Classification of harmonic homomorphisms between Riemannian three-dimensional unimodular Lie groups

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Received 19 January 2022
Revised 6 June 2022
Accepted 21 June 2022

Abstract

Purpose – The purpose of this study is to classify harmonic homomorphisms ϕ $: (G, g) \to (H, h)$, where G, H are connected and simply connected three-dimensional unimodular Lie groups and g, h are left-invariant Riemannian metrics.

Design/methodology/approach – This study aims the classification up to conjugation by automorphism of Lie groups of harmonic homomorphism, between twodifferent non-abelian connected and simply connected three-dimensional unimodular Lie groups (G, g) and (H, h), where g and h are two left-invariant Riemannian metrics on G and H, respectively.

Findings – This study managed to classify some homomorphisms between two different non-abelian connected and simply connected three-dimensional uni-modular Lie groups.

Originality/value – The theory of harmonic maps into Lie groups has been extensively studied related homomorphism in compact Lie groups by many mathematicians, harmonic maps into Lie group and harmonics inner automorphisms of compact connected semi-simple Lie groups and intensively study harmonic and biharmonic homomorphisms between Riemannian Lie groups equipped with a left-invariant Riemannian metric.

Keywords Harmonic homomorphisms, Unimodular Riemannian Lie groups, Invariant metrics

Paper type Research paper

1. Introduction

The theory of harmonic maps is old and rich and has gained a growing interest in the past decade (see Ref. [1] and others). The theory of harmonic maps into Lie groups has been extensively studied related homomorphism in compact Lie groups by many mathematicians (see for examples [2]), in particular, harmonic maps into Lie groups [3] and harmonic inner automorphisms of compact connected semi-simple Lie groups in Ref. [4] and intensively study harmonic and biharmonic homomorphisms between Riemannian Lie groups equipped with a left-invariant Riemannian metric in Ref. [5].

The investigations described here are motivated by the paper [6], the author studied the classification, up to conjugation by an automorphism of Lie groups, of harmonic and

2020 MSC Classification — 53C30, 53C43, 22E15

The authors thank the referee for many useful suggestions and corrections which improved the first version.

Osamnia Nada and Zegga Kaddour are contributed equally to this work.

Arab Journal of Mathematical Sciences
Vol. 30 No. 1, 2024
pp. 95-111
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-01-2022-0010

biharmonic maps $f:(G,g_1)\to(G,g_2)$, where G is non-abelian connected and simply connected three-dimensional unimodular Lie group, f is a homomorphism of Lie group and g_1, g_2 are two left-invariant Riemannian metrics. The Lie group is unimodular if every left Haar measure is a right Haar measure and vice versa. It is known that G is unimodular if and only if $|\det Ad_x| = 1$ for all $x\in G$ if and only if the trace$ad(X) = 0$ for all X in its Lie algebra $\mathfrak{g}$ if and only if $\mathfrak{g}$ is unimodular.

There are five non-abelian connected and simply connected three-dimensional unimodular Lie groups, the nilpotent Lie group (or the Heisenberg group), the special unitary group $SU(2)$, the universal covering group $\tilde{PSL}(2,\mathbb{R})$ of the special linear group, the solvable Lie groups *Sol* and the universal covering group $\tilde{E}_0(2)$ of the connected component of the Euclidean group, for more detail, see Ref. [7].

In this paper, we aim the classification up to conjugation by an automorphism of Lie groups of harmonic homomorphism, between two different non-abelian connected, and simply connected three-dimensional unimodular Lie groups $\phi:(G,g)\to(H,h)$, where g and h are two left-invariant Riemannian metrics on G and H, respectively.

2. Preliminaries

Let $\varphi:(M,g)\to(N,h)$ be a smooth map between two Riemannian manifolds with $m=\dim M$ and $n=\dim N$. We denote by ∇^M and ∇^N the Levi-Civita connexions associated, respectively, to g and h and by $T^\varphi N$ the vector bundle over M pull-back of TN by φ. It is a Euclidean vector bundle and the tangent map of φ is a bundle homomorphism $d\varphi: TM\to T^\varphi N$. Moreover, $T^\varphi N$ carries a connexion ∇^φ pull-back of ∇^N by φ and there is a connexion on the vector bundle $End(TM, T^\varphi N)$ given by

$$(\nabla_X A)(Y) = \nabla^\phi_X A(Y) - A\left(\nabla^M_X Y\right),\ X, Y\in\Gamma(TM),\ A\in\Gamma(End(TM, T^\varphi N)).$$

