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On copula moment: empirical likelihood based estimation method

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Abstract

Purpose – In this paper, the authors applied the empirical likelihood method, which was originally proposed by Owen, to the copula moment based estimation methods to take advantage of its properties, effectiveness, flexibility and reliability of the nonparametric methods, which have limiting chi-square distributions and may be used to obtain tests or confidence intervals. The authors derive an asymptotically normal estimator of the empirical likelihood based on copula moment estimation methods (ELCM). Finally numerical performance with a simulation experiment of ELCM estimator is studied and compared to the CM estimator, with a good result.

Design/methodology/approach – In this paper we applied the empirical likelihood method which originally proposed by Owen, to the copula moment based estimation methods.

Findings – We derive an asymptotically normal estimator of the empirical likelihood based on copula moment estimation methods (ELCM). Finally numerical performance with a simulation experiment of ELCM estimator is studied and compared to the CM estimator, with a good result.

Originality/value – In this paper we applied the empirical likelihood method which originally proposed by Owen 1988, to the copula moment based estimation methods given by Brahimi and Necir 2012. We derive an new estimator of copula parameters and the asymptotic normality of the empirical likelihood based on copula moment estimation methods.

Keywords Archimedean copulas, Asymptotic distribution, Copula models, Method of moments, Semi-parametric models, *Z*-estimator

Paper type Research paper

1. Introduction

One of the main topics in multivariate statistical analysis is the statistical inference on the dependence parameter $\boldsymbol{\theta}$. Many researchers investigated the copula parameter estimation, namely the methods of concordance [1, 2] fully and the pseudo maximum likelihood [3], inference function of margins [4, 5], minimum distance [6] and recently the copula moment and L-moment based estimation methods given in [7, 8].

In this paper we applied the empirical likelihood method to the copula moment based estimation methods which originally proposed by [9–11]. Several authors investigated the empirical likelihood see for instance [12–16].

The advantage of this method is that the empirical likelihood has both effectiveness and flexibility of the likelihood method, and reliability of the non-parametric methods, and it helps

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us to construct confidence intervals without estimating the asymptotic variance, so the complexity of the asymptotic variance for some estimator especially the CM based estimators and the construction of non-parametric confidence intervals via estimating the asymptotic variance is usually inaccurate.

2. Empirical likelihood for CM based estimation method

We consider the Archimedean copula family defined by $C(\mathbf{u}) = \varphi^{-1}\left(\sum_{j=1}^{d}\varphi(u_j)\right)$, where $\varphi : [0,1] \to \mathbb{R}$ is a twice differentiable function called the generator, satisfying: $\varphi(1) = 0$, $\varphi'(x) < 0$, $\varphi''(x) \geq 0$ for any $x \in (0,1)$ and $\mathbf{u} = (u_1, \ldots, u_d)$. The notation φ^{-1} stands for the inverse function of φ. Archimedean copulas are easy to construct and have nice properties. A variety of known copula families belong to this class, including the models of Gumbel, Clayton, Frank, . . .(see, Table 4.1 in [17], p. 116).

Let $\mathbf{K}_C(s) := P(C(\mathbf{U}) \leq s)$, $s \in [0,1]$, be the df of rv $C(\mathbf{U})$, where $\mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_d)$, then the kth-moment $M_k(C)$, called *copula moment,* of rv $C(\mathbf{U})$ given in [7] as the expectation of $(C(\mathbf{U}))^k$, that is

$$M_k(C) := \mathbb{E}\left[(C(\mathbf{U}))^k\right] = \int_{[0,1]^d} (C(\mathbf{u}))^k dC(\mathbf{u}),\ k = 1,2,\ldots \tag{2.1}$$

