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Arithmetic properties of singular overpartition pairs without multiples of k

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Abstract

Purpose – In this paper, the author defines the function $\overline{B}_{i,j}^{\delta,k}(n)$, the number of singular overpartition pairs of n without multiples of k in which no part is divisible by δ and only parts congruent to $\pm i, \pm j$ modulo δ may be overlined.

Design/methodology/approach – Andrews introduced to combinatorial objects, which he called singular overpartitions and proved that these singular overpartitions depend on two parameters δ and i can be enumerated by the function $\overline{C}_{\delta,i}(n)$, which gives the number of overpartitions of n in which no part divisible by δ and parts $\equiv \pm i(\text{Mod } \delta)$ may be overlined.

Findings – Using classical spirit of q-series techniques, the author obtains congruences modulo 4 for $\overline{B}_{2,4}^{8,3}(n)$, $\overline{B}_{2,4}^{8,5}$ and $\overline{B}_{2,4}^{12,3}$.

Originality/value – The results established in this work are extension to those proved in Andrews' singular overpatition pairs of n.

Keywords Congruences, Dissections, Singular overpartition pairs without multiples of k

Paper type Research paper

1. Introduction

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1.1}$$

where the product representations arise from Jacobi's triple product identity [1, p. 35, Entry 19].

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2}$$

Throughout the paper, we use the standard q-series notation, and f_k is defined as

$$f_k := (q^k;q^k)_\infty = \lim_{n\to\infty}\prod_{m=1}^{n}(1-q^{mk}).$$

The special cases of $f(a, b)$ are

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Dedicated to Prof. M. S. Mahadeva Naika on his 62nd birthday.

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$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \tag{1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{1.5}$$

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. An overpartition, introduced by Corteel and Lovejoy [2], of a nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined.

Recently, G. E. Andrews [3] defined combinatorial objects, which he called singular overpartitions and proved that these singular overpartitions depend on two parameters δ and i can be enumerated by the function $\overline{C}_{\delta,i}(n)$, which gives the number of overpartitions of n in which no part divisible by δ and parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{C}_{\delta,i}(n)$ is

$$\sum_{n=0}^{\infty} \overline{C}_{\delta,i}(n) q^n = \frac{(q^\delta;q^\delta)_\infty (-q^i;q^\delta)_\infty (-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty}. \tag{1.6}$$

He also proved that

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod 3. \tag{1.7}$$

Andrews [3] proves that, for all $n \geq 0$, $\overline{C}_{3,1}(n) = \overline{A}_3(n)$, where $\overline{A}_3(n)$ is the number of overpartitions of n into parts not divisible by 3. The function $\overline{A}_\ell(n)$, which counts the number of overpartitions of n into parts not divisible by ℓ, plays a key role in the work of Lovejoy [4].

Chen *et al.* [5] have generalized (1.7) and proved some congruences modulo 2, 3, 4 and 8 for $\overline{C}_{3,1}(n)$. They also proved some congruence for $\overline{C}_{4,1}(n)$, $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$ modulo powers of 2 and 3. More recently, Ahmed and Baruah [6] have found some new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{8,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$ modulo 18, 36. Chen [7] has also found some congruences modulo powers of 2 for $\overline{C}_{3,1}(n)$, $\overline{C}_{4,1}(n)$. Yao [8] has proved congruences modulo 16, 32, 64 for $\overline{C}_{3,1}(n)$. Naika and Gireesh [9] have found some congruences modulo 6, 12, 16, 18, 24, 48 and 72 for $\overline{C}_{3,1}(n)$. Naika and Nayaka [10] have proved some congruences for $\overline{CO}_{3,1}(n)$ modulo powers of 2 and 3. They have also proved in a paper [11] modulo 4 for $\overline{C}^3_{4,1}(n)$ and $\overline{C}^5_{4,1}(n)$.

