Lie subalgebras of $\mathfrak{so}(3,1)$ up to conjugacy

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Abstract

Purpose – This study aims to find all subalgebras up to conjugacy in the real simple Lie algebra $\mathfrak{so}(3,1)$.
Design/methodology/approach – The authors use Lie Algebra techniques to find all inequivalent subalgebras of $\mathfrak{so}(3,1)$ in all dimensions.
Findings – The authors find all subalgebras up to conjugacy in the real simple Lie algebra $\mathfrak{so}(3,1)$.
Originality/value – This paper is an original research idea. It will be a main reference for many applications such as solving partial differential equations. If $\mathfrak{so}(3,1)$ is part of the symmetry Lie algebra, then the subalgebras listed in this paper will be used to reduce the order of the partial differential equation (PDE) and produce non-equivalent solutions.

Keywords Simple Lie algebra, Lie subalgebra, Conjugate subalgebras
Paper type Research paper

1. Introduction

In the classification of real simple Lie algebras, $\mathfrak{so}(3,1)$ is the unique simple six-dimensional Lie algebra. The Lie algebra $\mathfrak{so}(3,1)$ and its associated Lie group *SO*(3, 1) are of fundamental importance in the theory of relativity, as is very well known. However, in terms of finding representations of $\mathfrak{so}(3,1)$, the situation is apt to become confusing because the usual approach is to complexify and $\mathfrak{so}(3,1)\otimes\mathbb{C}\approx\mathfrak{so}(3,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$. A closely related idea is to use Weyl's unitarian trick. In this regard, we refer to [1] where an apparently non-standard representation of $\mathfrak{so}(3,1)$ is given. We do not know at this time if it is of physical significance.

In [2] Dynkin studied the problem of finding maximal dimension subgroups of a simple Lie group and by extension, maximal dimension subalgebras of its Lie algebra. In [3], the subalgebras of $\mathfrak{gl}(3,\mathbb{R})$ were classified. In [4], subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ were studied that are not solvable. In [5], a slightly different direction provides minimal dimension representations of Levi decomposition Lie algebras up to and including dimension eight.

Our goal in this note is to find all Lie subalgebras of $\mathfrak{so}(3,1)$ up to conjugacy. Most of the Lie subalgebras concerned can be found from consideration of the Cartan subalgebras, $\mathfrak{so}(3,1)$ being a rank two algebra. Of course it is important to understand that when we say "conjugate," we mean equivalent under a change of basis that belongs to *SO*(3, 1). We study the case of

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one-dimensional subalgebras in Section 3, two-dimensional subalgebras in Section 4, three-dimensional subalgebras in Section 5, show that there are no five-dimensional subalgebras in Section 6 and consider subalgebras of dimension four in Section 7. In Section 8, we give a different representation of $\mathfrak{so}(3,1)$ and argue that it is not conjugate to the standard representation. Finally, in Section 9, we provide a table of proper subalgebras of $\mathfrak{so}(3,1)$ up to conjugacy.

2. The Lie algebra $\mathfrak{so}(3,1)$

The real simple Lie algebra $\mathfrak{so}(3,1)$ is defined by the following space of matrices:

$$S = \begin{bmatrix} 0 & -s_6 & -s_5 & s_1 \\ s_6 & 0 & s_4 & s_2 \\ s_5 & -s_4 & 0 & s_3 \\ s_1 & s_2 & s_3 & 0 \end{bmatrix}. \tag{1}$$

From equation (1), the Lie brackets of $\mathfrak{so}(3,1)$ are

$$\begin{aligned} &[e_1,e_2] = -e_6, [e_1,e_3] = -e_5, [e_1,e_5] = -e_3, [e_1,e_6] = -e_2, \\ &[e_2,e_3] = e_4, [e_2,e_4] = e_3, [e_2,e_6] = e_1, [e_3,e_4] = -e_2, \\ &[e_3,e_5] = e_1, [e_4,e_5] = e_6, [e_4,e_6] = -e_5, [e_5,e_6] = e_4. \end{aligned} \tag{2}$$

Our goal in this note is to find all Lie subalgebras of $\mathfrak{so}(3,1)$ up to conjugacy.

