoretical and analytical in nature.

Findings – The authors established some fixed-point theorems for a general class of mappings in modular G-metric spaces. The results generalize and extend several known results to modular G-metric spaces, including the results of Mutlu *et al.* [1]. An example was constructed to demonstrate the applicability of the results.

Research limitations/implications – Analytical and theoretical results.

Practical implications – The results of this paper can be applied in science and engineering.

Social implications – The results of this paper is applicable in certain social sciences.

Originality/value – The results of this paper are new and will open up new areas of research in mathematical sciences.

Keywords Fixed point, Lower semicontinuous function, Modular G-metric spaces

Paper type Research paper

1. Introduction

In search for the generalization of classical metric spaces, in 1966, Gahler [2], introduced the concept of 2-metric spaces and proved that its results exists. Dhage [3] extend the work in [2] in which D-metric spaces were introduced. These authors claimed that their results generalized the concept of metric spaces.

In 2003, Mustafa and Sims [4] claimed that the fundamental topological properties of D-metric spaces introduced by Dhage [3] were incorrect. To ameliorate the drawbacks about D-metric spaces, Mustafa and Sims [5] introduced a generalization of metric spaces, which they called G-metric spaces and proved some fixed-point theorems, and in [6], Mustafa *et al.* proved some fixed-point results on complete G-metric spaces.

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Modular theories on linear spaces were given by Nakano in his two monographs [7, 8], where he developed a spectral theory in semiordered linear spaces (vector lattices) and established the integral representation for projections acting in this modular spaces. Nakano [7] established some modulars on real linear spaces, which are convex functionals. Nonconvex modulars and the corresponding modular linear spaces were constructed by Musielak and Orlicz [9]. Orlicz spaces and modular linear spaces have already become classical tools in modern nonlinear functional analysis.

In 2010, a remarkable work of Chistyakov [10] introduced an aspect of metric spaces called modular metric spaces or parameterized metric spaces with the time parameter λ (say), and his purpose was to define the notion of a modular on an arbitrary set and developed the theory of metric spaces generated by modulars, called modular metric spaces and, on the basis of it, defined new metric spaces of (multi-valued) functions of bounded generalized variation of a real variable with values in metric semigroups and abstract convex cones.

In the same year, Chistyakov [11], as an application, presented an exhausting description of Lipschitz continuous and some other classes of superposition (Nemytskii) operators, acting in these modular metric spaces. Chistyakov developed the theory of metric spaces generated by modulars and extended the results given by Nakano [7], Musielak and Orlicz [9] and Musielak [12] to modular metric spaces. Modular spaces are extensions of Lebesgue, Riesz and Orlicz spaces of integrable functions.

The development of theory of metric spaces generated by modulars, called modular metric spaces attracted many research mathematicians still investigating fixed-point results in this area, including Chistyakov himself. Chistyakov [13] also established some fixed-point theorems for contractive maps in modular spaces. It is related to contracting, rather generalized average velocities than metric distances, and the successive approximations of fixed points converge to the fixed points in a weaker sense as compared to the metric convergence in [13] and other fixed-point results in modular metric spaces can be found in [1, 14]. Considering applicability, these fixed-point results are applied in finding the fixed-point solution of nonlinear integral equations see [14–16] and references therein, while [17] deals with application to partial differential equation in modular metric spaces. Interested readers may see [16, 18–21] and the references therein for further studies in modular function spaces.

In 2013, Azadifar *et al.* [22] introduced the concept of modular G-metric space and obtained some fixed-point theorems of contractive mappings defined on modular G-metric spaces. Our intention in this paper is to extend the fixed-point theorem of Mutlu *et al.* [1] from the setting of modular metric spaces to modular G-metric spaces. Our results extend and generalize several known results in the literature. For results in non-unique fixed-point theorems in modular metric spaces, readers should also see Hussain [23] and references therein.

Zhao [24] 2019 applied the exponential dichotomy, and Tikhonov and Banach fixed-point theorems are used to study the existence and uniqueness of pseudo almost periodic solutions of a class of iterative functional differential equations of the form $x'(t) = \sum_{n=1}^{k}\sum_{l=1}^{\infty} C_{l,n}(t) (x^{[n]}(t))^{l} + G(t)$, where $x^{[n]}(t)$ is the nth iterate of $x(t)$.

Recently, Combettes and Glaudin [25] constructed iteratively, a common fixed-point of nonexpansive operators by activating only a block of operators at each iteration. In the more challenging class of composite fixed-point problems involving operators that do not share common fixed points, current methods require the activation of all the operators at each iteration, and the question of maintaining convergence while updating only blocks of operators is open. They propose a method that achieves this goal and analyzed its asymptotic behavior. Weak, strong and linear convergence results are established by exploiting a connection with the theory of concentrating arrays. Applications to several nonlinear and nonsmooth analysis problems are presented, ranging from monotone inclusions and inconsistent feasibility problems to variational inequalities and minimization problems arising in data science.

2. Preliminaries

Definition 2.1. [22] *Let X be a nonempty set, and let $\omega^G : (0, \infty) \times X \times X \times X \to [0, \infty]$ be a function satisfying;*

(1) $\omega_\lambda^G(x,y,z) = 0$ *for all* $x, y, z \in X$ *and* $\lambda > 0$ *if* $x = y = z$,

(2) $\omega_\lambda^G(x,x,y) > 0$ *for all* $x, y \in X$ *and* $\lambda > 0$ *with* $x \neq y$,

(3) $\omega_\lambda^G(x,x,y) \leq \omega_\lambda^G(x,y,z)$ *for all* $x, y, z \in X$ *and* $\lambda > 0$ *with* $z \neq y$,

(4) $\omega_\lambda^G(x,y,z) = \omega_\lambda^G(x,z,y) = \omega_\lambda^G(y,z,x) = \cdots$ *for all* $\lambda > 0$ *(symmetry in all three variables),*

(5) $\omega_{\lambda+\mu}^G(x,y,z) \leq \omega_\lambda^G(x,a,a) + \omega_\mu^G(a,y,z)$, *for all* $x, y, z, a \in X$ *and* $\lambda, \nu > 0$,
then the function ω_λ^G *is called a modular G-metric on X.*

Remarks 2.1. *(a) The pair (X, ω^G) is called a modular G-metric space, and without any confusion, we will take X_{ω^G} as a modular G-metric space. From condition (5) above, if ω^G is convex, then we have a strong form as,*

(b) $\omega_{\lambda+\mu}^G(x,y,z) \leq \omega_{\frac{\lambda}{\lambda+\mu}}^G(x,a,a) + \omega_{\frac{\mu}{\lambda+\mu}}^G(a,y,z)$,

(c) If $x = a$, then (5) above becomes $\omega_{\lambda+\mu}^G(a,y,z) \leq \omega_\mu^G(a,y,z)$ and

(d) Condition (5) is called rectangle inequality.