Equation (2.1) may be rewritten into:

$$M_k(C) = \int_0^1 s^k d\mathbf{K}_C(s),\ k = 1,2,\ldots.$$

Suppose now, for unknown $\boldsymbol{\theta} \in \mathcal{O}$, that $\varphi = \varphi_{\boldsymbol{\theta}}$, it follows that $C = C_{\boldsymbol{\theta}}$, $\mathbf{K}_C = \mathbf{K}_{\boldsymbol{\theta}}$ and $M_k(C) = M_k(\boldsymbol{\theta})$, that is

$$M_k(\boldsymbol{\theta}) = \int_0^1 s^k d\mathbf{K}_{\theta}(s),\ k = 1,2,\ldots,$$

from Theorem 4.3.4 in [17] we have for any $s \in [0,1]$, $\mathbf{K}_{\boldsymbol{\theta}}(s) = s - \varphi_{\boldsymbol{\theta}}(s)/\varphi'_{\boldsymbol{\theta}}(s)$, it follows that the corresponding density is $\mathbf{K}'_{\boldsymbol{\theta}}(s) = \varphi''_{\boldsymbol{\theta}}(s)\varphi_{\boldsymbol{\theta}}(s)/\left(\varphi'_{\boldsymbol{\theta}}(s)\right)^2$. Therefore (2.1), may be rewritten into

$$M_k(\boldsymbol{\theta}) = \int_0^1 s^k \frac{\varphi''_{\boldsymbol{\theta}}(s)\varphi_{\boldsymbol{\theta}}(s)}{\left(\varphi'_{\boldsymbol{\theta}}(s)\right)^2} ds,\ k = 1,2,\ldots \tag{2.2}$$

In terms of $\varphi_{\boldsymbol{\theta}}$.

The non-parametric likelihood of distribution function $\mathbf{K}_C$ of rv $C(\mathbf{U})$ is defined by

$$\mathcal{L}(\mathbf{K}_C) = \prod_{i=1}^{n} \mathbf{K}'_C(C(\mathbf{U}_i)), \tag{2.3}$$

Where $\mathbf{U}_i := (U_{i1}, \ldots, U_{id})$. We restrict $\mathbf{K}_C$ to the one having the probability

$$p_i = \mathbf{K}'_C(C(\mathbf{U}_i)) > 0$$

on each observation $\mathbf{U}_i$. By a simple calculation, we find the maximizer of the non-parametric likelihood (2.3) turns to be the empirical distribution function $\mathbf{K}_{C_n}$, placing probability $1/n$ on each observation. Therefore, similar to the parametric case, non-parametric likelihood ratio of $\mathbf{K}_C$ to the maximizer $\mathbf{K}_{C_n}$ is defined by:

$$\mathcal{R}(\mathbf{K}_C) = \frac{\mathcal{L}(\mathbf{K}_C)}{\mathcal{L}(\mathbf{K}_{C_n})} = \frac{\prod_{i=1}^{n} p_i}{\prod_{i=1}^{n} 1/n} = \prod_{i=1}^{n} (np_i).$$

Suppose now that we are interested in a parameter $\boldsymbol{\theta} \in \mathcal{O} \subset \mathbb{R}^r$, $C = C_{\boldsymbol{\theta}}$ and $\mathbf{K}_C := \mathbf{K}_{\boldsymbol{\theta}}$ that the parameter $\boldsymbol{\theta}$ satisfies the following equations

$$\mathbb{E}[L_k(\mathbf{u}; \boldsymbol{\theta}_0)] = 0, k = 1, 2, \ldots, r \tag{2.4}$$

where

$$[L_k(\mathbf{u}; \boldsymbol{\theta})]_{k=1,2,\ldots,r} := \left[(C_{\boldsymbol{\theta}}(\mathbf{u}))^k - M_k(\boldsymbol{\theta}) \right]_{k=1,2,\ldots,r} \in \mathbb{R}^r \tag{2.5}$$

is a vector-valued function, called estimating function. Let the sample $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ from random vector $\mathbf{X} = (X_1, \ldots, X_d)$, we define the corresponding joint empirical df by

$$F_n(\mathbf{x}) = n^{-1} \sum_{i=1}^{n} \mathbf{1}\{X_{1i} \le x_1, \ldots, X_{di} \le x_d\},$$

with $\mathbf{x} := (x_1, \ldots, x_d)$, and the marginal empirical df's pertaining to the sample $(X_{j1}, \ldots, X_{jn})$, from rv X_j, by

$$F_{jn}(x_j) = n^{-1} \sum_{i=1}^{n} \mathbf{1}\{X_{ji} \le x_j\}, j = 1, \ldots, d. \tag{2.6}$$