3. One-dimensional Lie subalgebras

Starting from (1), there is a transformation in $SO(3,1)$ of the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$, where $A \in SO(3)$ such that we can reduce s_4 and s_5 to zero. Now consider the matrix

$$P = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Then conjugating S by P, we obtain

$$P^{-1}SP = \begin{bmatrix} 0 & -s_6 & 0 & s_1\cos\theta - s_2\sin\theta \\ s_6 & 0 & 0 & s_1\sin\theta + s_2\cos\theta s_2 \\ 0 & 0 & 0 & 0 \\ s_1\cos\theta - s_2\sin\theta & s_1\sin\theta + s_2\cos\theta & 0 & 0 \end{bmatrix}. \tag{4}$$

Note that $P \in \mathfrak{so}(3,1)$. As such, we can choose θ so that $s_2 = 0$. The matrix S has been reduced to

$$S = \begin{bmatrix} 0 & -s_6 & 0 & s_1 \\ s_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 \\ s_1 & 0 & s_3 & 0 \end{bmatrix}. \tag{5}$$

Now the characteristic polynomial of this reduced S is given by

$$\lambda^4 + (s_6^2 - s_1^2 - s_3^2)\lambda^2 - s_3^2 s_6^2 = 0. \tag{6}$$

3.1 Zero eigenvalues

If the four roots of (6) are all zero, we must have in the first instance, $s_3 s_6 = 0$. However, if $s_6 = 0$, then looking at the λ^2 term, we would have $s_1 = s_3 = 0$ and $S = 0$. Hence, for non-zero S, we must have $s_3 = 0$ and $s_6 = \pm s_1$. It appears as though we have two cases to consider now, but there is just one case as we shall now explain.

Conjugate S by the matrix $Q \in \mathfrak{so}(3,1)$, where

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Then we find that

$$Q^{-1}SQ = \begin{bmatrix} 0 & s_6 & 0 & 0 \\ -s_6 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 0 \\ 0 & s_1 & 0 & 0 \end{bmatrix}, \tag{8}$$

but we may conjugate again by P from (3) with $\theta = \frac{3\pi}{2}$, so as to restore s_1 to the (1, 4)-entry, without disturbing s_6 and arrive finally at

$$S = \begin{bmatrix} 0 & s_1 & 0 & s_1 \\ -s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

Since we require only a generator for a one-dimensional Lie subalgebra, we may further suppose that $s_1 = 1$ in (9).

3.2 Eigenvalues not all zero

From now on, we shall assume that the eigenvalues of S are not all zero. In this case, we introduce the matrix R that belongs to $\mathfrak{so}(3,1)$

$$R = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cosh\psi & 0 & \sinh\psi \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sinh\psi & 0 & \cosh\psi \end{bmatrix}. \tag{10}$$

In this case, matrix (5) may be conjugated to

$$R^{-1}SR = \begin{bmatrix} 0 & -t_6 & 0 & t_1 \\ t_6 & 0 & t_4 & 0 \\ 0 & -t_4 & 0 & t_3 \\ t_1 & 0 & t_3 & 0 \end{bmatrix} \tag{11}$$

where

$$t_1 = (b\cos\theta + c\sin\theta)\cosh\psi - a\sinh\psi\cos\theta \tag{12}$$

$$t_3 = a\sin\theta\sinh\psi + (c\cos\theta - b\sin\theta)\cosh\psi \tag{13}$$

$$t_4 = (b\sin\theta - c\cos\theta)\sinh\psi - a\sin\theta\cosh\psi \tag{14}$$

$$t_6 = a\cos\theta\cosh\psi - (b\cos\theta + c\sin\theta)\sinh\psi. \tag{15}$$

It is always possible to choose θ and ψ such that $t_1 = 0$ and $t_4 = 0$. Indeed (12) and (14) imply that

$$\tanh 2\psi = \frac{2ab}{a^2+b^2+c^2}, \ \tan 2\theta = \frac{2bc}{b^2-a^2-c^2}. \tag{16}$$

If $b^2 - a^2 - c^2 = 0$, we choose $\theta = \frac{\pi}{4}$. The conclusion is that if the eigenvalues of S are not all zero, then S may always be conjugated to the form

$$S = \begin{bmatrix} 0 & -s_6 & 0 & 0 \\ s_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 \\ 0 & 0 & s_3 & 0 \end{bmatrix}. \tag{17}$$

In terms of a one-dimensional Lie subalgebra, we may further suppose that either $s_3 = 1$ or $s_6 = 1$.