The map φ is called harmonic if it is a critical point of the energy

$$E(\varphi)=\frac{1}{2}\int_M |d\varphi^2| v_g.$$

The corresponding Euler-Lagrange equation for the energy is given by the vanishing of the tension field

$$\tau(\varphi)=\mathrm{tr}\nabla d\varphi=\sum_{i=1}^{m}(\nabla_{e_i}d\varphi)e_i,$$

where $(e_i)_{i=1}^m$ is a local frame of orthonormal vector fields.Let (G, g) be a Riemannian Lie group, i.e., a Lie group endowed with a left-invariant Riemannian metric. If $\mathfrak{g}=T_eG$ is its Lie algebra and $<\,,\,>_{\mathfrak{g}}=g(e)$, then there exists a unique bilinear map $A:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ called the Levi-Civita product associated with $(\mathfrak{g},<\,,\,>_{\mathfrak{g}})$ given by the formula:

$$2<A_u v, w>_{\mathfrak{g}} = <[u,v]^{\mathfrak{g}}, w>_{\mathfrak{g}} + <[w,u]^{\mathfrak{g}}, v>_{\mathfrak{g}} + <[w,v]^{\mathfrak{g}}, u>_{\mathfrak{g}}.$$

A is entirely determined by the following properties

(1) for any $u, v\in\mathfrak{g}, A_u v - A_v u = [u,v]^{\mathfrak{g}}$,

(2) for any $u,v,w\in\mathfrak{g}, <A_u v, w>_{\mathfrak{g}} + <v, A_u w>_{\mathfrak{g}} = 0$.

If we denote by u^ℓ the left-invariant vector field on G associated with $u \in \mathfrak{g}$ then the Levi-Civita connection associated with (G, g) satisfies $\nabla_{u^\ell} v^\ell = (A_u v)^\ell$, the couple $(\mathfrak{g}, <,> \mathfrak{g})$ defines a vector denoted $U^{\mathfrak{g}}$ by

$$< U^{\mathfrak{g}}, v > \mathfrak{g} = \mathrm{tr}(ad_v), \textit{for any } v \in \mathfrak{g}.$$

One can deduce easily that, for any orthonormal basis $(e_i)_{i=1}^m$ of $\mathfrak{g}$,

$$U^{\mathfrak{g}} = \sum_{i=1}^{m} A_{e_i} e_i.$$

Note that $\mathfrak{g}$ is unimodular if and only if $U^{\mathfrak{g}} = 0$.

Let $\varphi : (G, g) \to (H, h)$ be a Lie group homomorphism between two Riemannian Lie groups. The differential $\xi : \mathfrak{g} \to \mathfrak{h}$ of φ at e is a Lie algebra homomorphism. There is a left action of G on $\Gamma(T^\varphi H)$ given by

$$(a.X)(b) = T_{\varphi(ab)} L_{\varphi\left(a^{-1}\right)} X(ab), \; a, b \in G, \; X \in \Gamma(T^\varphi H).$$

A section X of $T^\varphi H$ is called left-invariant if, for any $a \in G$, $a.X = X$. For any left-invariant section X of $T^\varphi H$, we have for any $a \in G$, $X(a) = (X(e))^\ell(\varphi(a))$. Thus the space of left-invariant sections is isomorphic to the Lie algebra $\mathfrak{h}$. Since φ is a homomorphism of Lie groups, g and h are leftinvariant, one can see easily that $\tau(\varphi)$ is left invariant and hence φ is harmonic if and only if $\tau(\varphi)(e) = 0$. Now, one can see easily that

$$\tau(\xi) := \tau(\varphi)(e) = U^\xi - \xi(U^{\mathfrak{g}}),$$

where

$$U^\xi = \sum_{i=1}^{m} B_{\xi(e_i)} \xi(e_i),$$

where B is the Levi-Civita product associated with $(\mathfrak{h}, < , >_{\mathfrak{h}})$ and $(e_i)_{i=1}^m$ is an orthonormal basis of $\mathfrak{g}$. So we get the following proposition.

Proposition 2.1. Let $\phi : G \to H$ be a homomorphism between two Riemannian Lie groups. Then ϕ is harmonic if only if $\tau(\xi) = 0$, where $\xi : \mathfrak{g} \to \mathfrak{h}$ is the differential of ϕ at e. The classification of harmonic homomorphisms will be done up to conjugation.

Two homomorphisms between Euclidean Lie algebras:

$$\xi_1 : \left(\mathfrak{g}, < , >_{\mathfrak{g}}\right) \to \left(\mathfrak{h}, < , >_{\mathfrak{h}}\right) \quad \textit{and} \quad \xi_2 : \left(\mathfrak{g}, < , >_{\mathfrak{g}}\right) \to \left(\mathfrak{h}, < , >_{\mathfrak{h}}\right)$$

are conjugate if there exists two isometric automorphisms $\varphi_1 : (\mathfrak{g}, < , >_{\mathfrak{g}}) \to (\mathfrak{g}, < , >_{\mathfrak{g}})$ and $\varphi_2 : (\mathfrak{h}, < , >_{\mathfrak{h}}) \to (\mathfrak{h}, < , >_{\mathfrak{h}})$ such that

$$\varphi_2 \circ \xi_1 = \xi_2 \circ \varphi_1. \tag{1.1}$$

Proposition 2.2. Let $\xi : (\mathfrak{g}, < , >_{\mathfrak{g}}) \to (\mathfrak{h}, < , >_{\mathfrak{h}})$ be a homomorphism between unimodular Euclidean Lie algebras, the following formula was established in [5]

$$< \tau(\xi), X >_{\mathfrak{h}} = tr_{\mathfrak{g}}\left(\xi^{*}\circ ad_X \circ \xi\right) \qquad \forall X \in \mathfrak{h} \quad (1.2).$$

where $\xi^{*} : \mathfrak{h} \to \mathfrak{g}$ is given by

$$< \xi^{*} U, V >_{\mathfrak{g}} = < U, \xi V >_{\mathfrak{h}}, \ for \quad V \in \mathfrak{g} \ and \ U \in \mathfrak{h}. \quad (1.3).$$