According to [18]; the empirical copula function is defined by

$$C_n(\mathbf{u}) := F_n\left(F_{1n}^{-1}(u_1), \ldots, F_{dn}^{-1}(u_d)\right), \text{ for } \mathbf{u} \in [0,1]^d, \tag{2.7}$$

where $F_{jn}^{-1}(s) := \inf\{x : F_{jn}(x) \ge s\}$ denotes the empirical quantile function pertaining to df F_{jn}. For each $j = 1, \ldots, d$, we compute $\widehat{U}_{ji} := F_{jn}(X_{ji})$, then set

$$\widehat{\mathbf{U}}_i := \left(\widehat{U}_{1i}, \ldots, \widehat{U}_{di}\right), \ i = 1, \ldots, n. \tag{2.8}$$

and for each $k = 1, \ldots, r$, we compute

$$\widehat{M}_k := n^{-1} \sum_{i=1}^{n} \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k \tag{2.9}$$

By substitution of M_k by $\widehat{M}_k$ and solving system (2.4) in θ we obtain the solution $\widehat{\theta}^{CM} := \left(\widehat{\theta}_1, \ldots, \widehat{\theta}_r\right)$, called the CM estimator for $\boldsymbol{\theta}$.

Assume that the following assumptions $[H.1] - [H.3]$ hold.

(1) $[H.1]$ $\boldsymbol{\theta}_0 \in \mathcal{O} \subset \mathbb{R}^r$ is the unique zero of the mapping $\boldsymbol{\theta} \to \int_{[0,1]^d} \mathbf{L}(\mathbf{u}; \theta) dC_{\boldsymbol{\theta}_0}(\mathbf{u})$ which is defined from $\mathcal{O}$ to $\mathbb{R}^r$, where $\mathbf{L}(\mathbf{u}; \boldsymbol{\theta}) = (L_1(\mathbf{u}; \boldsymbol{\theta}), \ldots, L_r(\mathbf{u}; \boldsymbol{\theta}))$.

(2) $[H.2]$ $\mathbf{L}(\cdot; \boldsymbol{\theta})$ is differentiable with respect to $\boldsymbol{\theta}$ with the Jacobean matrix denoted by

$$\dot{\mathbf{L}}(\mathbf{u}; \boldsymbol{\theta}) := \left[\frac{\partial L_k(\mathbf{u}; \boldsymbol{\theta})}{\partial \theta_\ell} \right]_{r \times r},$$

$\dot{\mathbf{L}}(\mathbf{u}; \boldsymbol{\theta})$ is continuous both in $\mathbf{u}$ and $\boldsymbol{\theta}$, and the Euclidean norm $\left|\dot{\mathbf{L}}(\mathbf{u}; \boldsymbol{\theta})\right|$ is dominated by a $dC_{\boldsymbol{\theta}}$ -integrable function $h(\mathbf{u})$.

(3) $[H.3]$ The $r \times r$ matrix $A_0 := \int_{[0,1]^d} \dot{\mathbf{L}}(\mathbf{u};\boldsymbol{\theta}_0)dC_{\boldsymbol{\theta}_0}(\mathbf{u})$ is non-singular.

Theorem 2.1. *Assume that assumptions $[H.1]-[H.3]$ hold. Then with probability tending to one as $n \to \infty$, the solution $\widehat{\boldsymbol{\theta}}^{CM}$ converges to $\boldsymbol{\theta}_0$. Moreover*

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}^{CM} - \boldsymbol{\theta}_0\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(\mathbf{0}, A_0^{-1}D_0\left(A_0^{-1}\right)^T\right), \text{ as } n \to \infty,$$

where $D_0 := var\{\mathbf{L}(\xi;\boldsymbol{\theta}_0) + \mathbf{V}(\xi;\boldsymbol{\theta}_0)\}$ and $\mathbf{V}(\xi;\boldsymbol{\theta}_0) = (V_1(\xi;\boldsymbol{\theta}_0), \ldots, V_r(\xi;\boldsymbol{\theta}_0))$ with

$$V_k(\xi;\boldsymbol{\theta}_0) := \sum_{j=1}^{d} \int_{[0,1]^d} \frac{\partial\left(C_{\boldsymbol{\theta}_0}(\mathbf{u})\right)^k}{\partial u_j}\left(\mathbf{1}\{\xi_j \le u_j\} - u_j\right)dC_{\boldsymbol{\theta}_0}(\mathbf{u}),\ k = 1, \ldots, r,$$

where $\xi := (\xi_1, \ldots, \xi_d)$ is a $(0,1)^d$-uniform random vector with joint df $C_{\boldsymbol{\theta}_0}$.