Definition 2.2. [22] *Let (X, ω^G) be a modular G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is modular G-convergent to x, if it converges to x in the topology $\tau(\omega_\lambda^G)$.*
A function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called modular G-continuous if $\omega_\lambda^G(x_n, x, x) \to 0$ then $\omega_\lambda^G(Tx_n, Tx, Tx) \to 0$, for all $\lambda > 0$.

Remark 2.1. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is modular G-converges to x as $n \to \infty$, if $\lim_{n\to\infty} \omega_\lambda^G(x_n, x_m, x) = 0$. That is for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, x) < \epsilon$ for all $n, m \geq n_0$. Here we say that x is modular G-limit of $\{x_n\}_{n\in\mathbb{N}}$.*

Definition 2.3. [22] *Let (X, ω^G) be a modular G-metric space, then $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ is said to be modular G-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq n_\epsilon$ and $\lambda > 0$.*
A modular G-metric space X_{ω^G} is said to be modular G-complete if every modular G-Cauchy sequence in X_{ω^G} is modular G-convergent in X_{ω^G}.

Proposition 2.1. [22] *Let (X, ω^G) be a modular G-metric space, for any $x, y, z, a \in X$, it follows that*

(1) *If* $\omega_\lambda^G(x,y,z) = 0$ *for all* $\lambda > 0$, *then* $x = y = z$.

(2) $\omega_\lambda^G(x,y,z) \leq \omega_{\frac{\lambda}{2}}^G(x,x,y) + \omega_{\frac{\lambda}{2}}^G(x,x,z)$ *for all* $\lambda > 0$.

(3) $\omega_\lambda^G(x,y,y) \leq 2\omega_{\frac{\lambda}{2}}^G(x,x,y)$ *for all* $\lambda > 0$.

(4) $\omega^G_\lambda(x,y,z) \le \omega^G_{\frac{\lambda}{2}}(x,a,z) + \omega^G_{\frac{\lambda}{2}}(a,y,z)$ *for all* $\lambda > 0$.

(5) $\omega^G_\lambda(x,y,z) \le \frac{2}{3}(\omega^G_{\frac{\lambda}{2}}(x,y,a) + \omega^G_{\frac{\lambda}{2}}(x,a,z) + \omega^G_{\frac{\lambda}{2}}(a,y,z))$ *for all* $\lambda > 0$.

(6) $\omega^G_\lambda(x,y,z) \le \omega^G_{\frac{\lambda}{2}}(x,a,a) + \omega^G_{\frac{\lambda}{4}}(y,a,a) + \omega^G_{\frac{\lambda}{4}}(z,a,a)$ *for all* $\lambda > 0$.

Proposition 2.2. [22] *Let* (X, ω^G) *be a modular G-metric space and* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{ω^G}. *Then the following are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ *is* ω^G*-convergent to* x,

(2) $\omega^G_\lambda(x_n, x) \to 0$ *as* $n \to \infty$, *i.e.* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *relative to modular metric* $\omega^G_\lambda(.)$,

(3) $\omega^G_\lambda(x_n, x_n, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$,

(4) $\omega^G_\lambda(x_n, x, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$ *and*

(5) $\omega^G_\lambda(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$ *for all* $\lambda > 0$.

The following construction was motivated by conditions (3) and (4) of Proposition 2.2 above and [1].

Let $\omega^G : (0, \infty) \times X \times X \times X \to [0, \infty]$ be a modular G-metric on X, X_{ω^G} be a modular G-metric space, $B \subseteq X_{\omega^G}$ and $\kappa : B \to \mathbb{R}^+ \cup \{\infty\}$ be a function on B. κ is called lower semicontinuous on B if $\lim_{n\to\infty} \omega^G_\lambda(x, x_n, x_n) = 0 \Rightarrow \kappa(x) \le \liminf_{n\to\infty} \kappa(x_n)$, or $\lim_{n\to\infty} \omega^G_\lambda(x, x, x_n) = 0 \Rightarrow \kappa(x) \le \liminf_{n\to\infty} \kappa(x_n)$ for all $\lambda > 0$ and $\{x_n\}_{n\ge 1} \subseteq B$. B is closed, if the limit of a modular G-convergent sequence in B always belongs to B. Also B is modular G-bounded, if $\delta_{\omega^G}(B) = \sup\{\omega^G_\lambda(x, y, y) : x, y \in B, \ \forall\ \lambda > 0\}$ is finite.

3. Main results

We begin this section with the following results, which extends the results of Mutlu *et al.* [1] from the setting of modular metric spaces to modular G-metric spaces.

Theorem 3.1. *Let* ω^G *be a modular G-metric on X,* X_{ω^G} *be a complete modular G-metric space,* $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ *be a lower semicontinuous function on* X_{ω^G} *and* $T : X_{\omega^G} \to X_{\omega^G}$ *be a self-map such that*

$$\kappa(Tx) + \omega^G_\lambda(x, Tx, Tx) \le \kappa(x) \tag{3.1}$$

for all $x \in X_{\omega^G}$ *and* $\lambda > 0$. *Then T has a fixed point in* X_{ω^G}.