3. Riemanian three-dimensional unimodular Lie groups G

Definition 3.1. The Heisenberg group Nil

The nilpotent Lie group Nil known as Heisenberg group, whose Lie algebra will be denoted by $\mathfrak{n}$. We have

$$Nil = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ with \ a, b, c \in \mathbb{R} \right\}$$

and

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \ with \ x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{n}$ has a basis $\{X, Y, Z\}$, where $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where the non-vanishing Lie bracket is $[X, Y] = Z$.

Proposition 3.1. [7]

Any left-invariant metric on Nil is equivalent up to automorphism to a metric whose associated matrix is of the form

$$< , >_{\mathfrak{n}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ where \ \rho > 0. \quad (3.1)$$

Definition 3.2. The solvable Lie group Sol

The solvable Lie group Sol whose Lie algebra will be denoted by $\mathfrak{sol}$. We have $\mathfrak{sol} = \mathbb{R}^2 \rtimes_{\imath} \mathbb{R}$ where $\imath(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. We can choose a basis $\{X, Y, Z\}$ of $\mathfrak{sol}$, where $X = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0\right)$, $Y = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right)$ and $Z = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$.

and the non-vanishing Lie brackets are $[Z, X] = X$ and $[Y, Z] = Y$. The Lie group of the solvable Lie algebra $\mathfrak{sol} = \mathbb{R}^2 \rtimes_\iota \mathbb{R}$ is the solvable Lie group *Sol*, which is the semi-direct product $\mathbb{R}^2 \rtimes_\Theta \mathbb{R}$, where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by $\Theta(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Proposition 3.2. [7]

Any left-invariant metric on $Sol = \mathbb{R}^2 \rtimes_\Theta \mathbb{R}$ is equivalent up to automorphism to a metric whose associated matrix is of the form

$$< , >_{\mathfrak{sol}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}, \ \textit{where}\ \nu > 0, \tag{3.2}$$

Or

$$< , >_{\mathfrak{sol}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \ \textit{where}\ \nu > 0\ \textit{and}\ \mu > 1. \tag{3.3}$$

Definition 3.3. The solvable Lie group $\tilde{E}_0(2)$

The solvable Lie group $\tilde{E}_0$ whose Lie algebra will be denoted by $\mathfrak{e}_0(2)$, where $\mathfrak{e}_0(2) = \mathbb{R}^2 \rtimes \mathfrak{so}(2)$. We can choose a basis $\{X, Y, Z\}$ of $\mathfrak{e}_0(2)$ where $X = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$, $Y = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$, $Z = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$ and the non-vanishing Lie brackets are $[Z, X] = Y, [Y, Z] = X$.

The Lie algebra $\mathfrak{e}_0(2) = \mathbb{R}^2 \rtimes \mathfrak{so}(2)$ is Lie algebra of the Lie group $E_0(2) = \mathbb{R}^2 \rtimes SO(2)$. The group $E_0(2)$ is not simply connected. The unique simply connected Lie group corresponding to the Lie algebra $\mathfrak{e}_0 = \mathbb{R}^2 \rtimes \mathfrak{so}(2)$ is universal covering group $\tilde{E}_0(2)$ of $E_0(2)$.

The group $\tilde{E}_0(2)$ is the semi-direct product $\mathbb{C} \rtimes_{\mathbb{R}}$, where $(z, t).(z', t') = (z + z' e^{2i\pi t}, t + t')$ has a faithful matrix representation in $GL(3, \mathbb{C})$ by

$$(z, t) \mapsto \begin{pmatrix} e^{2i\pi t} & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix},$$

where $z \in \mathbb{C}$ and $t \in \mathbb{R}$.

Proposition 3.3. [7]

Any left-invariant metric on $\tilde{E}_0(2)$ is equivalent up to automorphism to a metric whose associated matrix is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varrho & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \ \textit{where}\ \sigma > 0\ \textit{and}\ 0 < \varrho \leq 1. \tag{3.4}$$

4. Harmonic homomorphisms between *Sol* and *Nil*

The following result gives a complete classification of harmonic homomorphisms between $\mathfrak{sol}$ equipped with the left-invariant metric defined in (3.2) or (3.3) and $\mathfrak{n}$ equipped with the left-invariant metric defined in (3.1).

Theorem 4.1.