4. Two-dimensional Lie subalgebras

4.1 Two-dimensional abelian Lie subalgebras

Now we proceed to examine the two-dimensional Lie subalgebras of $\mathfrak{so}(3,1)$. First of all, it is easy to check that, starting from matrix (9), a matrix in $\mathfrak{so}(3,1)$ that commutes with (9) other than (9) itself, must be of the form

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -s_2 & 0 \\ 0 & s_2 & 0 & s_2 \\ 0 & 0 & s_2 & 0 \end{bmatrix}. \tag{18}$$

Putting the matrices (9) and (18) together gives a two-dimensional abelian subalgebra.

Secondly, the only two-dimensional abelian Lie subalgebra to which the matrix (17) belongs is the Cartan subalgebra obtained by taking the span of the matrices $s_3 = 1, s_6 = 0$ and $s_3 = 0, s_6 = 1$ in (17). Hence, any two-dimensional abelian Lie subalgebra of $\mathfrak{so}(3,1)$ is a Cartan subalgebra, and all of them are conjugate: see [6, 7].

4.2 Two-dimensional non-abelian Lie subalgebras

4.2.1 One generator of type (9). Now we attempt to find two-dimensional non-abelian Lie subalgebras. We shall assume that one generator A is given by (9) and we take a second B in the form (1). In B, by subtracting a multiple of A from B, we may assume that $s_6 = 0$. Now we find that

$$[A,B] - \mu A - \nu B = \begin{bmatrix} 0 & s_2-\mu & \nu s_5+s_3+s_4 & s_2-\nu s_1-\mu \\ \mu - s_2 & 0 & -\nu s_4+s_5 & -\nu s_2 - s_1 \\ -(\nu s_5+s_3+s_4) & \nu s_4 - s_5 & 0 & -\nu s_3 - s_5 \\ s_2-\nu s_1-\mu & -\nu s_2 - s_1 & -\nu s_3 - s_5 & 0 \end{bmatrix}. \tag{19}$$

We begin to solve the conditions arising from setting to zero all entries in the matrix that appear on the right hand side of (19). We find

$$s_4 = \nu^2 s_3 - s_3,\ \mu = s_2,\ s_1 = -\nu s_2 - s_6, s_5 = -\nu s_3. \tag{20}$$

At this point, we see that if $\nu \neq 0$, then $B = 0$. However, if $\nu = 0$, then (19) is now satisfied. Furthermore, we have now that

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -s_3 & s_2 \\ 0 & s_3 & 0 & s_3 \\ 0 & s_2 & s_3 & 0 \end{bmatrix}. \tag{21}$$

If we assume that $s_2 = 0$, then we find that $[A, B] = 0$, whereas we are assuming that our two-dimensional subalgebra is non-abelian. Thus, we may suppose that $s_2 \neq 0$, and we find $P^{-1}BP$ where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \dfrac{s_3^2}{2s_2^2} & -\dfrac{s_3}{s_2} & -\dfrac{s_3^2}{2s_2^2} \\ 0 & \dfrac{s_3}{s_2} & 1 & \dfrac{s_3}{s_2} \\ 0 & \dfrac{s_3^2}{2s_2^2} & \dfrac{s_3}{s_2} & 1 + \dfrac{s_3^2}{2s_2^2} \end{bmatrix}. \tag{22}$$

We have chosen P so that it belongs to $\mathfrak{so}(3,1)$ and commutes with A. We find that

$$P^{-1}BP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \end{bmatrix} \tag{23}$$

and hence we may assume $s_2 = 1$. We now have our two-dimensional non-abelian Lie subalgebra with generators A, B

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{24}$$

and Lie bracket $[A, B] = A$. This subalgebra is unique up to conjugacy.

