Weierstrass points on modular curves $X_0(N)$ fixed by the Atkin–Lehner involutions

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Received 1 January 2021
Revised 9 June 2021
Accepted 18 June 2021

Abstract

Purpose – The authors have determined whether the points fixed by all the full and the partial Atkin–Lehner involutions W_Q on $X_0(N)$ for $N \leq 50$ are Weierstrass points or not.
Design/methodology/approach – The design is by using Lawittes's and Schoeneberg's theorems.
Findings – Finding all Weierstrass points on $X_0(N)$ fixed by some Atkin–Lehner involutions. Besides, the authors have listed them in a table.
Originality/value – The Weierstrass points have played an important role in algebra. For example, in algebraic number theory, they have been used by Schwartz and Hurwitz to determine the group structure of the automorphism groups of compact Riemann surfaces of genus $g \geq 2$. Whereas in algebraic geometric coding theory, if one knows a Weierstrass nongap sequence of a Weierstrass point, then one is able to estimate parameters of codes in a concrete way. Finally, the set of Weierstrass points is useful in studying arithmetic and geometric properties of $X_0(N)$.

Keywords Modular curves, Weierstrass points, Hyperelliptic curves, Bielliptic curves
Paper type Research paper

1. Introduction

Let $\mathcal{H}$ be the complex upper half plane and Γ be a congruence subgroup of the full modular group $SL_2(\mathbb{Z})$. Denote by $X(\Gamma)$ the modular curve obtained from compactification of the quotient space $\Gamma\backslash\mathcal{H}$ by adding finitely many points called cusps. Then $X(\Gamma)$ is a compact Riemann surface.

For each positive integer N, we have a subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ defined by:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 (mod\ N) \right\}$$

and let $X_0(N) = X_0(\Gamma(N))$.

A modular curve $X_0(N)$ of genus $g \geq 2$ is called hyperelliptic (respectively bielliptic) if it admits a map $\phi: X \to C$ of degree 2 onto a curve C of genus 0 (respectively 1). A point P of $X_0(N)$ is a Weierstrass point if there exists a non-constant function f on $X_0(N)$ which has a pole of order $\leq g$ at P and is regular elsewhere.

The Weierstrass points on modular curves have been studied by Lehner and Newman in [1]; they have given conditions when the cusp at infinity is a Weierstrass point on $X_0(N)$ for $N = 4n, 9n$, and Atkin [2] has given conditions for the case of $N = p^2n$ where p is a prime ≥ 5. Besides, Ogg [3], Kohnen [4, 5] and Kilger [6] have given some conditions when the cusp at

JEL Classification — 11G18, 14H55.

Arab Journal of Mathematical Sciences
Vol. 29 No. 1, 2023
pp. 63-72
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-01-2021-0001

infinity is not a Weierstrass point on $X_0(N)$ for certain N. Also, Ono [7] and Rohrlich [8] have studied Weierstrass points on $X_0(p)$ for some primes p. And Choi [9] has shown that the cusp $\frac{1}{2}$ is a Weierstrass point of $\Gamma_1(4p)$ when p is a prime > 7. In addition, Jeon [10, 11] has computed all Weierstrass points on the hyperelliptic curves $X_1(N)$ and $X_0(N)$. Recently Im, Jeon and Kim [12] have generalised the result of Lehner and Newman [1] by giving conditions when the points fixed by the partial Atkin–Lehner involution on $X_0(N)$ are Weierstrass points and have determined whether the points fixed by the full Atkin–Lehner involution on $X_0(N)$ are Weierstrass points or not. In this paper, we have determined which of the points fixed by W_Q on $X_0(N)$ are Weierstrass points and found Weierstrass points on modular curves $X_0(N)$ for $N \leq 50$ fixed by the partial and the full Atkin–Lehner involutions. The Weierstrass points have played an important role in algebra. For example, in algebraic number theory, they have been used by Schwartz and Hurwitz to determine the group structure of the automorphism groups of compact Riemann surfaces of genus $g \geq 2$. Whereas in algebraic geometric coding theory, if we know a Weierstrass nongap sequence of a Weierstrass point, then we are able to estimate parameters of codes in a concrete way. Finally, the set of Weierstrass points is useful in studying arithmetic and geometric properties of $X_0(N)$.