Proof. See [7]. □

Now we define the empirical likelihood ratio function for θ by

$$\mathcal{L}(\boldsymbol{\theta}) := \sup_{\mathbf{p}}\left\{\prod_{i=1}^{n}(np_i) : \sum_{i=1}^{n} p_i L_k(\mathrm{U}_i, \boldsymbol{\theta}) = \mathbf{0}, p_i > 0, \sum_{i=1}^{n} p_i = 1\right\}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$. This is the maximum of the non-parametric likelihood ratio with the restriction that the mean of the estimating function is zero under the distribution $\mathbf{K}_\theta$.

Let, for $k = 1, \ldots, r$

$$L_i^{(k)} = (C_{\boldsymbol{\theta}}(\mathbf{U}_i))^k - M_k(\boldsymbol{\theta})$$

and

$$\widehat{L}_{i,n}^{(k)} = \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k - M_k(\boldsymbol{\theta})$$

where $\widehat{M}_k$ is defined in (2.9). Then, the empirical likelihood evaluated at $\boldsymbol{\theta}$ is defined as

$$\tilde{\mathcal{L}}(\boldsymbol{\theta}) = \sup_{\mathbf{p}}\left\{\prod_{i=1}^{n}(np_i) : \sum_{i=1}^{n} p_i L_i^{(k)} = \mathbf{0}, \sum_{i=1}^{r} p_i = 1\right\}$$

Since the $L_i^{(k)}$'s depend on C_θ, for an unknown $\boldsymbol{\theta}$, we replace them by the $\widehat{L}_{i,n}^{(k)}$'s. Therefore, an estimated the empirical likelihood evaluated at $\boldsymbol{\theta}$ is defined by

$$\mathcal{L}(\boldsymbol{\theta}) = \sup_{\mathbf{p}}\left\{\prod_{i=1}^{n}(np_i) : \sum_{i=1}^{n} p_i \widehat{L}_{i,n}^{(k)} = \mathbf{0}, \sum_{i=1}^{n} p_i = 1\right\}$$

Now, by introducing a vector of Lagrange multipliers $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$, to find the optimal p_i's i.e. maximizing

$$G = \sum_{i=1}^{n} \log(np_i) - n\lambda_k \sum_{i=1}^{n} p_i \widehat{L}_{i,n}^{(k)} + \gamma\left(\sum_{i=1}^{n} p_i - 1\right).$$

So, setting $\frac{\partial G}{\partial p_i} = 0$ gives

$$\frac{\partial G}{\partial p_i} = \frac{1}{p_i} - n\lambda_k \widehat{L}_{i,n}^{(k)} + \gamma.$$

Therefore, the equation $\sum_{i=1}^{n} p_i \frac{\partial G}{\partial p_i} = 0$ gives $\gamma = -n$. Then, p_i is given by

$$p_i = \frac{1}{n}\left(1 + \lambda_k \widehat{L}_{i,n}^{(k)}\right)^{-1} =: p_{i,k}.$$

We have the problem that all the solutions $p_{1,k}, p_{2,k}, \ldots, p_{n,k}, \lambda_k$ and γ are not obtained in a closed form. Note that for $k = 1, 2, \ldots, r$: $\prod_{i=1}^{n} p_{i,k}$, subject to $\sum_{i=1}^{n} p_{i,k} = 1$, attains its maximum n^{-n} at $p_{i,k} = 1/n$. So we define the empirical likelihood ratio for $\boldsymbol{\theta}$ as

$$\mathcal{R}(\boldsymbol{\theta}) = \prod_{i=1}^{n}(np_{i,k}) = \prod_{i=1}^{n}\left(1 + \lambda_k \widehat{L}_{i,n}^{(k)}\right)^{-1},$$

and the corresponding empirical log-likelihood ratio is defined as

$$\mathcal{L}(\boldsymbol{\theta}) = -2\log\mathcal{R}(\theta) = 2\sum_{i=1}^{n}\log\left(1 + \lambda_k \widehat{L}_{i,n}^{(k)}\right), k = 1, 2, \ldots, r, \tag{2.10}$$

where the vector $\boldsymbol{\lambda}$ is the solution of the system of r equations given by