Proof. For any $x \in X_{\omega^G}$, let $F(x) = \{y \in X_{\omega^G} : \omega^G_\lambda(x, y, y) \le \kappa(x) - \kappa(y), \ \forall\ \lambda > 0\}$ and $\eta(x) = \inf\{\kappa(y) : y \in F(x)\}$. Since $x \in F(x)$, therefore, $F(x) \ne \emptyset$ and $0 \le \eta(x) \le \kappa(x)$. Let $x \in X_{\omega^G}$ be an arbitrary point. Now, we construct a sequence $\{x_n\}_{n\ge 1}$ in X_{ω^G} as follows. Let $x = x_1$ and when $x_1, x_2, \ldots, x_n$ have been chosen, choose $x_{n+1} \in F(x)$ such that $\kappa(x_{n+1}) \le \eta(x_n) + \frac{1}{2^n}$ for all $n \in \mathbb{N}$. By the process above, we get a sequence $\{x_n\}_{n\ge 1}$ satisfying the conditions.

$$\omega^G_\lambda(x_n, x_{n+1}, x_{n+1}) \le \kappa(x_n) - \kappa(x_{n+1}), \ \ \eta(x_n) \le \kappa(x_{n+1}) \le \eta(x_n) + \frac{1}{2^n} \tag{3.2}$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. Then $\{\kappa(x_n)\}_{n\ge 1}$ is a nonincreasing sequence in $\mathbb{R}$, and it is bounded blow by zero. So, the sequence $\{\kappa(x_n)\}_{n\ge 1}$ is convergent to a real number $M \ge 0$ (say). By inequality (3.2), we get

$$M = \lim_{n\to\infty}\kappa(x_n) = \lim_{n\to\infty}\eta(x_n) \tag{3.3}$$

Now, let $k\in\mathbb{N}$ be arbitrary, from inequalities (3.2) and (3.3), there exits at least a positive number N_k such that $\kappa(x_n) < M + \frac{1}{2^k}$ for all $n \geq N_k$. Since $\kappa(x_n)$ is monotone, we get $M \leq \kappa(x_m) \leq \kappa(x_n) < M + \frac{1}{2^k}$ for $m \geq n \geq N_k$. It follows that

$$\kappa(x_n) - \kappa(x_m) < \frac{1}{2^k} \text{ for all } m \geq n \geq N_k. \tag{3.4}$$

Without loss of generality, suppose that $m > n$ and $m, n \in \mathbb{N}$. From inequality (3.2), we get

$$\omega^G_{\frac{\lambda}{m-n}}(x_n, x_{n+1}, x_{n+1}) \leq \kappa(x_n) - \kappa(x_{n+1}), \text{ for } \frac{\lambda}{m-n} \geq \frac{\lambda}{n} > 0. \tag{3.5}$$

Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, i.e. condition (5) of Definition (2.1) we have

$$\begin{aligned}
\omega^G_\lambda(x_n, x_m, x_m) &\leq \omega^G_{\frac{\lambda}{m-n}}(x_n, x_{n+1}, x_{n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}, x_{n+2}) + \omega^G_{\frac{\lambda}{m-n}}(x_{n+2}, x_{n+3}, x_{n+3}) \\
&\quad + \omega^G_{\frac{\lambda}{m-n}}(x_{n+3}, x_{n+4}, x_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{m-n}}(x_{m-1}, x_m, x_m) \\
&\leq \omega^G_{\frac{\lambda}{n}}(x_n, x_{n+1}, x_{n+1}) + \omega^G_{\frac{\lambda}{n}}(x_{n+1}, x_{n+2}, x_{n+2}) + \omega^G_{\frac{\lambda}{n}}(x_{n+2}, x_{n+3}, x_{n+3}) \\
&\quad + \omega^G_{\frac{\lambda}{n}}(x_{n+3}, x_{n+4}, x_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{n}}(x_{m-1}, x_m, x_m) \\
&\leq \kappa(x_n) - \kappa(x_{n+1}) + \kappa(x_{n+1}) - \kappa(x_{n+2}) + \cdots + \kappa(x_{m-1}) - \kappa(x_m) \\
&= \kappa(x_n) - \kappa(x_m)
\end{aligned} \tag{3.6}$$

for all $m > n \geq N_k$ for some $N_k \in \mathbb{N}$. Then by inequality (3.4), we have

$$\omega^G_\lambda(x_n, x_m, x_m) < \frac{1}{2^k}, \tag{3.7}$$

for all $m, l, n \geq N_k$ for some $N_k \in \mathbb{N}$, so that by condition (2) of proposition (2.1), we have

$$\omega^G_\lambda(x_n, x_m, x_l) \leq \omega^G_{\frac{\lambda}{2}}(x_n, x_m, x_m) + \omega^G_{\frac{\lambda}{2}}(x_l, x_m, x_m), \tag{3.8}$$

so that

$$\begin{aligned}
\lim_{n,m,l\to\infty} \omega^G_\lambda(x_n, x_m, x_l) &\leq \lim_{n,m\to\infty} \omega^G_{\frac{\lambda}{2}}(x_n, x_m, x_m) + \lim_{l,m\to\infty} \omega^G_{\frac{\lambda}{2}}(x_l, x_m, x_m) \\
&\leq \lim_{n,m\to\infty} \omega^G_\lambda(x_n, x_m, x_m) + \lim_{l,m\to\infty} \omega^G_\lambda(x_l, x_m, x_m) \\
&< \frac{1}{2^k} + \frac{1}{2^k} \\
&= \frac{2}{2^k} = 2^{1-k}.
\end{aligned} \tag{3.9}$$

Thus, as $k \to \infty$, we have

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(x_n, x_m, x_l) = 0. \tag{3.10}$$

Therefore, we can say straightaway that $\{x_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence. The completeness of (X_ω, ω^G) implies that for any $\lambda > 0$, $\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, u) = 0$, i.e. for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim_{n\to\infty} x_n \to u \in X_{\omega^G}$ as $n \to \infty$. But $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ is a lower semicontinuous function on X_{ω^G}, using inequality (3.6), we get

$$\begin{aligned} \kappa(u) &\leq \lim_{m\to\infty} \inf(\kappa(x_m)) \\ &\leq \lim_{m\to\infty} \inf(\kappa(x_n) - \omega_\lambda^G(x_n, x_m, x_m)) \\ &= \kappa(x_n) - \omega_\lambda^G(x_n, u, u) \end{aligned} \tag{3.11}$$

Thus, we have that $\omega_\lambda^G(x_n, u, u) \leq \kappa(x_n) - \kappa(u)$. So that $u \in F(x_n)$ for all $n \in \mathbb{N}$ and hence $\eta(x_n) \leq \kappa(u)$. Then by inequality (3.3), we get $M \leq \kappa(u)$. Moreover, by lower semicontinuity of κ and inequality (3.3), we have $\kappa(u) \leq \lim_{n\to\infty} \inf \kappa(x_n) = M$. So $\kappa(u) = M$. From inequality (3.1), we know that $Tu \in F(u)$, such $u \in F(u)$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \omega_\lambda^G(x_n, Tu, Tu) &\leq \omega_{\frac{\lambda}{2}}^G(x_n, u, u) + \omega_{\frac{\lambda}{2}}^G(u, Tu, Tu) \\ &\leq \omega_\lambda^G(x_n, u, u) + \omega_\lambda^G(u, Tu, Tu) \\ &\leq \kappa(x_n) - \kappa(u) + \kappa(u) - \kappa(Tu) \\ &= \kappa(x_n) - \kappa(Tu). \end{aligned} \tag{3.12}$$