In [12, 13], Naika *et al.* have defined the Andrews' singular overpartition pairs of n. Let $\overline{C}^\delta_{i,j}(n)$ denote the number of Andrews' singular overpartition pairs of n in which no part is divisible by δ and only parts congruent to $\pm i, \pm j$ modulo δ may be overlined. Andrews' singular overpartition pair π of n is a pair of Andrews' singular overpartitions (λ, μ) such that the sum of all of the parts is n. They also established Ramanujan-like congruences for $\overline{A}^6_{1,2}(n)$ modulo 3,9, 27 and infinite families of congruences for $\overline{A}^{12}_{1,5}(n)$ modulo 4, 6 and 9.

In this paper, we define the function $\overline{B}_{i,j}^{\delta,k}(n)$, the number of singular overpartition pairs of n without multiples of k in which no part is divisible by δ and only parts congruent to $\pm i$, $\pm j$ modulo δ may be overlined. The generating function of $\overline{B}_{i,j}^{\delta,k}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{B}_{i,j}^{\delta,k}(n)q^n = \frac{f(q^i,q^{\delta-i})f(q^j,q^{\delta-j})(q^k;q^k)_\infty^2}{f(q^{ki},q^{k(\delta-i)})f(q^{kj},q^{k(\delta-j)})(q;q)_\infty^2}. \tag{1.8}$$

In this paper, we establish some congruences modulo 4 for $\overline{B}_{2,4}^{8,3}(n)$, $\overline{B}_{2,4}^{8,5}$ and $\overline{B}_{2,4}^{12,3}$. The main results of this paper can be stated as follows.

Theorem 1.1. For all integers $\alpha \geq 0$ *and* $n \geq 0$,

$$\overline{B}_{2,4}^{8,3}(16n+9) \equiv 0 \pmod 4, \tag{1.9}$$

$$\overline{B}_{2,4}^{8,3}(16n+13) \equiv 0 \pmod 4, \tag{1.10}$$

$$\overline{B}_{2,4}^{8,3}(32n+23) \equiv 0 \pmod 4, \tag{1.11}$$

$$\overline{B}_{2,4}^{8,3}(32n+31) \equiv 0 \pmod 4, \tag{1.12}$$

$$\overline{B}_{2,4}^{8,3}(64n+35) \equiv 0 \pmod 4, \tag{1.13}$$

$$\overline{B}_{2,4}^{8,3}(64n+51) \equiv 0 \pmod 4, \tag{1.14}$$

$$\overline{B}_{2,4}^{8,3}(256n+139) \equiv 0 \pmod 4, \tag{1.15}$$

$$\overline{B}_{2,4}^{8,3}(256n+203) \equiv 0 \pmod 4, \tag{1.16}$$

$$\overline{B}_{2,4}^{8,3}(4n+3) \equiv \overline{B}_{2,4}^{8,3}(64n+43) \pmod 4, \tag{1.17}$$

$$\overline{B}_{2,4}^{8,3}(8n+5) \equiv \overline{B}_{2,4}^{8,3}(32n+19) \equiv \overline{B}_{2,4}^{8,3}(128+75) \pmod 4. \tag{1.18}$$

Theorem 1.2. Let p be a prime ≥ 5, $\left(\frac{-6}{p}\right) = -1$. Then for all integers $\alpha \geq 1$, and $n \geq 0$,

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}\left(16p^{2\alpha}n + \frac{14p^{2\alpha}+1}{3}\right)q^n \equiv 2f_1f_6 \pmod 4. \tag{1.19}$$

Theorem 1.3. Let p be a prime ≥ 5, $\left(\frac{-6}{p}\right) = -1$. Then for all integers $\alpha \geq 0$, and $n \geq 0$,

$$\overline{B}_{2,4}^{8,3}\left(16p^{2\alpha+2}n + 16p^{2\alpha+1}i + \frac{14p^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod 4, \tag{1.20}$$

where i is an integer and $1 \leq i \leq p-1$.