2. Points fixed by the Atkin–Lehner involutions

For each divisor $Q|N$ with $(Q, \frac{N}{Q}) = 1$, consider the matrices of the form $\begin{pmatrix} Qx & y \\ Nz & Qw \end{pmatrix}$ with $x, y, z, w \in \mathbb{Z}$ and determinant Q. Then each of these matrices defines a unique involution on $X_0(N)$, which is called the Atkin–Lehner involution and denoted by W_Q. In particular, if $Q = N$, then W_N is called the full Atkin–Lehner involution (Fricke involution). We also denote by W_Q a matrix of the above form.

Let $X_0^Q(N)$ be the quotient space of $X_0(N)$ by W_Q. Let $g_0(N)$ and $g_0^Q(N)$ be the genus of $X_0(N)$ and $X_0^Q(N)$ respectively. Then $g_0^Q(N)$ is computed by the Riemann–Hurwitz formula as follows:

$$g_0^Q(N) = \frac{1}{4}(2g_0(N) + 2 - v(Q)),$$

where $v(Q) = v(Q; N)$ is the number of points on $X_0(N)$ fixed by W_Q. It is given by:

Proposition 2.1. [13] *For each $Q||N$, $v(Q)$ is given by*

$$\begin{aligned} v(Q) &= \left(\prod_{p|N/Q} c_1(p)\right) h(-4Q) \\ &+ \left(\prod_{p|N/Q} c_2(p)\right) h(-Q), && \text{if } Q \geq 4 \text{ and } Q \equiv 3 (\mathrm{mod}\, 4), \\ &+ \prod_{p|N/2} \left(1 + \left(\frac{-4}{p}\right)\right), && \text{if } Q = 2, \\ &+ \prod_{p|N/3} \left(1 + \left(\frac{-3}{p}\right)\right), && \text{if } Q = 3, \\ &+ \prod_{p^k|N/4} \left(p^{\left\lfloor \frac{k}{2} \right\rfloor} + p^{\left\lfloor \frac{k-1}{2} \right\rfloor}\right), && \text{if } Q = 4, \end{aligned}$$

where $h(-Q)$ is the class number of primitive quadratic forms of discriminant $-Q$, $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol and the functions $c_i(p)$ are defined as follows: for $i = 1, 2$,

$$c_i(p) = \begin{cases} 1 + \left(\dfrac{-Q}{p}\right), & \text{if } p \neq 2 \text{ and } Q \equiv 3 (\text{mod } 4), \\ 1 + \left(\dfrac{-4Q}{p}\right), & \text{if } p \neq 2 \text{ and } Q \not\equiv 3 (\text{mod } 4), \end{cases}$$

$$c_1(2) = \begin{cases} 1, & \text{if } Q \equiv 1 (\text{mod } 4) \text{ and } 2 \| N, \\ 0, & \text{if } Q \equiv 1 (\text{mod } 4) \text{ and } 4 | N, \\ 2, & \text{if } Q \equiv 3 (\text{mod } 4) \text{ and } 2 \| N, \\ 3 + \left(\dfrac{-Q}{2}\right), & \text{if } Q \equiv 3 (\text{mod } 4) \text{ and } 4 \| N, \\ 3\left(1 + \left(\dfrac{-Q}{2}\right)\right), & \text{if } Q \equiv 3 (\text{mod } 4) \text{ and } 8 | N, \end{cases}$$

$$c_2(2) = 1 + \left(\frac{-Q}{2}\right), \text{ if } Q \equiv 3 (\text{mod } 4).$$