Thus, $Tu \in F(x_n)$, and this implies that $\eta(x_n) \leq \kappa(Tu)$. Hence, we obtain $M \leq \kappa(Tu)$. From inequality (3.1), we get $\kappa(Tu) \leq \kappa(u)$. As $\kappa(u) = M$, we have $\kappa(u) = M \leq \kappa(Tu) \leq \kappa(u)$. Therefore, $\kappa(Tu) = \kappa(u)$. Then from inequality (3.1), we get $\omega_\lambda^G(u, Tu, Tu) \leq \kappa(u) - \kappa(Tu) = \kappa(u) - \kappa(u) = 0$. Thus, $Tu = u$. Therefore, T has a fixed point in X_{ω^G}. □

Remark 3.1. *Suppose that ω^G is a modular G-metric on X, X_{ω^G} be a complete modular G-metric space, $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ be a lower semicontinuous function on X_{ω^G} and $T : X_{\omega^G} \to X_{\omega^G}$ be a self-map. To get inequality (3.1) of Theorem 3.1 in Mutlu et al.* [1]*, we invoke the definition of modular G metric space as follows for any $\lambda > 0$, define $\omega_\lambda(x, y, z) = \frac{1}{2\lambda}\{|x-y| + |y-z| + |x-z|\}$. Take $y = Tx$ and $z = Tx$, then inequality (3.1) transform into*

$$\omega_\lambda(x, Tx) \leq \kappa(x) - \kappa(Tx) \tag{3.13}$$

for all $x \in X_\omega$ and $\lambda > 0$. Then T has a fixed point in X_ω, which is clearly the result in Mutlu et al. [1].

Theorem 3.2. *Let ω^G be a modular G-metric on X, X_{ω^G} be a complete modular G-metric space, $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ be a lower semicontinuous function on X_{ω^G} and $T : X_{\omega^G} \to X_{\omega^G}$ be a self-map such that for some positive integer, $m \geq 1$,*

$$\omega_\lambda^G(x, T^m x, T^m x) \leq \kappa(x) - \kappa(T^m x) \tag{3.14}$$

for all $x \in X_{\omega^G}$ and $\lambda > 0$. Then T has a fixed point in X_{ω^G} for some positive integer $m \geq 1$.

Proof. By Theorem 3.1, T^m has a fixed point say $u \in X_{\omega^G}$ for some positive integer $m \geq 1$, by using inequality (3.14) for some positive integer $m \geq 1$. Now $T^m(Tu) = T^{m+1}u = T(T^m u) = Tu$, so Tu is a fixed point of T^m. Hence, we have $Tu = u$. Therefore, u is a fixed point of T because fixed point of T is also fixed point of T^m for some positive integer $m \geq 1$. □

Next, we produce the following example to demonstrate the applicability of our results.

Example 3.1. *Let $X_{\omega^G} = \mathbb{R}$ and we define the mapping $\omega^G : (0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,\infty]$ by $\omega^G_\lambda(x,y,y) = \frac{2}{\lambda}|x-y|$ for all $x,y \in \mathbb{R}$ and $\lambda > 0$. So we can see that $(\mathbb{R},\omega^G)$ is a complete modular G-metric space and let us define $T : (\mathbb{R},\omega^G) \to (\mathbb{R},\omega^G)$ by $Tx = \frac{1}{x}$ for $x \in \mathbb{R}^+ \setminus \{0\}$ and $\kappa : (\mathbb{R},\omega^G) \to \mathbb{R}^+ \cup \{\infty\}$ by $\kappa(x) = \frac{3}{2}|x|$ for which $\kappa(x)$ defined above is lower semicontinuous. Now we verify the inequality (3.1) of Theorem 3.1 as follows; For $x \in \mathbb{R}^+ \setminus \{0\}$ and $\lambda > 0$, we have*

$$\begin{aligned}
\omega^G_\lambda(x,Tx,Tx) &= \omega^G_\lambda\left(x,\frac{1}{x},\frac{1}{x}\right)\\
&= \frac{1}{\lambda}\left\{\left|x-\frac{1}{x}\right| + \left|\frac{1}{x}-\frac{1}{x}\right| + \left|x-\frac{1}{x}\right|\right\}\\
&= \frac{2}{\lambda}\left|x-\frac{1}{x}\right|\\
&= \frac{2}{\lambda}\left|\frac{x^2-1}{x}\right| = \frac{2}{\lambda}\left|\frac{(x-1)(x+1)}{x}\right|\\
&= \frac{2}{\lambda}\left(\frac{|x-1||x+1|}{|x|}\right)\\
&\leq \frac{2}{\lambda}\left(\frac{|x-1||x+1|}{|x+1|}\right)\\
&= \frac{2}{\lambda}|x-1| \leq |x|.
\end{aligned}$$

And

$$\begin{aligned}
\kappa(x) - \kappa(Tx) &= \frac{3}{2}|x| - \frac{3}{2}\frac{1}{|x|}\\
&= \frac{3}{2}\left(|x| - \frac{1}{|x|}\right)\\
&= \frac{3}{2}\left(\frac{|x|^2-1}{|x|}\right)\\
&= \frac{3}{2}\left(\frac{(|x|-1)(|x|+1)}{|x|}\right)\\
&\leq \frac{3}{2}\left(\frac{(|x|-1)(|x|+1)}{|x|+1}\right)\\
&= \frac{3}{2}(|x|-1)\\
&\leq \frac{3}{2}|x|.
\end{aligned}$$

Therefore, $\omega_\lambda^G(x, Tx, Tx) \le \kappa(x) - \kappa(Tx)$ *for all* $\lambda > 0$. *Hence, the mapping* T *has a fixed point. The trivial fixed point of this map,* T *is* 1.

Remark 3.2. *As we can see clearly in this Example 3.1 that the map* T *has a trivial fixed point at* 1.

Proposition 3.3. *Let* ω^G *be a modular G-metric on X, and* X_{ω^G} *be a complete modular G-metric space,* $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ *be a lower semicontinuous function on* X_{ω^G}, *which is bounded from below, then there exists a point* $u \in X_{\omega^G}$ *such that* $\kappa(u) < \kappa(z) + \omega_\lambda^G(u, z, z)$ *for each* $z \in X_{\omega^G}, z \neq u$ *and for all* $\lambda > 0$.