Theorem 1.4. For all integers $\alpha \geq 0$ *and* $n \geq 0$,

$$\overline{B}_{2,4}^{8,5}(4n+3) \quad \equiv 0 \pmod 4, \tag{1.21}$$

$$\overline{B}^{8,5}_{2,4}(8n+5) \equiv 0 \pmod 4. \tag{1.22}$$

Theorem 1.5. For any prime $p \geq 5$, $\alpha \geq 0$ *and* $n \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{B}^{8,5}_{2,4}\left(8p^{2\alpha}n + \frac{p^{3\alpha}+2}{3}\right)q^n \equiv 2f_1 \pmod 4. \tag{1.23}$$

Theorem 1.6. For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, \ldots p-1$, we have

$$\overline{B}^{8,5}_{2,4}\left(8p^{2\alpha+1}(pn+l) + \frac{p^{3\alpha}+2}{3}\right) \equiv 0 \pmod 4. \tag{1.24}$$

Theorem 1.7. For all integers $\alpha \geq 0$ and $n \geq 0$,

$$\overline{B}^{12,3}_{2,4}(6n+5) \equiv 0 \pmod 4, \tag{1.25}$$

$$\overline{B}^{12,3}_{2,4}(12n+9) \equiv 0 \pmod 4, \tag{1.26}$$

$$\overline{B}^{12,3}_{2,4}(36n+27) \equiv 0 \pmod 4, \tag{1.27}$$

$$\overline{B}^{12,3}_{2,4}(108n+51) \equiv 0 \pmod 4, \tag{1.28}$$

$$\overline{B}^{12,3}_{2,4}(108n+87) \equiv 0 \pmod 4, \tag{1.29}$$

$$\overline{B}^{12,3}_{2,4}\left(4\cdot 3^{2\alpha+3}n + \frac{32\cdot 3^{2\alpha+3}+3}{2}\right) \equiv 0 \pmod 4. \tag{1.30}$$

Theorem 1.8. For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{B}^{12,3}_{2,4}\left(36p^{2\alpha}n + \frac{3(p^{3\alpha}+1)}{2}\right)q^n \equiv 2f_1 \pmod 4. \tag{1.31}$$

Theorem 1.9. For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, \ldots p-1$, we have

$$\overline{B}^{12,3}_{2,4}\left(36p^{2\alpha+1}(pn+l) + \frac{3(p^{3\alpha}+1)}{2}\right) \equiv 0 \pmod 4. \tag{1.32}$$

2. Preliminary results

We need the following few dissection formulas to prove our main results,

Lemma 2.1. The following two dissections hold:

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.1}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{2.2}$$

Hirschhorn, Garvan and Borwein [14] have proved Eqn (2.1). For proof of (2.2), see [15].

Lemma 2.2. The following two dissections hold:

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.3}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{2.4}$$

Eqn (2.3) was proved by Baruah and Ojah [16]. Replacing q by $-q$ in (2.3) and using the fact that $(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}$, we get (2.4).

Lemma 2.3. The following two dissections hold:

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.5}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}. \tag{2.6}$$

Xia and Yao [17] gave a proof of Lemma (2.3). Replacing q by $-q$ in (2.5) and using the fact that $(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}$, we get (2.6).

Lemma 2.4. The following two dissections hold:

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{2.7}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \tag{2.8}$$

Xia and Yao [18] proved (2.7) by employing an addition formula for theta functions.

Lemma 2.5. The following two dissections hold:

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.9}$$

Eqn (2.9) was proved by Hirschhorn and Sellers [19].

Lemma 2.6. The following three dissections hold:

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{2.10}$$

One can see this identity in [20].

Lemma 2.7. (Cui and Gu [21, Theorem 2.2]). For any prime $p \geq 5$,

$$f_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2},$$

where

$$\frac{\pm p - 1}{6} := \begin{cases} \dfrac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Lemma 2.8. For any prime p and positive integer n,

$$f_1^{p^n} \equiv f_p^{p^{n-1}} \pmod{p^n}. \tag{2.11}$$

3. Proof of Theorem (1.1)

Setting $i = 2, j = 4, \delta = 8$ and $k = 3$ in (1.8), we see that

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(n) q^n = \frac{f(q^2, q^6) f(q^4, q^4) (q^3; q^3)_\infty^2}{f(q^6, q^{18}) f(q^{12}, q^{12}) (q; q)_\infty^2}. \tag{3.1}$$

By the definition of $f(a, b)$ and the well-known Jacobi triple product identity, we get

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(n) q^n = \frac{f_3^2 f_6 f_8^5 f_{48}^2}{f_1^2 f_2 f_{16}^2 f_{24}^5}. \tag{3.2}$$