A homomorphism from $\mathfrak{sol}$ to $\mathfrak{n}$ is conjugate to $\xi : \mathfrak{sol} \to \mathfrak{n}$, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} \quad \textit{with } a, b, c \in \mathbb{R}.$$

Proof.

The basis of $\mathfrak{sol}$ is $\{X, Y, Z\}$ where $[Z, X] = X$, and $[Y, Z] = Y$.

The basis of $\mathfrak{n}$ is $\{E, F, H\}$ with $[E, F] = H$. we put

$$\begin{aligned} \xi: \quad & X \mapsto a_1 E + b_1 F + c_1 H \\ & Y \mapsto a_2 E + b_2 F + c_2 H \\ & Z \mapsto a_3 E + b_3 F + c_3 H. \end{aligned}$$

Thus, we obtain

$$\begin{cases} [\xi X, \xi Y] = \xi[X, Y] = 0 \\ [\xi X, \xi Z] = \xi[X, Z] = -\xi X \\ [\xi Y, \xi Z] = \xi[Y, Z] = \xi Y \end{cases} \Leftrightarrow a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0.$$

Theorem 4.2.

Let $\xi : \mathfrak{sol} \to \mathfrak{n}$ a homomorphism, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \tag{4.1}$$

the Lie algebra $\mathfrak{sol}$ equipped with the left-invariant metric defined in (3.2) or (3.3) and $\mathfrak{n}$ equipped with the left-invariant metric defined in (3.1). Then

$$\tau(\xi) = \frac{bc}{\nu} E - \frac{ac}{\nu} F. \tag{4.2}$$

Proof.

We have

$$ad_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ ad_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \textit{ and } ad_H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formula (1.3) where $U \in \mathfrak{n}$ and $V \in \mathfrak{sol}$, we obtain

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho a & \rho b & c \end{pmatrix}.$$

Using formula (1.2), a simple calculation gives us

$$< \tau(\xi), E >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_E \circ \xi\right) = \frac{bc}{\nu},$$

$$< \tau(\xi), F >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_F \circ \xi\right) = \frac{-ac}{\nu}$$

and

$$< \tau(\xi), H >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_H \circ \xi\right) = 0$$

Corollary 4.1.

$\xi : (\mathfrak{sol}, < , >_{\mathfrak{sol}}), \to (\mathfrak{n} < , >_{\mathfrak{sol}})$ is harmonic if and only if ($a = b = 0$ or $c = 0$).

Theorem 4.3.

A homomorphism from $\mathfrak{n}$ to $\mathfrak{sol}$ is conjugate to $\xi_{i=1,2} : \mathfrak{n} \to \mathfrak{sol}$, where

$$\xi_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \; with \; a, b \in \mathbb{R}, \tag{4.3}$$

Or

$$\xi_2 = \begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; with \; a_i, b_i \in \mathbb{R} \; for \; i = 1, 2. \tag{4.4}$$

Proof.

The basis of $\mathfrak{sol}$ is $\{X, Y, Z\}$, where $[Z, X] = X$ and $[Y, Z] = Y$, the basis of $\mathfrak{n}$ is $\{E, F, H\}$ with $[E, F] = H$, then we can suppose

$$E \mapsto a_1 X + b_1 Y + c_1 Z,$$

$$F \mapsto a_2 X + b_2 Y + c_2 Z$$

and

$$H \mapsto a_3 X + b_3 Y + c_3 Z.$$

Thus we obtain

$$\begin{cases} \xi[E,F] = [\xi E, \xi F] = \xi H \\ \xi[E,H] = [\xi E, \xi H] = 0 \\ \xi[F,H] = [\xi E, \xi H] = 0 \end{cases} \Leftrightarrow (c_3 = a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0 \; or \; a_3 = b_3 = c_3 = c_1 = c_2 = 0).$$

Theorem 4.4.

Let $\xi_1, \xi_2 : \mathfrak{n} \to \mathfrak{sol}$ be homomorphisms, where ξ_1 and ξ_2 are defined in formulas (4.3) and (4.4), the Lie algebra $\mathfrak{sol}$ is equipped with the left-invariant metric defined in formula (3.2). Then

$$\tau(\xi_1) = (a^2 - b^2)Z \tag{4.5}$$

and

$$\tau(\xi_2) = \frac{\left((a_1^2 - b_1^2) + (a_2^2 - b_2^2)\right)}{\rho}Z. \tag{4.6}$$

Proof.