Now, we recall the algorithms for finding $\Gamma_0(N)$-inequivalent points fixed by W_Q on $X_0(N)$ [14]. For a negative integer D congruent to 0 or 1 modulo 4, we denote by $\mathcal{Q}_D$ the set of positive definite integral binary quadratic forms:

$$Q(x, y) = [p, q, r] = px^2 + qxy + ry^2$$

with discriminant $D = q^2 - 4pr$. Then $\Gamma(1)$ acts on $\mathcal{Q}_D$ by

$$Q \circ \gamma(x, y) = Q(sx + ty, ux + vy)$$

where $\gamma = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$. A primitive positive definite form $[p, q, r]$ is said to be in reduced form if

$$|q| \leq p \leq r, \text{and } q \geq 0 \text{ if either } |q| = p \text{ or } p = r.$$

Let $\mathcal{Q}_D^{\circ} \subset \mathcal{Q}_D$ be the subset of primitive forms, that is,

$$\mathcal{Q}_D^{\circ} := \{[p, q, r] \in \mathcal{Q}_D;\ gcd(p, q, r) = 1\}.$$

Then $\Gamma(1)$ also acts on $\mathcal{Q}_D^{\circ}$. As is well known [15], there is a 1-1 correspondence between the set of classes $\Gamma(1) \backslash \mathcal{Q}_D^{\circ}$ and the set of reduced primitive definite forms.

Proposition 2.2. *[14] for each $\beta \in \mathbb{Z}/2N\mathbb{Z}$, we define*

$$\mathcal{Q}^{\circ}{}_{D,N,\beta} = \{[pN, q, r] \in \mathcal{Q}_D; \beta \equiv q (\text{mod } 2N),\ gcd(p, q, r) = 1\}.$$

Then we have the following:

(1) *Define* $m = gcd(N, \beta, \frac{\beta^2 - D}{4N})$ *and fix a decomposition* $m = m_1 m_2$ *with* $m_1, m_2 > 0$ *and* $gcd(m_1, m_2) = 1$. *Let*

$$\mathcal{Q}^{\circ}_{D,N,\beta,m_1,m_2} = \{[pN, q, r] \in \mathcal{Q}^{\circ}_{D,N,\beta};\ gcd(N, p, q) = m_1, gcd(N, q, r) = m_2\}.$$

Then $\Gamma_0(N)$ *acts on* $\mathcal{Q}^{\circ}_{D,N,\beta,m_1,m_2}$ *and there is an 1-1 correspondence between*

$$\begin{array}{ll} \mathcal{Q}^{\circ}_{D,N,\beta,m_1,m_2}/\Gamma_0(N) & \to \mathcal{Q}^{\circ}_{D}/\Gamma(1) \\ [pN, q, r] & \to [pN_1, q, rN_2] \end{array}$$

where N_1N_2 *is any decomposition of* N *into coprime factors such that* $gcd(m_1, N_2) = gcd(m_2, N_1) = 1$. *Moreover we have a* $\Gamma_0(N)$*-invariant decomposition as follows:*

$$\mathcal{Q}^{\circ}_{D,N,\beta} = \bigcup_{\substack{m = m_1 m_2 \\ m_1 m_2 > 0 \\ gcd(m_1, m_2) = 1}} \mathcal{Q}^{\circ}_{D,N,\beta,m_1,m_2}. \tag{1}$$

(2) *The inverse image* $[pN_2, q, r/N_2]$ *of any primitive form* $[\bar{p}, \bar{q}, \bar{r}]$ *of discriminant* D *under the 1-1 correspondence in (1) is obtained by solving the following equations:*

$$\begin{aligned} p &= \bar{p}s^2 + \bar{q}su + ru^2 \\ q &= 2\bar{p}st + \bar{q}(sv + tu) + 2\bar{r}uv \\ r &= \bar{p}t^2 + \bar{q}tv + \bar{r}v^2. \end{aligned}$$

satisfying $p \equiv 0 (mod\ N_1)$, $q \equiv \beta (mod\ 2N)$, $r \equiv 0 (mod\ N_2)$ *and* $\begin{pmatrix} s & t \\ u & v \end{pmatrix} \in \Gamma(1)$.