4.2.2 One generator of type (17). Now we shall show that there can be no two-dimensional non-abelian Lie subalgebra when one generator is of type (17). Thus, we assume that

$$A = \begin{bmatrix} 0 & -s_6 & 0 & 0 \\ s_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 \\ 0 & 0 & s_3 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -t_6 & -t_5 & t_1 \\ t_6 & 0 & t_4 & t_2 \\ t_5 & -t_4 & 0 & t_3 \\ t_1 & t_2 & t_3 & 0 \end{bmatrix}. \tag{25}$$

Now supposing there exist μ, ν such that $[A, B] - \mu A - \nu B = 0$, leads to the following system of equations:

$$\mu s_6 + \nu t_6 = 0$$
$$\nu t_5 - s_3 t_1 - s_6 t_4 = 0$$
$$\nu t_1 - s_3 t_5 + s_6 t_2 = 0$$
$$\nu t_4 + s_3 t_2 + s_6 t_5 = 0$$
$$\nu t_2 + s_3 t_4 - s_6 t_1 = 0$$
$$\mu s_3 + \nu t_3 = 0.$$

However, it is easy to see that solving this system leads to an *abelian* subalgebra.

5. Three-dimensional Lie subalgebras

There are, depending how one counts, perhaps six classes of real, solvable, three-dimensional Lie algebras. In this context, we are referring to *abstract* Lie algebras, and not at the moment necessarily subalgebras of $\mathfrak{so}(3,1)$. They are comprised of the algebras $A_{3.1}, \ldots A_{3.7}$ and $A_{2.1}\oplus$ in [8], as well as the abelian three-dimensional Lie subalgebra. Each of these algebras has a two-dimensional abelian ideal. We saw in the previous Section that two-dimensional abelian subalgebras can occur in just two ways, up to isomorphism. One such way is as a Cartan subalgebra. However, we know that Cartan subalgebras are self-normalizing [7]. Therefore, the only possibility for a three-dimensional solvable subalgebra of $\mathfrak{so}(3,1)$ to have a two-dimensional abelian ideal is if it the subalgebra spanned by the matrices (9) and (18), up to isomorphism.

Next we take a matrix of the form (1) that we call C, and find the conditions on C such that $[A, C]$ and $[B, C]$ are linear combinations of A and B, where A is a matrix of the form (9) and B of the form (18). We may ease the working by assuming that $s_1 = 0$ and $s_6 = 0$ in P. A straightforward calculation reveals that in P we must have $s_3 = s_4 = 0$. If we set A, B, C equal to e_1, e_2, e_3 and $s_5 = a$ and $s_2 = b$, respectively, we obtain the non-zero Lie brackets:

$$[e_1, e_3] = ae_1 - be_2, [e_2, e_3] = be_1 + ae_2. \tag{26}$$

Assuming that $a^2 + b^2 \neq 0$ so that the matrix C does not vanish, we may scale C by a non-zero factor, so we can suppose that either $b = 1$ or $a = 1, b = 0$. As abstract Lie algebras, they are $A_{3.3}$ and $A_{3.6/7}$ in [8].

It remains only to discuss the cases of subalgebras that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$. Concerning $\mathfrak{sl}(2,\mathbb{R})$, we see from (2), that we can take the brackets in the form

$$[e_2 + e_6, e_1] = e_2 + e_6, [e_1, e_2 - e_6] = e_2 - e_6, [e_2 - e_6, e_2 + e_6] = 2e_1. \tag{27}$$