We have

$$ad_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; ad_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; and \; ad_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the homomorphism ξ_1, using formula (1.3), where $V \in \mathfrak{n}$ and $U \in \mathfrak{sol}$, we obtain

$$\xi_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix}.$$

Using formula (1.2), a simple calculation gives us

$$< \tau(\xi_1), X >_{\mathfrak{sol}} = tr\left(\xi_1^* \circ ad_X \circ \xi_1\right) = 0,$$

$$< \tau(\xi_1), Y >_{\mathfrak{sol}} = tr\left(\xi_1^* \circ ad_Y \circ \xi_1\right) = 0$$

and

$$< \tau(\xi_1), Z >_{\mathfrak{sol}} = tr\left(\xi_1^* \circ ad_Z \circ \xi_1\right) = a^2 - b^2.$$

For the homomorphism ξ_2, we have

$$\xi_2^* = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using formula (1.2), we obtain

$$< \tau(\xi_2), X >_{\mathfrak{sol}} = tr\left(\xi_2^* \circ ad_X \circ \xi_2\right) = 0,$$

$$< \tau(\xi_2), Y >_{\mathfrak{sol}} = tr\left(\xi_2^* \circ ad_Y \circ \xi_2\right) = 0$$

and

$$< \tau(\xi_2), Z >_{\mathfrak{sol}} = tr\left(\xi_2^* \circ ad_Z \circ \xi_2\right) = \frac{\left((a_1^2 - b_1^2) + (a_2^2 - b_2^2)\right)}{\rho}.$$

Corollary 4.2.

$\xi_1 : (\mathfrak{n}, <\ ,\ >_{\mathfrak{n}}) \rightarrow (\mathfrak{sol}, <\ ,\ >_{\mathfrak{sol}})$ is harmonic if and only if $a = \pm b$.

$\xi_2 : (\mathfrak{n}, <\ ,\ >_{\mathfrak{n}}) \rightarrow (\mathfrak{sol}, <\ ,\ >_{\mathfrak{sol}})$ is harmonic if and only if $a_1^2 + a_2^2 = b_2^2 + b_1^2$.

Theorem 4.5.

Let $\xi_1, \xi_2 : \mathfrak{n} \rightarrow \mathfrak{sol}$ be homomorphisms, where ξ_1 and ξ_2 are defined in formulas (4.3), (4.4) and the Lie algebra $\mathfrak{sol}$ is equipped with the left-invariant metric defined in formula (3.3). Then

$$\tau(\xi_1) = \left(a^2 - \mu b^2\right) Z, \tag{4.7}$$

$$\tau(\xi_2) = \frac{\left(\left(a_1^2 - \mu b_1^2\right) + \left(a_2^2 - \mu b_2^2\right)\right)}{\rho} Z. \tag{4.8}$$

Proof.

By using formula (1.3), where $V \in \mathfrak{n}$ and $U \in \mathfrak{sol}$, we obtain:

For ξ_1

$$\xi_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a+b & a+\mu b & 0 \end{pmatrix}.$$

Using formula (1.2), we get

$$< \tau(\xi_1), X >_{\mathfrak{sol}} = tr\left(\xi_1^* \circ ad_X \circ \xi_1\right) = 0,$$

$$< \tau(\xi_1), Y >_{\mathfrak{sol}} = tr\left(\xi_1^* \circ ad_Y \circ \xi_1\right) = 0$$

and

$$< \tau(\xi_1), Z >_{\mathfrak{sol}} = tr\left(\xi_1^* \circ ad_Z \circ \xi_1\right) = a^2 - \mu b^2.$$

For ξ_2, we have

$$\xi_2^* = \begin{pmatrix} a_1 + b_1 & a_1 + \mu b_1 & 0 \\ a_2 + b_2 & a_2 + \mu b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

furthermore

$$< \tau(\xi_2), X >_{\mathfrak{sol}} = tr\left(\xi_2^* \circ ad_X \circ \xi_2\right) = 0,$$

$$< \tau(\xi_2), Y >_{\mathfrak{sol}} = tr\left(\xi_2^* \circ ad_Y \circ \xi_2\right) = 0$$

and

$$< \tau(\xi_2), Z >_{\mathfrak{sol}} = tr\left(\xi_2^* \circ ad_Z \circ \xi_2\right) = \frac{\left(\left(a_1^2 - \mu b_1^2\right) + \left(a_2^2 - \mu b_2^2\right)\right)}{\rho}.$$

Corollary 4.3.

$\xi_1 : (\mathfrak{n} <\,,>_{\mathfrak{n}}) \to (\mathfrak{sol}, <\,,>_{\mathfrak{sol}})$ is harmonic if and only if $a = \pm\sqrt{\mu}b$.

$\xi_2 : (\mathfrak{n} <\,,>_{\mathfrak{n}}) \to (\mathfrak{sol}, <\,,>_{\mathfrak{sol}})$ is harmonic if and only if $a_1^2 + a_2^2 = \sqrt{\mu}(b_2^2 + b_1^2)$.

5. Harmonic homomorphisms between *Sol* and $\tilde{E}_0(2)$

The following result gives a complete classification of harmonic homomorphisms between $\mathfrak{sol}$ equipped with the left-invariant metric defined in (3.2), (3.3) and $\mathfrak{e}_0(2)$ equipped with the left-invariant metric defined in (3.4).

Theorem 5.1.

Any homomorphism from $\mathfrak{sol}$ to $\mathfrak{e}_0(2)$ is conjugate to $\xi : \mathfrak{sol} \to \mathfrak{e}_0(2)$, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \textit{ such that } a, b, c \in \mathbb{R}.$$

Proof.