Proof. Following the proof of Theorem 3.1, we get a sequence $\{z_n\}_{n\ge1}$ such that $z_n \to u \in X_{\omega^G}$ as $n \to \infty$. Now for any $u \in X_{\omega^G}$, define $F(u) = \{z \in X_{\omega^G} : \omega_\lambda^G(u, z, z) \le \kappa(u) - \kappa(z)\ \forall\ \lambda > 0\}$ and $\eta(u) = \inf\{\kappa(z) : z \in F(u)\}$. We will show that $u \notin F(u)$ as $z \neq u$. Suppose, if possible, otherwise. Let $v \in F(u)$ for some $v \neq u$. Then we have that for all $\lambda > 0$, $0 < \omega_\lambda^G(u, v, v) \le \kappa(u) - \kappa(v)$ implies $\kappa(v) < \kappa(u) = M$, since

$$\begin{aligned}
\omega_\lambda^G(z_n, v, v) &\le \omega_{\frac{\lambda}{2}}^G(z_n, u, u) + \omega_{\frac{\lambda}{2}}^G(u, v, v) \\
&\le \omega_\lambda^G(z_n, u, u) + \omega_\lambda^G(u, v, v) \\
&\le \kappa(z_n) - \kappa(u) + \kappa(u) - \kappa(v) \\
&= \kappa(z_n) - \kappa(v).
\end{aligned} \tag{3.15}$$

for all $\lambda > 0$, $v \in F(z_n)$ for $n \ge 1$. So $\eta(z_n) \le \kappa(v)$ for all $n \ge 1$. Therefore, $M = \lim_{n\to\infty} \eta(z_n) \le \kappa(v)$. Hence, $M \le \kappa(v)$, which is a contradiction to the fact that $\kappa(v) < \kappa(u) = M$. Therefore, for each $z \in X_{\omega^G}, z \neq u \Rightarrow z \notin F(u)$, that is $z \neq u \Rightarrow \omega_\lambda^G(u, z, z) > \kappa(u) - \kappa(z)$. Hence, $\kappa(u) < \kappa(z) + \omega_\lambda^G(u, z, z)$ for each $z \in X_{\omega^G}, z \neq u$ and for all $\lambda > 0$. □

Proposition 3.4. *Let* ω^G *be a modular G-metric on X, and* X_{ω^G} *be a complete modular G-metric space,* $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ *be a lower semicontinuous function on* X_{ω^G}, *which is bounded from below, then for every* $y \in X_{\omega^G}$ *and* $\gamma > 0$, *there exists* $x_0 \in X_{\omega^G}$ *such that* $\kappa(x_0) < \kappa(x) + \gamma\omega_\lambda^G(x, x, x_0)$ *on* $X_{\omega^G} \setminus \{x_0\}$ *and* $\kappa(x_0) \le \kappa(y) - \gamma\omega_\lambda^G(x_0, y, y)$, *for all* $\lambda > 0$.

Proof. Define $X_{\omega^G}^\gamma = \{z \in X_{\omega^G} : \kappa(z) \le \kappa(y) - \gamma\omega_\lambda^G(z, y, y),\ \forall\ \lambda > 0\}$. Then $X_{\omega^G}^\gamma$ is a nonempty complete modular G-metric space and $\kappa : X_{\omega^G} \to \mathbb{R}^+ \cup \{\infty\}$ be a lower semicontinuous function on X_{ω^G}, which is bounded from below. Let $F(x) = \{z \in X_{\omega^G}^\gamma : \kappa(x) \ge \kappa(z) + \gamma\omega_\lambda^G(z, x, x),\ \forall\ \lambda > 0\}$. Then for every $x \in X_{\omega^G}^\gamma$, $F(x) \neq \emptyset$ and closed. Also $z \in F(x)$ implies $F(z) \subseteq F(x)$. Choose $x_1 \in X_{\omega^G}^\gamma$ with $\kappa(x_1) < \infty$ and when $x_1, x_2, \ldots, x_n$ have been chosen, we can find $x_{n+1} \in F(x)$ such that $\kappa(x_{n+1}) < \inf\{\kappa(u) : u \in F(x_n)\} + \frac{1}{2^n}$ for $n \ge 1$. For any $z \in F(x_{n+1}) \subseteq F(x_n)$, we get that for all $\lambda > 0$,

$$\begin{aligned}
\gamma\omega_\lambda^G(z, x_{n+1}, x_{n+1}) &\le \gamma\omega_{\frac{\lambda}{2}}^G(z, x_n, x_n) + \gamma\omega_{\frac{\lambda}{2}}^G(x_n, x_{n+1}, x_{n+1}) \\
&\le \gamma\omega_\lambda^G(z, x_n, x_n) + \gamma\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}) \\
&\le \kappa(x_n) - \kappa(z) + \kappa(x_{n+1}) - \kappa(x_n) \\
&= \kappa(x_{n+1}) - \kappa(z) \\
&\le \inf\{\kappa(u) : u \in F(x_n)\} - \kappa(z) + \frac{1}{2^n} \\
&\le \frac{1}{2^n}.
\end{aligned} \tag{3.16}$$

So that $\delta_{\omega^G}(F(x_n)) \to 0$ as $n \to \infty$, for all $\lambda > 0$. Since $X^{\gamma}_{\omega^G}$ be a complete modular G-metric space, $\cap_{n=1}^{\infty} F(x_n) = \{x_0\}$. Since the intersection is a singleton set, we proceed as follows. Now, $x_0 \in F(x_n)$ implies that $F(x_0) \subseteq F(x_n)$ for $n \geq 1$, we get $F(x_0) = \{x_0\}$, so that for all $\lambda > 0$, $\kappa(x_0) < \kappa(x) + \gamma\omega^G_\lambda(x, x, x_0)$ on $X^{\gamma}_{\omega^G} \backslash \{x_0\}$. Again, the inequality $\kappa(x_0) < \kappa(x) + \gamma\omega^G_\lambda(x, x, x_0)$ hold on $X_{\omega^G} \backslash X^{\gamma}_{\omega^G}$ since for $z \notin X^{\gamma}_{\omega^G}$, for all $\lambda > 0$, we have $\kappa(y) - \gamma\omega^G_\lambda(z, y, y) < \kappa(z)$ and thus, together with the fact that $x_0 \in X^{\gamma}_{\omega^G}$, we have

$$\begin{aligned}\kappa(x_0) &\leq \kappa(y) - \gamma\omega^G_\lambda(x_0, y, y) \\ &\leq \kappa(y) - \gamma\omega^G_{\frac{\lambda}{2}}(z, y, y) - \gamma\omega^G_{\frac{\lambda}{2}}(x_0, z, z) \\ &\leq \kappa(y) - \gamma\omega^G_\lambda(z, y, y) - \gamma\omega^G_\lambda(x_0, z, z) \\ &< \kappa(z) - \gamma\omega^G_\lambda(x_0, z, z).\end{aligned} \tag{3.17}$$