Substituting (2.7) into (3.2), we have

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(n) q^n = \frac{f_6^2 f_8^4 f_{48}^2 f_4^4 f_{12}^2}{f_2^6 f_{16}^2 f_{24}^6} + 2q \frac{f_6^3 f_8^6 f_4 f_{48}^2}{f_2^5 f_{16}^2 f_{24}^4 f_{12}}. \tag{3.3}$$

Equating odd parts of the aforementioned equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(2n+1) q^n = 2 \frac{f_3^3 f_4^6 f_2 f_{24}^2}{f_1^5 f_8^2 f_{12}^4 f_6}. \tag{3.4}$$

Involving (2.11) in (3.4), we get

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(2n+1) q^n \equiv 2 \frac{f_3 f_4^2}{f_1^3} \pmod 4. \tag{3.5}$$

Employing (2.2) into (3.5), we arrive at

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(2n+1) q^n \equiv 2 \frac{f_4^8 f_6^3}{f_2^9 f_{12}^2} + 2q \frac{f_4^4 f_6 f_{12}}{f_2^7} \pmod 4. \tag{3.6}$$

Extracting the terms involving q^{2n} from both sides of (3.6), we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(4n+1)q^n \equiv 2\frac{f_2^8 f_3^3}{f_1^9 f_6^2} \pmod 4. \tag{3.7}$$

Using (2.11) in the aforementioned equation, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(4n+1)q^n \equiv 2\frac{f_3^3 f_2^4}{f_1 f_6^2} \pmod 4. \tag{3.8}$$

Substituting (2.1) into (3.8), we find that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(4n+1)q^n \equiv 2\frac{f_2^2 f_4^3}{f_{12}} + 2q\frac{f_2^4 f_{12}^3}{f_4 f_6^2} \pmod 4, \tag{3.9}$$

which implies

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+1)q^n \equiv 2\frac{f_1^2 f_2^3}{f_6} \pmod 4. \tag{3.10}$$

Invoking (2.11) in (3.10), the equation reduces to

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+1)q^n \equiv 2\frac{f_1^2 f_2^3}{f_3^2} \pmod 4. \tag{3.11}$$

Employing (2.8) into (3.11), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+1)q^n \equiv 2\frac{f_2^4 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} \pmod 4. \tag{3.12}$$

Congruence (1.9) easily follows from the aforementioned equation.

From (3.6), we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(4n+3)q^n \equiv 2\frac{f_2^4 f_3 f_6^2}{f_1^7} \pmod 4. \tag{3.13}$$

Using (2.11) in (3.6), we found

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(4n+3)q^n \equiv 2\frac{f_3 f_2 f_{12}}{f_1} \pmod 4. \tag{3.14}$$

Substituting (2.5) into (3.14), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(4n+3)q^n \equiv 2\frac{f_6^3 f_4 f_{16} f_{24}^2}{f_2 f_8 f_{12} f_{48}} + 2q\frac{f_6^3 f_8^2 f_{48}}{f_2 f_{16} f_{24}} \pmod 4. \tag{3.15}$$

Extracting the terms involving q^{2n+1} from (3.15), dividing by q and replacing q^2 by q, we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+7)q^n \equiv 2\frac{f_3^3 f_4^2 f_{24}}{f_1 f_8 f_{12}} \pmod 4. \tag{3.16}$$

Invoking (2.11) in (3.16), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+7)q^n \equiv 2\frac{f_3^3 f_{12}}{f_1} \pmod 4. \tag{3.17}$$

Employing (2.1) into (3.17), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+7)q^n \equiv 2\frac{f_4^3 f_6^2}{f_2^2} + 2q\frac{f_{12}^4}{f_4} \pmod 4. \tag{3.18}$$

Extracting the terms involving q^{2n} from both sides of (3.18), we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+7)q^n \equiv 2\frac{f_2^3 f_3^2}{f_1^2} \pmod 4. \tag{3.19}$$

Using (2.11) in (3.19), we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+7)q^n \equiv 2f_2 f_6 \pmod 4. \tag{3.20}$$

Congruence (1.11) easily follows from (3.20).