The basis of $\mathfrak{sol}$ is $\{X, Y, Z\}$ where $[Z, X] = X, [Y, Z] = Y$ and the basis of $\mathfrak{e}_0(2)$ is $\{A, B, C\}$ with $[A, B] = 0, [C, A] = B$ and $[B, C] = A$, we suppose

$$\xi(X) = a_1A + b_1B + c_1C,$$

$$\xi(Y) = a_2A + b_2B + c_2C$$

and

$$\xi(Z) = a_3A + b_3B + c_3C.$$

Thus we obtain

$$\begin{cases} [\xi X, \xi Y] = \xi[X, Y] = 0 \\ [\xi X, \xi Z] = \xi[X, Z] = -\xi X \\ [\xi Y, \xi Z] = \xi[Y, Z] = \xi Y \end{cases} \Leftrightarrow (a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0).$$

Theorem 5.2.

Let $\xi : \mathfrak{sol} \to \mathfrak{e}_0(2)$ be a homomorphism, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \tag{5.1}$$

and $\mathfrak{sol}$ equipped with the left-invariant metric defined in (3.2) or in (3.3), then

$$\tau(\xi) = \frac{1}{\nu}(-\varrho bcA + acB + (\varrho - 1)abC). \tag{5.2}$$

Proof.

We have

$$ad_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \ ad_B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ ad_C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using formula (1.3), where $U \in \mathfrak{e}_0(2)$ and $V \in \mathfrak{sol}$, we obtain

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & \varrho b & \sigma c \end{pmatrix}.$$

Use formula (1.2), we get

$$< \tau(\xi), A >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_A \circ \xi\right) = \frac{-\varrho bc}{\nu},$$

$$< \tau(\xi), B >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_B \circ \xi\right) = \frac{ac}{\nu}$$

and

$$< \tau(\xi), C >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_C \circ \xi\right) = \frac{(\varrho - 1)ab}{\nu}.$$

Corollary 5.1.

$\xi : (\mathfrak{sol}, < , >_{\mathfrak{sol}}), \rightarrow (\mathfrak{e}_0(2) < , >_{\mathfrak{e}_0(2)})$ is harmonic if and only if ($\varrho = 1$ *and* $c = 0$) or ($a = b = 0$).

Theorem 5.3.

A homomorphism from $\mathfrak{e}_0(2)$ to $\mathfrak{sol}$ is conjugate to $\xi : \mathfrak{e}_0(2) \rightarrow \mathfrak{sol}$, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \quad \textit{such that } a, b, c \in \mathbb{R}.$$

Proof. $\xi : \mathfrak{e}_0(2) \rightarrow \mathfrak{sol}$, we have

$$A \mapsto a_1 X + b_1 Y + c_1 Z,$$

$$B \mapsto a_2 X + b_2 Y + c_2 Z$$

and

$$C \mapsto a_3 X + b_3 Y + c_3 Z.$$

Thus we obtain

$$\begin{cases} [\xi A, \xi B] = \xi[A, B] = 0 \\ [\xi A, \xi C] = \xi[A, C] = -\xi B \\ [\xi B, \xi C] = \xi[B, C] = \xi A \end{cases} \Leftrightarrow \ (a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0).$$

Theorem 5.4.

Let $\xi : \mathfrak{e}_0(2) \to \mathfrak{sol}$ be a homomorphism, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \tag{5.3}$$

and $\mathfrak{sol}$ equipped with the left-invariant metric defined in (3.2). Then

$$\tau(\xi) = \frac{1}{\sigma}\left(-acX + bcY + \left(a^2 - b^2\right)Z\right). \tag{5.4}$$

Proof.

By using formula (1.3) where $V \in \mathfrak{e}_0(2)$ and $U \in \mathfrak{sol}$, we obtain

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & \nu c \end{pmatrix}.$$

By direct calculation and we use formula (1.2), we obtain

$$< \tau(\xi), X >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_X \circ \xi\right) = \frac{-ac}{\sigma},$$

$$< \tau(\xi), Y >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_Y \circ \xi\right) = \frac{bc}{\sigma}$$

and

$$< \tau(\xi), Z >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_Z \circ \xi\right) = \frac{\left(a^2 - b^2\right)}{\sigma}.$$

Corollary 5.2.

$\xi : (\mathfrak{e}_0(2) < , >_{\mathfrak{e}_0(2)}) \to (\mathfrak{sol}, < , >_{\mathfrak{sol}})$ is harmonic if and only if ($c = 0$ and $a = \pm b$ or $a = b = 0$).

Theorem 5.5.

Let $\xi : \mathfrak{e}_0(2) \to \mathfrak{sol}$ be a homomorphism, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}. \tag{5.5}$$

Where $\mathfrak{sol}$ equipped with left-invariant metric define in (3.3).
Then

$$\tau(\xi) = \frac{1}{\sigma}\left(-(a + b)cX + \mu bcY + \left(a^2 - \mu b^2 + ab\right)Z\right). \tag{5.6}$$

Proof.

by a similar calculation, we get $\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a+b & \mu b & \nu c \end{pmatrix}$.