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{L}_{i,n}^{(k)}}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = 0, k = 1, 2, \ldots, r. \tag{2.11}$$

Since (2.11) is an implicit function of $\boldsymbol{\lambda}$, we may solve (2.11) with respect to $\boldsymbol{\lambda}$ by the iterative procedure such as the Newton-Raphson optimization method or a simple grid search.

Theorem 2.2. *Assume conditions* $[H.1]$–$[H.3]$ *hold. Then the limiting distribution of* $\mathcal{L}(\boldsymbol{\theta})$ *is a scaled chi-square distribution with r degrees of freedom, that is,*

$$\mathcal{L}(\boldsymbol{\theta}) \to \chi^2_{(r)}$$

3. Illustrative example and simulation study

We consider the transformed Gumbel copula given by

$$C_{\alpha,\beta}(u_1, u_2) := \left(\left(\left(u_1^{-\alpha} - 1\right)^{\beta} + \left(u_2^{-\alpha} - 1\right)^{\beta}\right)^{1/\beta} + 1\right)^{-1/\alpha}, \tag{3.12}$$

which is also a two-parameter Archimedean copula with generator $\varphi_{\alpha,\beta}(t) := (t^{-\alpha} - 1)^{\beta}$. Here $\boldsymbol{\theta} = (\alpha, \beta)$ then $r = 2$, and $\mathbf{U} = (U_1, U_2)$. By an elementary calculation we get the kth CM:

$$M_k(\alpha, \beta) = \frac{(k+1)\beta + \alpha\beta - k}{(k+1)^2\beta + (k+1)\alpha\beta}.$$

In particular the first two CM's are

$$M_1(\alpha, \beta) := \frac{2\beta + \alpha\beta - 1}{4\beta + 2\alpha\beta} \text{ and } M_2(\alpha, \beta) := \frac{3\beta + \alpha\beta - 2}{9\beta + 3\alpha\beta}.$$

Then

$$L_{in}^{(1)}(\mathbf{U}_i;\alpha,\beta) := (C_n(U_{1i},U_{2i})) - M_1(\alpha,\beta)$$

and

$$L_{in}^{(2)}(\mathbf{U}_i;\alpha,\beta) := (C_n(U_{1i},U_{2i}))^2 - M_2(\alpha,\beta)$$

Then

$$p_{i,1} = \frac{1}{n}\,(1+\lambda_1(C_n(U_{1i},U_{2i}) - M_1(\alpha,\beta)))^{-1}$$

and

$$p_{i,2} = \frac{1}{n}\,\Big(1+\lambda_2\Big((C_n(U_{1i},U_{2i}))^2 - M_2(\alpha,\beta)\Big)\Big)^{-1}$$

where the vector $\lambda = (\lambda_1,\lambda_2)$ satisfies two equations given by

$$\begin{aligned}\sum_{i=1}^{n}(1+\lambda_1(C_n(U_{1i},U_{2i}) - M_1(\alpha,\beta)))^{-1} &= 0,\\ \sum_{i=1}^{n}\Big(1+\lambda_2\Big((C_n(U_{1i},U_{2i}))^2 - M_2(\alpha,\beta)\Big)\Big)^{-1} &= 0\end{aligned}$$

Finally, we get

$$\begin{aligned}&\mathcal{R}(\alpha,\beta)\\ &= \left(\sum_{i=1}^{n}(1+\lambda_1(C_n(U_{1i},U_{2i})-M_1(\alpha,\beta)))^{-1},\sum_{i=1}^{n}\Big(1+\lambda_2\Big((C_n(U_{1i},U_{2i}))^2-M_2(\alpha,\beta)\Big)\Big)^{-1}\right)\end{aligned}$$