From (3.18), we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+15)q^n \equiv 2\frac{f_6^4}{f_2} \pmod 4. \tag{3.21}$$

Congruence (1.12) follows by extracting the terms involving q^{2n+1} from (3.21).

From (3.15), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+3)q^n \equiv 2\frac{f_3^3 f_2 f_8 f_{12}^2}{f_1 f_4 f_6 f_{24}} \pmod 4. \tag{3.22}$$

Using (2.11) in (3.22), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+3)q^n \equiv 2\frac{f_3^3 f_2 f_4}{f_1 f_6} \pmod 4. \tag{3.23}$$

Employing (2.1) into (3.23), we reduce that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+3)q^n \equiv 2\frac{f_4^4 f_6}{f_2 f_{12}} + 2q\frac{f_2 f_{12}^3}{f_6} \pmod 4. \tag{3.24}$$

Extracting the terms involving q^{2n} from both sides of (3.24), we find that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+3)q^n \equiv 2\frac{f_3 f_2^4}{f_1 f_6} \pmod 4. \tag{3.25}$$

Substituting (2.5) into (3.25), we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+3)q^n \equiv 2\frac{f_2^2 f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + 2q\frac{f_2^2 f_8^2 f_{48}}{f_{16} f_{24}} \pmod 4, \tag{3.26}$$

which implies

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(32n+3)q^n \equiv 2\frac{f_1^2 f_2 f_8 f_{12}^2}{f_4 f_6 f_{24}} \pmod 4. \tag{3.27}$$

Invoking (2.11) in (3.27), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(32n+3)q^n \equiv 2\frac{f_8}{f_6} \pmod 4. \tag{3.28}$$

Congruence (1.13) follows by extracting the terms involving q^{2n+1} from (3.28).

Extracting the terms involving q^{2n+1} from (3.26), dividing by q and replacing q^2 by q, we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(32n+19)q^n \equiv 2\frac{f_1^2 f_4^2 f_{24}}{f_8 f_{12}} \pmod 4. \tag{3.29}$$

Using (2.11) in (3.29), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(32n+19)q^n \equiv 2f_2 f_{12} \pmod 4. \tag{3.30}$$

Congruence (1.14) follows from (3.30).

Extracting the terms involving q^{2n} from both sides of (3.24), we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+11)q^n \equiv 2\frac{f_1 f_6^3}{f_3} \pmod 4. \tag{3.31}$$

Employing (2.6) into (3.31), we found

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(16n+11)q^n \equiv 2\frac{f_2 f_6 f_{16} f_{24}^2}{f_8 f_{48}} + 2q\frac{f_2 f_6 f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \pmod 4, \tag{3.32}$$

which implies

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(32n+11)q^n \equiv 2\frac{f_1 f_3 f_8 f_{12}^2}{f_4 f_{24}} \pmod 4. \tag{3.33}$$

Substituting (2.4) into (3.33), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(32n+11)q^n \equiv 2\frac{f_2 f_8^3 f_{12}^6}{f_4^3 f_6 f_{24}^3} + 2q\frac{f_4^3 f_6 f_{24}}{f_2 f_8} \pmod 4, \tag{3.34}$$

which implies

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(64n+11)q^n \equiv 2\frac{f_1 f_4^3 f_6^6}{f_2^3 f_3 f_{12}^3} \pmod 4. \tag{3.35}$$

Using (2.11) in (3.35), we find that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(64n+11)q^n \equiv 2\frac{f_1}{f_3 f_2^3} \pmod 4. \tag{3.36}$$

Employing (2.6) into (3.36), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(64n+11)q^n\equiv 2\frac{f_2^4f_{16}f_{24}^2}{f_6^2f_8f_{48}}+2q\frac{f_2^4f_8^2f_{12}f_{48}}{f_4f_6^2f_{16}f_{24}} \pmod 4. \tag{3.37}$$

Extracting the terms involving q^{2n} from (3.37) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(128n+11)q^n\equiv 2\frac{f_1^4f_8f_{12}^2}{f_3^2f_4f_{24}} \pmod 4. \tag{3.38}$$

Invoking (2.11) in (3.38), we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(128n+11)q^n\equiv 2\frac{f_2^2f_8}{f_4f_6} \pmod 4. \tag{3.39}$$

Congruence (1.15) follows by extracting the terms involving q^{2n+1} from (3.39).