Accordingly, following the discussion at the end of the previous Section, we may put $e_2 + e_6 = A$ and $e_1 = B$ from (25) so that the bracket $[e_2 + e_6, e_1] = e_2 + e_6$ is satisfied. We will use the remaining brackets to determine $e_2 - e_6$ and hence e_2 and e_6 separately. However, it is quite straightforward to check that we obtain precisely the span of the three matrices obtained from (2) by putting in turn $s_1 = 1$, $s_2 = s_3 = s_4 = s_5 = s_6 = 0$, $s_2 = 1$, $s_1 = s_3 = s_4 = s_5 = s_6 = 0$, $s_1 = s_2 = s_3 = s_4 = s_5 = 0$, $s_6 = 1$. In particular, all subalgebras of $\mathfrak{so}(3,1)$ that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ are conjugate. It is interesting to note that the representation of $\mathfrak{sl}(2,\mathbb{R})$ appearing in $\mathfrak{so}(3,1)$ is conjugate via a transformation of $\mathfrak{gl}(4,\mathbb{R})$ (not $\mathfrak{so}(3,1)$!) to the direct sum of the adjoint and a one-dimensional trivial representation, as we invite the reader to show: see also the end of Section 8 below.

As regards $\mathfrak{so}(3)$, there are only two possible representations in $\mathfrak{gl}(4,\mathbb{R})$, coming from the irreducible 4×4 and standard 3×3 representations. However, the former is by 4×4 skew-symmetric matrices and so cannot be found in (1). Thus, the only possibility of obtaining $\mathfrak{so}(3)$ at all in (1), is the obvious one, that is, the upper left 3×3 block using s_4, s_5, s_6 in (1).

6. Four-dimensional Lie subalgebras

A *Borel subalgebra* in a semi-simple Lie algebra is a solvable subalgebra of maximal dimension. We may construct a Borel subalgebra by using the positive roots in a Cartan decomposition. Referring to (1), we use the Cartan subalgebra that corresponds to s_3 and s_6. Then we use the positive simple roots $\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$ with root vectors $e_1 + \mp ie_2 + \pm ie_4 + e_5$.

We can obtain the Borel subalgebra from the following set of matrices:

$$T = \begin{bmatrix} 0 & -t_4 & t_1 & t_1 \\ t_4 & 0 & t_2 & t_2 \\ -t_1 & -t_2 & 0 & t_3 \\ t_1 & t_2 & t_{30} & 0 \end{bmatrix}. \tag{28}$$

The matrix T engenders the following Lie algebra

$$[e_1, e_3] = e_1, [e_1, e_4] = -e_2, [e_2, e_3] = e_2, [e_2, e_4] = e_1, \tag{29}$$

which is precisely algebra $A_{4.12}$ in [8]. We could also arrive at the same conclusion by revisiting the calculation of the previous Section and allowing the parameters s_2 and s_5 to generate independent matrices. It is known [7] that all such Borel subalgebras are conjugate.

There can be no four-dimensional Lie subalgebras of $\mathfrak{so}(3,1)$ that have a necessarily trivial Levi decomposition, that is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ or $\mathfrak{so}(3) \oplus \mathbb{R}$, for in both cases the centralizers consist of diagonal matrices and do not belong to $\mathfrak{so}(3,1)$.

7. Five-dimensional Lie subalgebras

Finally, we shall show that $\mathfrak{so}(3,1)$ does not possess any five-dimensional Lie subalgebras. Since the Borel subalgebras are four-dimensional, there can be no five-dimensional solvable subalgebras. For the same reason as in dimension four, there can be no Levi decomposition subalgebras that have a trivial Levi decomposition. Thus, we have only to show that we cannot obtain the five-dimensional indecomposable Lie algebra, denoted by $A_{5.40}$ in [8], which is a semi-direct product of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}^2$. The $\mathbb{R}^2$ factor here is the radical, which is an ideal. Now according to Section 5, we may assume that the Levi factor $\mathfrak{sl}(2,\mathbb{R})$ is determined by s_1, s_2, s_6 in (1). However, as such, we have a representation of $\mathfrak{sl}(2,\mathbb{R})$ that reduces as an irreducible three-dimensional representation and a trivial one-dimensional representation. Hence, there can be no two-dimensional invariant subspace that would be needed to accommodate the radical of the Lie subalgebra $A_{5.40}$.