To evaluate and compare the performance of empirical likelihood for CM's estimator is called the empirical likelihood copula moment (ELCM) estimator with the CM's and PML's estimator, a simulation study is carried out by considering the above example of bivariate Gumbel copula family $C_{\alpha,\beta}$. The evaluation of the performance is based on the bias and the RMSE defined as follows:

$$\text{Bias} = \frac{1}{R}\sum_{i=1}^{R}\widehat{\theta}_i - \theta,\ \text{RMSE} = \left(\frac{1}{R}\sum_{i=1}^{R}\left(\widehat{\theta}_i - \theta\right)^2\right)^{1/2}, \tag{3.13}$$

where $\widehat{\theta}_i$ is an estimator (from the considered method) of θ from the ith samples for R generated samples from the underlying copula. In both parts, we selected $R = 1000$. To assess the improvement in the bias and RMSE of the estimators we repeat the following steps:

Step 1: For a given sample $(\mathbf{X}_1,\ldots,\mathbf{X}_n)$ from random vector $\mathbf{X} = (X_1,\ldots,X_d)$, we define the corresponding joint empirical df by

$$C_n(\mathbf{x}) = n^{-1}\sum_{i=1}^{n}\mathbf{1}\{X_{1i}\le x_1,\ldots,X_{di}\le x_d\}.$$

with $\mathbf{x} := (x_1,\ldots,x_d)$. For each $j = 1, \ldots, d$, compute (2.8).

Step 2: Solve the following system for $k = 1, \ldots, r$,

$$\sum_{i=1}^{n}\left(1+\lambda_k\left(\left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k - M_k(\boldsymbol{\theta})\right)\right)^{-1} = 0,$$

The obtained solution $\widehat{\boldsymbol{\theta}}^{ELCM} := \left(\widehat{\theta}_1, \ldots, \widehat{\theta}_r\right)$.

For different sample sizes n with $n = 50, 100, 200, 500$ with increasing sample size and a large set of parameters of the true copula $C_{\alpha,\beta}$. The choice of the true values of the parameter (α, β) has to be meaningful, in the sense that each couple of parameters assigns a value of one of the dependence measure, that is weak, moderate and strong dependence. The selected values of the true parameters are summarized in Table 1, the results are summarized in Table 2.

4. Comments and conclusions

From Table 2, by considering three dependence cases: weak ($\tau = 0.01$), moderate ($\tau = 0.5$) and strong ($\tau = 0.8$), the performance of the ELCM estimator remains quite good in small sample size. We show that the ELCM estimator is performs better than the CM based estimator in large one. Moreover, in time-consuming point of view, we observe that for a sample size $n = 30$ with $N = 1000$ replications, the central processing unit (CPU) time to apply ELCM method took 1.442 hours, which takes approximately the same time with the PLM method and is relatively big to the CM method, which is measured in seconds 22.013. For only one replication, the CPU times (in seconds), for different sample sizes, are summarized as follows: (n, *CPU*) = (30, 5.2613), (100, 10.891), (200, 16.965), (500, 25.995), (see Table 3). Which opens the door to new applications in copulas estimation framework.

5. Proofs

5.1 Proof of Theorem 2.2

For the proof we need the following Lemmas

Lemma 5.1. *Under the same conditions as in Theorem 2.1*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{L}_{i,n}^{(k)} \to \mathcal{N}\left(0, \sigma_1^2(\boldsymbol{\theta})\right).$$

Proof. Follows straightaway from Theorem 2.1, see [7].

Lemma 5.2. *Under the same conditions as in Theorem 2.1, for $k = 1, 2, \ldots, r$ we have*

$$(1)\ \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2 = O_p(1), \quad (2)\ \widehat{\sigma}_k^2(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2 + o_p(1).$$

	τ	α	β
Weak	0.01	0.1	1.059
Moderate	0.5	0.5	1.600
Strong	0.8	0.9	3.450

Table 1. The true parameters of transformed gumbel copula used for the simulation study

Table 2.
Bias and RMSE of ELCM estimator of two-parameter transformed gumbel copula