From (3.37), we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(128n+75)q^n\equiv 2\frac{f_1^4f_4^2f_6f_{24}}{f_2f_3^2f_8f_{12}} \pmod 4. \tag{3.40}$$

Invoking (2.11) in (3.40), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(128n+75)q^n\equiv 2f_2f_{12} \pmod 4. \tag{3.41}$$

Congruence (1.16) follows from (3.41).

From (3.34), we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(64n+43)q^n\equiv 2\frac{f_2^3f_3f_{12}}{f_1f_4} \pmod 4. \tag{3.42}$$

Using (2.11) in (3.42), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(64n+43)q^n\equiv 2\frac{f_3f_2f_{12}}{f_1} \pmod 4. \tag{3.43}$$

From the equations (3.14) and (3.43), we obtain (1.17).

Extracting the terms involving q^{2n+1} from (3.9), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+5)q^n\equiv 2\frac{f_1^4f_6^3}{f_2f_3^2} \pmod 4. \tag{3.44}$$

Invoking (2.11) in (3.44), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{8,3}(8n+5)q^n\equiv 2f_2f_{12} \pmod 4. \tag{3.45}$$

Congruence (1.10) follows by extracting the terms involving q^{2n+1} from (3.45).

From equations (3.45), (3.30) and (3.40), we obtain (1.18).

4. Proof of Theorem (1.2)

Extracting the terms involving q^{2n} from (3.45) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(16n+5)q^n \equiv 2f_1 f_6 \pmod 4. \tag{4.1}$$

define

$$\sum_{n=0}^{\infty} g(n)q^n = f_1 f_6. \tag{4.2}$$

Combining (4.1) and (4.2), we find that

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,3}(16n+5)q^n \equiv 2\sum_{n=0}^{\infty} g(n)q^n \pmod 4. \tag{4.3}$$

For a prime, $p \geq 5$ or $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$, consider

$$\frac{3k^2+k}{2} + 6\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p, \tag{4.4}$$

therefore,

$$(6k+1)^2 + 6\cdot(6m+1)^2 \equiv 0 \pmod p,$$

Since $\left(\frac{-6}{p}\right) = -1$ the congruence relation (4.4) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute Lemma (2.7) into (4.2) and then extract the terms in which the powers of q are congruent to $\frac{7p^2-7}{24}$ modulo p and then divide by $q^{\frac{7p^2-7}{24}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn + \frac{7p^2-7}{24}\right)q^{pn} = f_{p^2} f_{6p^2},$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2 n + \frac{7p^2-7}{24}\right)q^n = f_1 f_6 \tag{4.5}$$

and for $n \geq 0$,

$$g\left(p^2 n + pi + \frac{7p^2-7}{24}\right) = 0, \tag{4.6}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha} n + \frac{7p^{2\alpha}-7}{24}\right) = g(n). \tag{4.7}$$

Replacing n by $p^{2\alpha}n + \frac{7p^{2\alpha}-7}{24}$ in (4.3), we arrive at (1.19).

5. Proof of Theorem (1.3)

Replacing n by $p^2n + pi + \frac{7p^2-7}{24}$ in (4.7) and using (4.6), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{7p^{\alpha+2}-7}{24}\right) = 0. \tag{5.1}$$

Comparing coefficients of q^n from both sides of (4.3), we see that for $n \geq 0$,

$$\overline{B}_{2,4}^{8,3}(16n+5) \equiv 2g(n) \pmod 4. \tag{5.2}$$

The required result follows from (5.1) and (5.2).