Using formula (1.2), a direct calculation gives us

$$< \tau(\xi), X >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_X \circ \xi\right) = -\frac{(a+b)c}{\sigma},$$

$$< \tau(\xi), Y >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_Y \circ \xi\right) = \frac{\mu bc}{\sigma}$$

and

$$< \tau(\xi), Z >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_Z \circ \xi\right) = \frac{a^2 - \mu b^2 + ab}{\sigma}.$$

Corollary 5.3.

$\xi : (\mathfrak{e}_0(2) <\, ,\, >_{\mathfrak{e}_0(2)}) \rightarrow (\mathfrak{sol}, <\, ,\, >_{\mathfrak{sol}})$ is harmonic if and only if ($a = b = 0$) or ($b = c = 0$).

6. Harmonic homomorphisms between *Nil* and $\tilde{E}_0(2)$

The following result gives a complete classification of harmonic homomorphisms between $\mathfrak{n}$ equipped with the left-invariant metric defined in (3.1) and $\mathfrak{e}_0(2)$ equipped with the left-invariant metric defined in (3.4).

Theorem 6.1.

A homomorphism from $\mathfrak{e}_0(2)$ to $\mathfrak{n}$ is conjugate to $\xi : \mathfrak{e}_0(2) \rightarrow \mathfrak{n}$, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \ \textit{such that } a, b, c \in \mathbb{R}.$$

Proof.

The basis of $\mathfrak{e}_0(2)$ is $\{A, B, C\}$ with $[A, B] = 0, [C, A] = B, [B, C] = A$ and the basis of $\mathfrak{n}$ is $\{E, F, H\}$ with $[E, F] = H$. Suppose that

$$A \mapsto a_1 E + b_1 F + c_1 H,$$

$$B \mapsto a_2 E + b_2 F + c_2 H,$$

and

$$C \mapsto a_3 E + b_3 F + c_3 H.$$

Thus, we obtain

$$\begin{cases} [\xi A, \xi B] = \xi[A, B] = 0 \\ [\xi A, \xi C] = \xi[A, C] = -\xi B \\ [\xi B, \xi C] = \xi[B, C] = \xi A \end{cases} \Leftrightarrow (a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0).$$

Theorem 6.2.

Let $\xi : \mathfrak{e}_0(2) \to \mathfrak{n}$ a homomorphism, where

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}. \tag{6.1}$$

Then

$$\tau(\xi) = \frac{b}{c}\nu E - \frac{ac}{\nu}F. \tag{6.2}$$

Proof.

We have $ad_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $ad_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, and $ad_H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

using formula (1.3), where $U \in \mathfrak{n}$ and $V \in \mathfrak{e}_0(2)$, we get

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho a & \rho b & c \end{pmatrix}.$$

Using formula (1.2), a simple calculation gives us

$$< \tau(\xi), E >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_E \circ \xi\right) = \frac{bc}{\sigma}$$

and

$$< \tau(\xi), F >_{\mathfrak{n}} = tr\left(\xi^* \circ ad_F \circ \xi\right) = -\frac{ac}{\sigma}.$$

Corollary 6.1.

$\xi : (\mathfrak{e}_0(2), < , >_{\mathfrak{e}_0(2)}) \to (\mathfrak{n} < , >_{\mathfrak{n}})$ is harmonic if and only if ($a = b = 0$ or $c = 0$).

Theorem 6.3.

A homomorphism from $\mathfrak{n}$ to $\mathfrak{e}_0(2)$ is conjugate to $\xi_i : \mathfrak{n} \to \mathfrak{e}_0(2)$, with $i = 1, 2, 3$ where

$$\xi_1 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } a, b, c, d \in \mathbb{R},$$

$$\xi_2 = \begin{pmatrix} 0 & a & 0 \\ 0 & b & 0 \\ 0 & c & 0 \end{pmatrix} \text{ where } a, b, c \in \mathbb{R}$$

and

$$\xi_3 = \begin{pmatrix} a & d & 0 \\ b & \frac{cd}{a} & 0 \\ c & \frac{bd}{a} & 0 \end{pmatrix} \textit{ with } b, c, d \in \mathbb{R} \textit{ and } a \in \mathbb{R}^*.$$

Proof.

The basis of $\mathfrak{e}_0(2)$ is $\{A, B, C\}$ such that $[A, B] = 0, [C, A] = B, [B, C] = A$ and the basis of $\mathfrak{n}$ is $\{E, F, H\}$ with $[E, F] = H$. We put

$$E \mapsto a_1 A + b_1 B + c_1 C,$$

$$F \mapsto a_2 A + b_2 B + c_2 C$$

and

$$H \mapsto a_3 A + b_3 B + c_3 C.$$

Thus, we obtain

$$\begin{cases} [\xi E, \xi F] = \xi[E, F] = \xi H \\ [\xi E, \xi H] = 0 \\ [\xi F, \xi H] = \xi[F, H] = 0 \end{cases} \Leftrightarrow \begin{cases} a_3 = b_3 = c_3 = 0 \\ c_1 = c_2 = 0 \end{cases} or \begin{cases} a_3 = b_3 = c_3 = 0 \\ a_1 = b_1 = c_1 = 0 \end{cases} or \begin{cases} a_3 = b_3 = c_3 = 0 \\ c_1 \times a_2 = b_2 \times a_1 \\ b_1 \times a_2 = c_2 \times a_1 \end{cases}$$

Theorem 6.4.