We are now at home since for all $\lambda > 0$, $\kappa(x_0) < \kappa(z) - \gamma\omega^G_\lambda(x_0, z, z)$. □

Theorem 3.5. *Let ω^G be a modular G-metric on X, and $X_{\omega^G}, Y_{\omega^G}$ are complete modular G-metric spaces. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an arbitrary self mapping. Suppose that there exists a closed mapping $L : X_{\omega^G} \to Y_{\omega^G}$, and $\kappa : L(X_{\omega^G}) \to \mathbb{R}^+ \cup \{\infty\}$ be a lower semicontinuous function on X_{ω^G}, which is bounded from below, and for every $\gamma > 0$, there exists $x \in X_{\omega^G}$ such that*

$$\omega^G_\lambda(x, Tx, Tx) \leq \kappa(Lx) - \kappa(LTx), \tag{3.18}$$

$$\gamma\omega^G_\lambda(Lx, LTx, LTx) \leq \kappa(Lx) - \kappa(LTx), \tag{3.19}$$

for all $\lambda > 0$. Then, T has a fixed point in X_{ω^G}.

Proof. For any $x \in X_{\omega^G}$, put $Tx = y$ and let $F(x) = \{y \in X_{\omega^G} : \omega^G_\lambda(x, y, y) \leq \kappa(x) - \kappa(Ly)$ and $\gamma\omega^G_\lambda(Lx, Ly, Ly) \leq \kappa(Lx) - \kappa(Ly)\ \forall\ \lambda > 0\}$ and $\eta(x) = \inf\{\kappa(Ly) : y \in F(x)\}$. Since $x \in F(x)$, therefore, $F(x) \neq \emptyset$ and $0 \leq \eta(x) \leq \kappa(Lx)$. Let $x \in X_{\omega^G}$ be an arbitrary point. Now, we construct a sequence $\{x_n\}_{n\geq 1}$ in X_{ω^G} as follows. Let $x = x_1$ and when $x_1, x_2, \ldots, x_n$ have been chosen, choose $x_{n+1} \in F(x)$ such that $\kappa(Lx_{n+1}) \leq \eta(x_n) + \frac{1}{2^n}$ for all $n \in \mathbb{N}$. By the process above, we get a sequence $\{x_n\}_{n\geq 1}$ satisfying the conditions.

$$\omega^G_\lambda(x_n, x_{n+1}, x_{n+1}) \leq \kappa(x_n) - \kappa(Lx_{n+1}), \tag{3.20}$$

$$\gamma\omega^G_\lambda(Lx_n, Lx_{n+1}, Lx_{n+1}) \leq \kappa(Lx_n) - \kappa(Lx_{n+1}), \tag{3.21}$$

and

$$\kappa(Lx_{n+1}) - \frac{1}{2^n} \leq \eta(x_n) \leq \kappa(Lx_{n+1}), \tag{3.22}$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. Then from inequalities, (3.20), (3.21), $\{\kappa(Lx_n)\}_{n\geq 1}$ is a nonincreasing sequence in $\mathbb{R}$, and it is bounded blow by zero. So, the sequence $\{\kappa(Lx_n)\}_{n\geq 1}$ is a modular G-convergent and converges to a real number $\beta \geq 0$ (say). By inequality (3.22), we get

$$\beta = \lim_{n\to\infty} \kappa(Lx_n) = \lim_{n\to\infty} \eta(x_n) \tag{3.23}$$

Now, let $k \in \mathbb{N}$ be arbitrary, from inequalities (3.20),(3.21) and (3.23), there exits at least a positive number N_k such that $\kappa(Lx_n) < \beta + \frac{1}{2^k}$ for all $n \geq N_k$. Since $\kappa(Lx_n)$ is monotone for

$m \geq n \geq N_k$, we get $\beta \leq \kappa(Lx_m) \leq \kappa(Lx_n) < \beta + \frac{1}{2^k}$ for $m \geq n \geq N_k$. It follows that

$$\kappa(Lx_n) - \kappa(Lx_m) < \frac{1}{2^k} \text{ for all } m \geq n \geq N_k. \tag{3.24}$$

Without loss of generality, suppose that $m > n$ and $m, n \in \mathbb{N}$. From inequalities (3.20) and (3.21), we get

$$\omega^G_{\frac{\lambda}{m-n}}(x_n, x_{n+1}, x_{n+1}) \leq \kappa(Lx_n) - \kappa(Lx_{n+1}), \tag{3.25}$$

$$\omega^G_{\frac{\lambda}{m-n}}(Lx_n, Lx_{n+1}, Lx_{n+1}) \leq \kappa(Lx_n) - \kappa(Lx_{n+1}), \text{ for } \frac{\lambda}{m-n} > 0, \tag{3.26}$$

or since $\frac{\lambda}{m-n} \geq \frac{\lambda}{n}$, we have

$$\omega^G_{\frac{\lambda}{n}}(x_n, x_{n+1}, x_{n+1}) \leq \kappa(Lx_n) - \kappa(Lx_{n+1}), \tag{3.27}$$

$$\omega^G_{\frac{\lambda}{n}}(Lx_n, Lx_{n+1}, Lx_{n+1}) \leq \kappa(Lx_n) - \kappa(Lx_{n+1}), \text{ for } \frac{\lambda}{n} > 0. \tag{3.28}$$

Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Using rectangle inequality repeatedly, i.e. condition 5 of Definition (2.1), we have

$$\begin{aligned}
\omega^G_\lambda(x_n, x_m, x_m) &\leq \omega^G_{\frac{\lambda}{m-n}}(x_n, x_{n+1}, x_{n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}, x_{n+2}) + \omega^G_{\frac{\lambda}{m-n}}(x_{n+2}, x_{n+3}, x_{n+3}) \\
&+ \omega^G_{\frac{\lambda}{m-n}}(x_{n+3}, x_{n+4}, x_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{m-n}}(x_{m-1}, x_m, x_m) \\
&\leq \omega^G_{\frac{\lambda}{n}}(x_n, x_{n+1}, x_{n+1}) + \omega^G_{\frac{\lambda}{n}}(x_{n+1}, x_{n+2}, x_{n+2}) + \omega^G_{\frac{\lambda}{n}}(x_{n+2}, x_{n+3}, x_{n+3}) \\
&+ \omega^G_{\frac{\lambda}{n}}(x_{n+3}, x_{n+4}, x_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{n}}(x_{m-1}, x_m, x_m) \\
&\leq \kappa(Lx_n) - \kappa(Lx_{n+1}) + \kappa(Lx_{n+1}) - \kappa(Lx_{n+2}) + \cdots + \kappa(Lx_{m-1}) - \kappa(Lx_m) \\
&= \kappa(Lx_n) - \kappa(Lx_m),
\end{aligned} \tag{3.29}$$

for all $m > n \geq N_k$ for some $N_k \in \mathbb{N}$. Then by inequality (3.24), we have

$$\omega^G_\lambda(x_n, x_m, x_m) < \frac{1}{2^k}, \tag{3.30}$$

for all $m, l, n \geq N_k$ for some $N_k \in \mathbb{N}$, so that by condition (2) of proposition 2.1, we have

$$\omega^G_\lambda(x_n, x_m, x_l) \leq \omega^G_{\frac{\lambda}{2}}(x_n, x_m, x_m) + \omega^G_{\frac{\lambda}{2}}(x_l, x_m, x_m), \tag{3.31}$$

so that

$$\lim_{n,m,l\to\infty}\omega_\lambda^G(x_n,x_m,x_l)\le\lim_{n,m\to\infty}\omega_{\frac{\lambda}{2}}^G(x_n,x_m,x_m)+\lim_{l,m\to\infty}\omega_{\frac{\lambda}{2}}^G(x_l,x_m,x_m)$$
$$\le\lim_{n,m\to\infty}\omega_\lambda^G(x_n,x_m,x_m)+\lim_{l,m\to\infty}\omega_\lambda^G(x_l,x_m,x_m)$$
$$<\frac{1}{2^k}+\frac{1}{2^k}$$
$$=\frac{2}{2^k}=2^{1-k}. \tag{3.32}$$

Thus, as $k\to\infty$, we have

$$\lim_{n,m,l\to\infty}\omega_\lambda^G(x_n,x_m,x_l)=0. \tag{3.33}$$

Therefore, we can say straightaway that $\{x_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence in X_{ω^G}. Again, using the same procedure, we get

$$\begin{aligned}\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_m)&\le\gamma\omega_{\frac{\lambda}{m-n}}^G(Lx_n,Lx_{n+1},Lx_{n+1})+\gamma\omega_{\frac{\lambda}{m-n}}^G(Lx_{n+1},Lx_{n+2},Lx_{n+2})\\&+\gamma\omega_{\frac{\lambda}{m-n}}^G(Lx_{n+2},Lx_{n+3},Lx_{n+3})+\gamma\omega_{\frac{\lambda}{m-n}}^G(Lx_{n+3},Lx_{n+4},Lx_{n+4})\\&+\cdots+\gamma\omega_{\frac{\lambda}{m-n}}^G(Lx_{m-1},Lx_m,Lx_m)\\&\le\gamma\omega_{\frac{\lambda}{n}}^G(Lx_n,Lx_{n+1},Lx_{n+1})+\gamma\omega_{\frac{\lambda}{n}}^G(Lx_{n+1},Lx_{n+2},Lx_{n+2})\\&+\gamma\omega_{\frac{\lambda}{n}}^G(Lx_{n+2},Lx_{n+3},Lx_{n+3})+\gamma\omega_{\frac{\lambda}{n}}^G(Lx_{n+3},Lx_{n+4},Lx_{n+4})\\&+\cdots+\gamma\omega_{\frac{\lambda}{n}}^G(Lx_{m-1},Lx_m,Lx_m)\\&\le\kappa(Lx_n)-\kappa(Lx_{n+1})+\kappa(Lx_{n+1})-\kappa(Lx_{n+2})\\&+\cdots+\kappa(Lx_{m-1})-\kappa(Lx_m)\\&=\kappa(Lx_n)-\kappa(Lx_m),\end{aligned} \tag{3.34}$$

for all $m>n\ge N_k$ for some $N_k\in\mathbb{N}$. Then by inequality (3.24), we have

$$\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_m)<\frac{1}{2^k}, \tag{3.35}$$

for all $m,l,n\ge N_k$ for some $N_k\in\mathbb{N}$, so that by condition (2) of proposition 2.1, we have

$$\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_l)\le\gamma\omega_{\frac{\lambda}{2}}^G(Lx_n,Lx_m,Lx_m)+\gamma\omega_{\frac{\lambda}{2}}^G(Lx_l,Lx_m,Lx_m), \tag{3.36}$$

so that

$$\begin{aligned}\lim_{n,m,l\to\infty}\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_l) &\le \lim_{n,m\to\infty}\gamma\omega_{\frac{\lambda}{2}}^G(Lx_n,Lx_m,Lx_m)+\lim_{l,m\to\infty}\gamma\omega_{\frac{\lambda}{2}}^G(Lx_l,Lx_m,Lx_m)\\ &\le \lim_{n,m\to\infty}\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_m)+\lim_{l,m\to\infty}\gamma\omega_\lambda^G(Lx_l,Lx_m,Lx_m)\\ &< \frac{1}{2^k}+\frac{1}{2^k}\\ &= \frac{2}{2^k}=2^{1-k}.\end{aligned}\tag{3.37}$$

Thus, as $k\to\infty$, we have

$$\lim_{n,m,l\to\infty}\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_l)=0.\tag{3.38}$$