6. Proof of Theorem (1.4)

Setting $i = 2, j = 4$, $\delta = 8$ and $k = 5$ in (1.8), we see that

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}(n)q^n = \frac{f(q^2,q^6)f(q^4,q^4)(q^5;q^5)_\infty^2}{f(q^{10},q^{30})f(q^{20},q^{20})(q;q)_\infty^2}. \tag{6.1}$$

By the definition of $f(a, b)$ and the well-known Jacobi triple product identity, we get

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}(n)q^n = \frac{f_5^2 f_8^5 f_{10} f_{80}^2}{f_1^2 f_2 f_{16}^2 f_{40}^5}. \tag{6.2}$$

Substituting (2.9) into (6.2), we have

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}(n)q^n = \frac{f_8^7 f_{10} f_{20}^4 f_{80}^2}{f_2^5 f_{16}^2 f_{40}^7} + 2q\frac{f_4^3 f_8^5 f_{10}^2 f_{20} f_{80}^2}{f_2^6 f_{16}^2 f_{40}^5} + q^2\frac{f_4^6 f_8^3 f_{10}^3 f_{80}^2}{f_2^7 f_{16}^2 f_{40}^3 f_{20}^2}. \tag{6.3}$$

Equating odd parts of the aforementioned equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}(2n+1)q^n = 2\frac{f_2^3 f_4^5 f_5^2 f_{10} f_{40}^2}{f_1^6 f_8^2 f_{20}^5}. \tag{6.4}$$

Involving (2.11) in (6.4), we get

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}(2n+1)q^n \equiv 2f_4 \pmod 4. \tag{6.5}$$

Congruences (1.21) and (1.22) follow from the aforementioned equation.

7. Proof of Theorem (1.5)

From (6.5), we have

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}(8n+1)q^n \equiv 2f_1 \pmod 4. \tag{7.1}$$

Employing Lemma (2.7) into (7.1), it can be see that

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}\left(8\left(pn + \frac{p^2-1}{24}\right) + 1\right)q^n \equiv 2f_p \pmod 4, \tag{7.2}$$

which implies

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}\left(8p^2 n+\frac{p^3+2}{3}\right)q^n \equiv 2f_1 \pmod 4. \tag{7.3}$$

Therefore,

$$\overline{B}_{2,4}^{8,5}\left(8p^2 n+\frac{p^3+2}{3}\right) \equiv \overline{B}_{2,4}^{8,5}(8n+1) \pmod 4.$$

Using the aforementioned relation and by induction on α, we arrive at (1.23).

8. Proof of Theorem (1.6)

Combining (7.2) with Theorem (1.5), we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}\left(8p^{2\alpha+1} n+\frac{p^{3\alpha}+2}{3}\right) \equiv 2f_p \pmod 4.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{8,5}\left(8p^{2\alpha+1}(pn+l)+\frac{p^{3\alpha}+2}{3}\right) \equiv 0 \pmod 4.$$

where $l = 1, 2, \ldots, p-1$, we obtain (1.24).

9. Proof of Theorem (1.7)

Setting $i = 2$, $j = 4$, $\delta = 12$ and $k = 3$ in (1.8), we see that

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{12,3}(n)q^n = \frac{f(q^2,q^{10})f(q^4,q^8)(q^3;q^3)_\infty^2}{f(q^6,q^{30})f(q^{12},q^{24})(q;q)_\infty^2}. \tag{9.1}$$

By the definition of $f(a, b)$ and the well-known Jacobi triple product identity, we get

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{12,3}(n)q^n = \frac{f_3^2 f_4 f_6^2}{f_1^2 f_2 f_{18} f_{36}}. \tag{9.2}$$

Substituting (2.7) into (9.2), we have

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{12,3}(n)q^n = \frac{f_4^5 f_6^3 f_{12}^2}{f_2^6 f_8 f_{18} f_{24} f_{36}} + 2q\frac{f_4^2 f_6^4 f_8 f_{24}}{f_2^5 f_{12} f_{18} f_{36}}. \tag{9.3}$$

Equating odd parts of the aforementioned equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{12,3}(2n+1)q^n = 2\frac{f_2^2 f_3^4 f_4 f_{12}}{f_1^5 f_6 f_9 f_{18}}. \tag{9.4}$$

Involving (2.11) in (9.4), we get

$$\sum_{n=0}^{\infty} \overline{B}_{2,4}^{12,3}(2n+1)q^n \equiv 2\frac{f_2^2 f_6 f_{12}}{f_1 f_9 f_{18}} \pmod 4. \tag{9.5}$$