Let $\xi_i : \mathfrak{n} \to \mathfrak{e}_0(2)$ be homomorphisms, where $(\xi_i)_{i=1,2,3}$ are defined in (Theorem 5.3.), then

$$\tau(\xi_1) = (\varrho - 1)\frac{ac + bd}{\rho} C.$$

$$\tau(\xi_2) = \frac{1}{\rho}(-\varrho bcA + acB + ab(\varrho - 1)C).$$

$$\tau(\xi_3) = \frac{-\varrho bc}{\rho}\left(1 + \frac{d^2}{a^2}\right)A + \frac{1}{\rho}\left(ac + \frac{b^2 d}{a}\right)B + \frac{1}{\rho}\left(\frac{cd}{a}(\varrho d - 1) + ab(\varrho - 1)\right)C.$$

Proof.

We have $ad_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $ad_B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $ad_C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

By using formula (1.3), where $U \in \mathfrak{e}_0(2)$ and $V \in \mathfrak{n}$, we obtain

$$\xi_1^* = \begin{pmatrix} a & \varrho c & 0 \\ b & \varrho d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We use formula (1.2), we obtain

$$< \tau(\xi_1), A >_{\mathfrak{e}_0(2)} = tr\left(\xi^* \circ ad_A \circ \xi_1\right) = 0,$$

$$< \tau(\xi_1), B >_{\mathfrak{e}_0(2)} = tr\left(\xi^* \circ ad_B \circ \xi\right) = 0,$$

and

$$< \tau(\xi_1), C >_{\mathfrak{e}_0(2)} = tr\left(\xi^* \circ ad_C \circ \xi\right) = (\varrho - 1)\frac{ad + bd}{\rho}.$$

For $\xi = \xi_2$, we have $\xi_2^* = \begin{pmatrix} 0 & 0 & 0 \\ a & \varrho b & \sigma c \\ 0 & 0 & 0 \end{pmatrix}$.

We use formula (1.2), we obtain

$$< \tau(\xi_2), A >_{\mathfrak{e}_0(2)} = tr\left(\xi_2^* \circ ad_A \circ \xi_2\right) = -\frac{1}{\rho}\varrho bc,$$

$$< \tau(\xi_2), B >_{\mathfrak{e}_0(2)} = tr\left(\xi_2^* \circ ad_B \circ \xi_2\right) = \frac{1}{\rho}ac$$

and

$$< \tau(\xi_2), C >_{\mathfrak{e}_0(2)} = tr\left(\xi_2^* \circ ad_C \circ \xi_2\right) = \frac{1}{\rho}ab(\varrho - 1).$$

For $\xi = \xi_3$, we have

$$\xi_3^* = \begin{pmatrix} a & b\varrho & c\sigma \\ d & \varrho\dfrac{cd}{a} & \sigma\dfrac{bd}{a} \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.3}$$

We use formula (1.2), we obtain

$$< \tau(\xi_3), A >_{\mathfrak{e}_0(2)} = tr\left(\xi_3^* \circ ad_A \circ \xi_3\right) = \frac{-\varrho bc}{\rho}\left(1 + \frac{d^2}{a^2}\right),$$

$$< \tau(\xi_3), B >_{\mathfrak{e}_0(2)} = tr\left(\xi_3^* \circ ad_B \circ \xi_3\right) = \frac{1}{\rho}\left(ac + \frac{bd^2}{a}\right)$$

and

$$< \tau(\xi_3), C >_{\mathfrak{e}_0(2)} = tr\left(\xi_3^* \circ ad_C \circ \xi_3\right) = \frac{\varrho - 1}{\rho}\left(\frac{cd^2}{a} + ab\right).$$

Corollary 6.2.

$\xi_1 : (\mathfrak{n} <\, ,\, >_{\mathfrak{n}}) \to (\mathfrak{e}_0(2), <\, ,\, >_{\mathfrak{e}_0(2)})$, is harmonic if and only if ($\varrho = 1$ or $ac + bd = 0$).

$\xi_2 : (\mathfrak{n} <\, ,\, >_{\mathfrak{n}}) \to (\mathfrak{e}_0(2), <\, ,\, >_{\mathfrak{e}_0(2)})$, is harmonic if and only if ($b = c = 0$ or $\varrho = 1$, **and** $c = 0$).

$\xi_3 : (\mathfrak{n} <\, ,\, >_{\mathfrak{n}}) \to (\mathfrak{e}_0(2), <\, ,\, >_{\mathfrak{e}_0(2)})$, is harmonic if and only if ($b = c = 0$ or $c = d = 0$ and $\varrho = 1$).

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