Therefore, we can say straightaway that $\{Lx_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence in Y_{ω^G}. The completeness of (X_ω,ω^G) and (Y_ω,ω^G) implies that for any $\lambda>0$, $\lim_{n,m\to\infty}\omega_\lambda^G(x_n,x_m,u)=0$, i.e. for any $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\omega_\lambda^G(x_n,x_m,u)<\epsilon$ for all $n,m\in\mathbb{N}$ and $n,m\ge n_0$, which implies that $\lim_{n\to\infty}x_n\to u\in X_{\omega^G}$ as $n\to\infty$ and for any $\lambda>0$, $\lim_{n,m\to\infty}\omega_\lambda^G(Lx_n,Lx_m,v)=0$, i.e. for any $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\omega_\lambda^G(Lx_n,Lx_m,v)<\epsilon$ for all $n,m\in\mathbb{N}$ and n, $m\ge n_0$, which implies that $\lim_{n\to\infty}Lx_n\to v\in X_{\omega^G}$ as $n\to\infty$. The fact that L is closed mapping implies that $Lu=v$. But $\kappa: X_{\omega^G}\to\mathbb{R}^+\cup\{\infty\}$ is a lower semicontinuous function on X_{ω^G}, using inequality (3.29), we get

$$\begin{aligned}\kappa(v)=\kappa(Lu) &\le \lim_{m\to\infty}\inf(\kappa(Lx_m))\\ &\le \lim_{m\to\infty}\inf(\kappa(Lx_n)-\omega_\lambda^G(x_n,x_m,x_m))\\ &= \kappa(Lx_n)-\omega_\lambda^G(x_n,u,u)\end{aligned}\tag{3.39}$$

Thus, we have that $\omega_\lambda^G(x_n,u,u)\le\kappa(Lx_n)-\kappa(Lu)$ for all $\lambda>0$. Again, using inequality, (3.34), we have

$$\begin{aligned}\kappa(v)=\kappa(Lu) &\le \lim_{m\to\infty}\inf(\kappa(Lx_m))\\ &\le \lim_{m\to\infty}\inf(\kappa(Lx_n)-\gamma\omega_\lambda^G(Lx_n,Lx_m,Lx_m))\\ &= \kappa(Lx_n)-\gamma\omega_\lambda^G(Lx_n,u,u)\end{aligned}\tag{3.40}$$

Thus, we have that $\gamma\omega_\lambda^G(Lx_n,u,u)\le\kappa(Lx_n)-\kappa(Lu)$ for all $\lambda>0$. So that $u\in F(x_n)$ for all $n\in\mathbb{N}$, and hence, $\eta(x_n)\le\kappa(Lu)$. So by inequality (3.23), we get $\beta\le\kappa(Lu)$. Meanwhile, by lower semicontinuity of κ and inequality (3.23), we have $\kappa(v)=\kappa(Lu)\le\lim_{n\to\infty}\inf\kappa(x_n)=\beta$. Therefore, $\kappa(Lu)=\beta$. By Proposition 3.3, we have that $x\ne u\Rightarrow x\notin F(u)$ and Proposition 3.4 for $y\notin X^\gamma_{\omega^G}$. From inequalities (3.18), (3.19), we know that $LTu\in F(u)$, such $u\in F(u)$. For $n\in\mathbb{N}$, we have

$$
\begin{aligned}
\omega_\lambda^G(x_n, Tu, Tu) &\le \omega_{\frac{\lambda}{2}}^G(x_n, u, u) + \omega_{\frac{\lambda}{2}}^G(u, Tu, Tu) \\
&\le \omega_\lambda^G(x_n, u, u) + \omega_\lambda^G(u, Tu, Tu) \\
&\le \kappa(Lx_n) - \kappa(Lu) + \kappa(Lu) - \kappa(LTu) \\
&= \kappa(Lx_n) - \kappa(LTu).
\end{aligned} \tag{3.41}
$$

Thus, $LTu \in F(x_n)$, and this implies that $\eta(Lx_n) \le \kappa(LTu)$. Hence, we obtain $\beta \le \kappa(LTu)$. From inequalities (3.18), (3.19), we get $\kappa(LTu) \le \kappa(Lu)$. As $\kappa(Lu) = \beta$, we have $\kappa(Lu) = \beta \le \kappa(LTu) \le \kappa(Lu)$. Therefore, $\kappa(LTu) = \kappa(Lu)$. Then from inequality (3.18) and (3.19), we get $\omega_\lambda^G(u, Tu, Tu) \le \kappa(Lu) - \kappa(LTu) = \kappa(Lu) - \kappa(Lu) = 0$. Thus, $Tu = u$. Therefore, T has a fixed point in X_{ω^G}. □

Theorem 3.6. *Let ω^G be a modular G-metric on X, and $X_{\omega^G}, Y_{\omega^G}$ are complete modular G-metric spaces. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an arbitrary self-mapping for some positive integer $m \ge 1$. Suppose that there exists a closed mapping $L : X_{\omega^G} \to Y_{\omega^G}$ for each integer $m \ge 1$, and $\kappa : L(X_{\omega^G}) \to \mathbb{R}^+ \cup \{\infty\}$ be a lower semicontinuous function on X_{ω^G}, which is bounded from below, and for every $\gamma > 0$, there exists $x \in X_{\omega^G}$ such that*

$$\omega_\lambda^G(x, T^m x, T^m x) \le \kappa(L^m x) - \kappa(L^m T^m x), \tag{3.42}$$

$$\gamma \omega_\lambda^G(L^m x, L^m T^m x, L^m T^m x) \le \kappa(L^m x) - \kappa(L^m T^m x), \tag{3.43}$$

for all $\lambda > 0$. Then, T has a fixed point in X_{ω^G} for some positive integer $m \ge 1$.

Proof. By Theorem 3.5, T^m has a fixed point say $u \in X_{\omega^G}$ for some positive integer $m \ge 1$, by using inequalities (3.42) and (3.43) for some positive integer $m \ge 1$. Now $T^m(Tu) = T^{m+1}u = T(T^m u) = Tu$, so Tu is a fixed point of T^m. Thus, we have $Tu = u$. Therefore, u is a fixed point of T. because fixed point of T is also fixed point of T^m for some positive integer $m \ge 1$. □

Remark 3.3. *The results of Theorem 3.6 improve and generalize several known results in the literature, including the results of Mutlu et al. [1].*

4 . Conclusion and future work

All fixed-point results obtained in this paper do not require the uniqueness of the fixed point of mappings under consideration. As a future direction of study, it will be of interest to prove some new fixed-point results for the nonunique fixed-point theorems established in this paper. More precisely, geometric properties of the set *Fix*(*T*) can be investigated as a future problem for a self-mapping T on a modular G-metric space in the case of nonunique fixed point.

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Further reading

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Corresponding author

Godwin Amechi Okeke can be contacted at: gaokeke1@yahoo.co.uk