	$\tau = 0.01$				$\tau = 0.5$				$\tau = 0.8$				
	$\alpha = 0.1$		$\beta = 1.059$		$\alpha = 0.5$		$\beta = 1.6$		$\alpha = 0.9$		$\beta = 3.45$		
n	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	CPU
50	0.047	0.103	0.024	0.138	0.033	0.118	0.012	0.213	0.036	0.113	0.073	0.349	1.462 hours
100	0.023	0.088	0.022	0.100	0.019	0.057	0.009	0.126	0.021	0.071	0.014	0.294	3.025 hours
200	0.009	0.079	0.011	0.057	0.007	0.046	0.008	0.097	0.004	0.064	0.026	0.105	4.710 hours
500	0.003	0.035	0.009	0.040	0.001	0.029	0.003	0.092	0.002	0.018	0.007	0.112	7.221 hours

Table 3. Bias and RMSE of ELCM, CM and PML estimators of two-parameter transformed gumbel copula

	$\tau = 0.01$				$\tau = 0.5$				$\tau = 0.8$			
	$\alpha = 0.1$		$\beta = 1.059$		$\alpha = 0.5$		$\beta = 1.6$		$\alpha = 0.9$		$\beta = 3.45$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$n = 50$												
ELCM	0.049	0.093	0.028	0.137	0.031	0.101	0.031	0.229	0.039	0.109	0.088	0.356
CM	−0.075	0.224	0.042	0.121	−0.078	0.452	0.061	0.326	−0.061	0.742	−0.251	1.051
PML	−0.059	0.098	−0.324	0.333	0.060	0.299	−0.273	0.482	−0.052	0.525	0.241	1.011
$n = 100$												
ELCM	0.023	0.087	0.021	0.103	0.021	0.077	0.029	0.111	0.031	0.073	0.016	0.299
CM	−0.035	0.234	−0.010	0.130	−0.025	0.494	0.014	0.353	−0.022	0.721	0.197	0.881
PML	−0.049	0.051	−0.466	0.470	−0.061	0.231	−0.327	0.433	0.014	0.300	0.319	0.731
$n = 200$												
ELCM	0.011	0.057	0.015	0.064	0.013	0.045	0.013	0.099	0.017	0.069	0.027	0.109
CM	−0.020	0.121	0.011	0.100	0.029	0.367	−0.015	0.257	−0.025	0.551	0.155	0.704
PML	−0.041	0.038	−0.282	0.285	−0.081	0.164	−0.304	0.469	−0.022	0.157	−0.018	0.227
$n = 500$												
ELCM	0.005	0.037	0.007	0.044	0.009	0.031	0.009	0.091	0.011	0.010	0.012	0.101
CM	−0.010	0.102	0.008	0.046	−0.014	0.233	0.011	0.120	−0.011	0.301	−0.063	0.423
PML	−0.040	0.061	−0.203	0.210	−0.046	0.100	−0.302	0.225	−0.019	0.244	0.028	0.200

Proof. (1) From the law of large number, it follows that

$$\frac{1}{n}\sum_{i=1}^{n}\left(L_i^{(k)}\right)^2 = \mathbb{E}\left[\left(L_1^{(k)}\right)^2\right] + o_p(1) = O_p(1)$$

(2) Let

$$\mathbb{T} = \left|\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\left(L_i^{(k)}\right)^2\right|.$$

So we can write

$$\begin{aligned}\mathbb{T} &= \left|\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)}\right)\left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)} + 2L_i^{(k)}\right)\right| \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)}\right)^2 + 2\left|\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)}\right)L_i^{(k)}\right| \equiv \mathbb{T}_1 + 2\mathbb{T}_2.\end{aligned}$$

We have

$$\mathbb{T}_1 = \frac{1}{n}\sum_{i=1}^{n}\left(\left(C_{\boldsymbol{\theta}}\left(\widehat{\mathbf{U}}_i\right)\right)^k - \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k\right)^2 = O_p(1),$$

and

$$\begin{aligned}\mathbb{T}_2 &= \left|\frac{1}{n}\sum_{i=1}^{n}\left(\left(C_{\boldsymbol{\theta}}\left(\widehat{\mathbf{U}}_i\right)\right)^k - \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k\right)L_i^{(k)}\right| \\ &\leq \sup_{\mathbf{t}}\left((C_{\boldsymbol{\theta}}(\mathbf{t}))^k - (C_n(\mathbf{t}))^k\right)\left|\frac{1}{n}\sum_{i=1}^{n}L_i^{(k)}\right| = O_p(1).\end{aligned}$$