(3) *we have the following* $\Gamma_0(N)$*-invariant decomposition:*

$$\mathcal{Q}_{D,N,\beta} = \bigcup_{\substack{l > 0 \\ l^2 | D}} \bigcup_{\substack{\lambda (2N) \\ l\lambda \equiv \beta (2N) \\ \lambda^2 \equiv D / l^2 (4N)}} l\mathcal{Q}^{\circ}_{D/l^2,N,\lambda}. \tag{2}$$

Suppose $Q \geq 5$. Since W_Q has a non-cuspidal fixed point on $X_0(N)$, then W_Q is given by an elliptic element, that is,

$$W_Q = \begin{pmatrix} Qx & y \\ Nz & -Qw \end{pmatrix}.$$

Then

$$\tau = \frac{2Qx + \sqrt{-4Q}}{2Nz} \tag{3}$$

is a point fixed by W_Q. Conversely, every point fixed by W_Q has the form (3).

We note that each fixed point in (3) can be considered as the Hegneer point of a quadratic form $[Nz, -2Qx, -y]$. So, if we can find $\Gamma_0(N)$-inequivalent quadratic forms $[Nz, -2Qx, -y]$ (by using Proposition 2.2), then we can produce $\Gamma_0(N)$-inequivalent points which are fixed points as in (3).

Regarding the computation of points of $X_0(N)$ fixed by W_Q, we can follow the next algorithms:

Algorithm 2.3. *[14] The following steps implement as algorithm to find $\Gamma_0(N)$-inequivalent points fixed by W_Q where $Q \neq N$:*

Step I *We search $\beta (mod\ 2N)$ such that $\beta^2 \equiv -4Q (mod\ 4N)$ with $\beta \equiv -2Qx (mod\ 2N)$ where $x \in \mathbb{Z}$.*

Step II *We set the decomposition as in (1) and (2) with $D = -4Q$.*

Step III *For each factor in the decomposition in Step II, we find the quadratic form representations and taking the inverse of reduced form under the map which is described in Proposition 2.2(2).*

Step IV *We form the elliptic elements corresponding to quadratic form representations obtained in Step III and find their Heegner points.*

Algorithm 2.4. *[12] When $Q = N$, the four steps above come as the following:*

Step I *Set $(Q, \beta) = (4N, 0)$ or $(Q, \beta) = (N, N)$ when $(N \equiv 3 (mod\ 4))$.*

Step II *Starting from a reduced form $\mathcal{Q}^{red}$, we first find a quadratic form $[a, b, c]$ which in $SL_2(\mathbb{Z})$-inquivalent with $\mathcal{Q}^{red}$ and $gcd(a, N) = 1$.*

Step III *Set $[A, B, C] = [a, b, c] \circ \begin{pmatrix} K & -1 \\ 1 & 0 \end{pmatrix}$ where K is a solution to the linear congruence equation $2aX + b \equiv -\beta (mod\ 2N)$. Then $[A, B, C]$ belongs to $\mathcal{Q}_{Q,N,\beta}$.*

Step IV *Let $\tau = \frac{-B+\sqrt{-Q}}{2A}$. Then $\Gamma_0(N)\tau$ gives a point fixed by W_N.*

3. Weierstrass points

In this section, we have computed Weierstrass points on $X_0(N)$ for $N \leq 50$ fixed by all the partial and the full Atkin–Lehner involutions in three cases:

(1) Modular curves of genus $g_0(N) \leq 1$.

(2) Hyperelliptic modular curves.

(3) Modular curves for $N = 34, 38, 42, 43, 44, 45$.

The number n of Weierstrass points is finite and satisfies

$$2g + 2 \leq n \leq g^3 - g,$$

with $n = 2g + 2$ if and only if X is hyperelliptic.

Next theorems help us to find Weierstrass points on modular curves $X_0(N)$.