8. Another representation of $\mathfrak{so}(3,1)$

In equation (1), we have given the definition of the Lie algebra $\mathfrak{so}(3,1)$. We now wish to exhibit another 4×4 representation of $\mathfrak{so}(3,1)$, which is not conjugate to the standard representation. Thus, we introduce the following matrix U.

$$U = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & s_4 & -s_3 \\ s_5 & s_6 & -s_1 & s_2 \\ s_6 & -s_5 & -s_2 & -s_1 \end{bmatrix}. \tag{30}$$

In the same way as in (1), we obtain the following Lie brackets:

$$\begin{aligned} &[e_1, e_3] = 2e_3, [e_1, e_4] = 2e_4, [e_1, e_5] = -2e_5, [e_1, e_6] = -2e_6, \\ &[e_2, e_3] = -2e_4, [e_2, e_4] = 2e_3, [e_2, e_5] = -2e_6, [e_2, e_6] = 2e_5, \\ &[e_3, e_5] = e_1, [e_3, e_6] = e_2, [e_4, e_5] = -e_2, [e_4, e_6] = e_1. \end{aligned} \tag{31}$$

If we make the following change of basis

$$-\frac{e_1}{2}, -\frac{(e_4+e_6)}{2}, -\frac{(e_3+e_5)}{2}, -\frac{e_2}{2}, \frac{(e_5-e_3)}{2}, \frac{(e_6-e_4)}{2} \tag{32}$$

then we will obtain precisely the same Lie brackets as in (1), and so we know that (30) is a representation of $\mathfrak{so}(3,1)$. The subalgebra of (30) given by putting $s_2 = s_4 = s_6 = 0$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. It appears in the "diagonal" representation of $\mathfrak{sl}(2,\mathbb{R})$.

Referring to (1), the subalgebra given by putting $s_3 = s_4 = s_5 = 0$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Clearly this representation is equivalent to

$$S = \begin{bmatrix} 0 & -s_6 & s_1 & 0 \\ s_6 & 0 & s_2 & 0 \\ s_1 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

It may be shown that (33) is equivalent to the direct sum of the adjoint representation and a one-dimensional trivial representation, that is,

$$S = \begin{bmatrix} 2s_6 & 2s_1 & 0 & 0 \\ s_2 & 0 & s_1 & 0 \\ 0 & 2s_2 & -2s_6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

Begin by finding a linear combination of the matrices (33) that are nilpotent, which inevitably necessitates the introduction of some $\sqrt{2}$'s. Thus, the representations (1) and (30) are not conjugate.

9. Table of proper subalgebras of $\mathfrak{so}(3,1)$ up to conjugacy

9.1 One-dimensional Lie subalgebras

$$\begin{bmatrix} 0 & s_1 & 0 & s_1 \\ -s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -as_1 & 0 & 0 \\ as_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & bs_1 \\ 0 & 0 & bs_1 & 0 \end{bmatrix} (a = 1 or b = 1).$$

9.2 Two-dimensional Lie subalgebras

$$\begin{bmatrix} 0 & s_1 & 0 & s_1 \\ -s_1 & 0 & -s_2 & 0 \\ 0 & s_2 & 0 & s_2 \\ s_1 & 0 & s_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -s_1 & 0 & 0 \\ s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 \\ 0 & 0 & s_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & s_1 & 0 & s_1 \\ -s_1 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 \\ s_1 & s_2 & 0 & 0 \end{bmatrix}.$$

9.3 Three-dimensional Lie subalgebras

$$\begin{bmatrix} 0 & s_3 & -s_2 & 0 \\ -s_3 & 0 & s_1 & 0 \\ s_2 & -s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -s_3 & -0 & s_1 \\ s_3 & 0 & 0 & s_2 \\ 0 & -0 & 0 & 0 \\ s_1 & s_2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & s_1 & -as_3 & s_1 \\ -s_1 & 0 & -s_2 & bs_3 \\ as_3 & s_2 & 0 & s_2 \\ s_1 & bs_3 & s_2 & 0 \end{bmatrix} (a = 1, b = 0 or b = 1).$$

9.4 Four-dimensional Lie subalgebras

$$\begin{bmatrix} 0 & -s_4 & s_1 & s_1 \\ s_4 & 0 & s_2 & s_2 \\ -s_1 & -s_2 & 0 & s_3 \\ s_1 & s_2 & s_3 & 0 \end{bmatrix}.$$

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