Hence $\mathbb{T} = O_p(1)$. It follows

$$\begin{aligned}\left|\widehat{\sigma}_k^2(\boldsymbol{\theta}) - \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2\right| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left(\left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k - \widehat{M}_k\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\left(\left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k - M_k(\boldsymbol{\theta})\right)^2\right| \\ &= \left|\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{M}_k - M_k(\boldsymbol{\theta})\right)\left(\widehat{M}_k - 2\left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k + M_k(\boldsymbol{\theta})\right)\right| \\ &\leq \left|\widehat{M}_k - M_k(\boldsymbol{\theta})\right|\left|\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{M}_k - \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k\right)\right. \\ &\quad \left. + \frac{1}{n}\sum_{i=1}^{n}\left(M_k(\boldsymbol{\theta}) - \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k\right)\right| \\ &\leq \left(\widehat{M}_k - M_k(\boldsymbol{\theta})\right)^2 = o_p(1).\end{aligned}$$

The proof of Lemma 5.2 is completed. □

As showing in [19]; for $k = 1, 2, \ldots, r$,

$$\lambda_k = O_p\left(n^{-1/2}\right).$$

Now applying Taylor's expansion to $\mathcal{L}(\boldsymbol{\theta})$, we have

$$\mathcal{L}(\boldsymbol{\theta}) \simeq 2\left(\sum_{i=1}^{n}\left(\lambda_k \widehat{L}_{i,n}^{(k)} - \frac{1}{2}\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^2\right)\right) \tag{5.14}$$

Note that from (2.11), for $k = 1, 2, \ldots, r$

$$\begin{aligned} \sum_{i=1}^{n} \frac{\widehat{L}_{i,n}^{(k)}}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} &= 0 = \sum_{i=1}^{n} \widehat{L}_{i,n}^{(k)}\left(1 - \lambda_k \widehat{L}_{i,n}^{(k)} + \frac{\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^2}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}}\right) \\ &= \sum_{i=1}^{n} \widehat{L}_{i,n}^{(k)} - \lambda_k \sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2 + \sum_{i=1}^{n} \frac{\lambda_k^2\left(\widehat{L}_{i,n}^{(k)}\right)^3}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} \end{aligned} \tag{5.15}$$

From, Lemma 5.2 it follows that for $k = 1, 2, \ldots, r$

$$\lambda_k = \left(\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2\right)^{-1} \sum_{i=1}^{n} \widehat{L}_{i,n}^{(k)} + o_p\left(n^{-1/2}\right). \tag{5.16}$$

By (5.15) we get

$$\sum_{i=1}^{n} \frac{\lambda_k \widehat{L}_{i,n}^{(k)}}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = \sum_{i=1}^{n} \lambda_k \widehat{L}_{i,n}^{(k)} - \sum_{i=1}^{n}\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^2 + \sum_{i=1}^{n} \frac{\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^3}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = 0$$

Note that

$$\sum_{i=1}^{n} \frac{\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^3}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = O_p\left(n^{-1/2}\right)$$

then

$$\sum_{i=1}^{n} \lambda_k \widehat{L}_{i,n}^{(k)} = \sum_{i=1}^{n}\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^2 + o_p(1),$$

Therefore, it follows from (5.14) and Lemmas 5.1 and (5.16) that

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta}) &= 2\left(\sum_{i=1}^{n}\left(\lambda_k \widehat{L}_{i,n}^{(k)} - \frac{1}{2}\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^2\right)\right) \\ &= \left(\sum_{i=1}^{n}\left(\lambda_k \widehat{L}_{i,n}^{(k)}\right)^2\right) + o_p(1) \\ &= \left(\sum_{i=1}^{n}\left(\widehat{L}_{i,n}^{(k)}\right)^2\right)^{-1}\left(\sum_{i=1}^{n} \widehat{L}_{i,n}^{(k)}\right)^2 + o_p(1) \\ &\to \chi_1^2. \end{aligned}$$

The proof of Theorem 2.2 is completed.

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