Ramanujan recorded the following identity in his third note book:

$$\psi(q)=\frac{f_2^2}{f_1}=\frac{f_6f_9^2}{f_3f_{18}}+q\frac{f_{18}^2}{f_9}. \tag{9.6}$$

Substituting (9.6) into (9.5), we deduce that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(2n+1)q^n\equiv 2\frac{f_6^2f_9f_{12}}{f_3f_{18}^2}+2q\frac{f_6f_{12}f_{18}}{f_9^2}\pmod 4. \tag{9.7}$$

Congruence (1.25) easily follows from the aforementioned equation.

Extracting the terms involving q^{3n+1} from (9.7), dividing by q and replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(6n+3)q^n\equiv 2\frac{f_2f_4f_6}{f_3^2}\pmod 4. \tag{9.8}$$

Using (2.11) in (9.8), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(6n+3)q^n\equiv 2f_2f_4\pmod 4. \tag{9.9}$$

Congruence (1.26) follows by extracting the terms involving q^{2n+1} from (9.9).

From (9.9), we can reduce that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(12n+3)q^n\equiv 2f_1f_2\pmod 4. \tag{9.10}$$

Employing (2.10) into (9.10), we get

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(12n+3)q^n\equiv 2\frac{f_6f_9^4}{f_3f_{18}^2}+2qf_9f_{18}\pmod 4. \tag{9.11}$$

Congruence (1.27) follows from (9.11).

Extracting the terms involving q^{3n+1} from (9.11), dividing by q and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(36n+15)q^n\equiv 2f_3f_6\pmod 4. \tag{9.12}$$

Congruences (1.28) and (1.29) follow by extracting the terms involving q^{3n+1} and q^{3n+2} from (9.9).

From (9.12), we find that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(108n+15)q^n\equiv 2f_1f_2\pmod 4. \tag{9.13}$$

Combining (9.10) and (9.13), we get

$$\overline{B}_{2,4}^{12,3}(108n+15)\equiv\overline{B}_{2,4}^{12,3}(12n+3)\pmod 4. \tag{9.14}$$

Using the aforementioned relation and by induction on α, we have

$$\overline{B}_{2,4}^{12,3}\left(4\cdot 3^{2\alpha+3}n+\frac{3\cdot(3^{2\alpha+2}+1)}{2}\right)\equiv\overline{B}_{2,4}^{12,3}(12n+3)\pmod 4. \tag{9.15}$$

Using (1.28) in (9.15), we obtain (1.30).

10. Proof of Theorem (1.8)

From (9.11), we find that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(36n+3)q^n \equiv 2\frac{f_2f_3^4}{f_1f_6^2} \pmod 4. \tag{10.1}$$

Invoking (2.11) in (10.1), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}(36n+3)q^n \equiv 2f_1 \pmod 4. \tag{10.2}$$

Employing Lemma (2.7) into (10.2), it can be see that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}\left(36\left(pn+\frac{p^2-1}{24}\right)+3\right)q^n \equiv 2f_p \pmod 4, \tag{10.3}$$

which implies

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}\left(36p^2n+\frac{3(p^3+1)}{2}\right)q^n \equiv 2f_1 \pmod 4. \tag{10.4}$$

Therefore,

$$\overline{B}_{2,4}^{12,3}\left(36p^2n+\frac{3(p^3+1)}{2}\right) \equiv \overline{B}_{2,4}^{12,3}(36n+3) \pmod 4.$$

Using the aforementioned relation and by induction on α, we arrive at (1.31).

11. Proof of Theorem (1.9)

Combining (10.3) with Theorem (1.8), we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}\left(36p^{2\alpha+1}n+\frac{3(p^{3\alpha}+1)}{2}\right) \equiv 2f_p \pmod 4.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty}\overline{B}_{2,4}^{12,3}\left(36p^{2\alpha+1}(pn+l)+\frac{3(p^{3\alpha}+1)}{2}\right) \equiv 0 \pmod 4.$$

where $l = 1, 2, \ldots, p-1$, we obtain (1.32).

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