Theorem 3.1. (Schoeneberg). *[16] Let X be a Riemann surface of genus $g \geq 2$. Let P be a point fixed by an automorphism T of X, of order $p > 1$, let g^T be the genus of $X^T = X/(T)$. If $g^T \neq \left\lfloor \frac{g}{p} \right\rfloor$, the greatest integer of $\frac{g}{p}$, then P is a Weierstrass point of X.*

Theorem 3.2. *[17] Let X be a Riemann surface of genus $g \geq 2$. Let T be an automorphism with 5 or more fixed points. Then, each fixed point is a Weierstrass point.*

Theorem 3.3. *[17] If P is not a Weierstrass point and $T(P) = P$, then there are at least two and at most four points fixed by T and the genus g^T of $X^T = X/(T)$ is given by $g^T = \left\lfloor \frac{g}{p} \right\rfloor$, the greatest integer of $\frac{g}{p}$. Writing $g = g^T p + r$ there are only three possible cases:*

(1) $r = 0, g = g^T p, v(T) = 2.$

(2) $r = \frac{p-1}{2}, g = (g^T + \frac{1}{2})p - \frac{1}{2}, v(T) = 3.$

(3) $r = p - 1, g = (g^T + 1)p - 1, v(T) = 4.$

where $v(T)$ is the number of points fixed by T.

Theorem 3.4. *[12] The points fixed by W_N for $N \leq 50$ are Weierstrass points on $X_0(N)$ with $g_0(N) > 1$ except possibly for the following values*

$$N = 22, 28, 30, 33, 34, 37, 40, 42, 43, 45, 46, 48.$$

First, only a finite number of Weierstrass points can exist on $X_0(N)$, and if $g_0(N) \leq 1$, then are no such points at all. So we have the following theorem:

Theorem 3.5. *The modular curves $X_0(N)$ for $N = 1 - 21$, 24, 25, 27, 32, 36, 49 have no Weierstrass points.*

Second, Let $g_0(N) \geq 2$ and $X_0(N)$ be hyperelliptic modular curves. Then there are 19 values of N, which belong to the set

$$\{22, 23, 26, 28, 29, 30, 31, 33, 35, 37, 39, 40, 41, 46, 47, 48, 50, 59, 71\}.$$

Lewittes [17] proved that if $X_0(N)$ is a hyperelliptic modular curve, then any involution on $X_0(N)$ either has no fixed points or has only non Weierstrass fixed points or is the hyperelliptic involution. Jeon [11] found all Weierstrass points on the hyperelliptic modular curves $X_0(N)$ fixed by the hyperelliptic involution. So we have the following theorem:

Theorem 3.6. *If $X_0(N)$ is a hyperelliptic modular curve of genus $g_0(N) \geq 2$, then only $2g_0(N) + 2$ points fixed by the hyperelliptic involution are Weierstrass points on $X_0(N)$.*

Third, in this case, we will study the modular curves $X_0(N)$ for $N = 34$, 38, 42, 43, 44, 45.

Theorem 3.7. *Let $X_0(N)$ be bielliptic modular curves for $N = 34, 43, 45$. Then, all points fixed by any bielliptic involution W_Q are not Weierstrass points.*

Proof: Since W_Q is a bielliptic involution of $X_0(N)$ of genus 3, it has $4 = 2g_0(N) - 2$ points fixed by W_Q on $X_0(N)$. And $g_0^Q(N) = \left\lfloor \frac{g_0(N)}{2} \right\rfloor = 1$, thus by theorems 3.3 and 3.4, each of these points is not a Weierstrass point.

Theorem 3.8. *The modular curves $X_0(N)$ for $N = 38, 42, 44$ have Weierstrass points fixed by some W_Q.*

Proof: Since W_{19} is a bielliptic involution of $X_0(38)$ of genus 4, it has six points fixed by W_{19} on $X_0(38)$. So, by theorem 3.2, all these points are Weierstrass points (similarly $X_0^{38}(38), X_0^{14}(42), X_0^{11}(44), X_0^{44}(44)$). While $g_0^2(38) = \left\lfloor \frac{g_0(38)}{2} \right\rfloor = 2$. So, by theorem 3.3, the

modular curve $X_0(38)$ has non Weierstrass points fixed by W_2 (similarly $X_0^3(42), X_0^6(42), X_0^{21}(42), X_0^{42}(42), X_0^4(44)$). Finally, the modular curve $X_0(42)$ has no points fixed by W_2 and W_7. Therefore, $X_0(42)$ has no Weierstrass points fixed by W_2 and W_7.

Now we will give an example by using Proposition 2.2 and Algorithm 2.3 to find Weierstrass points on $X_0(44)$ fixed by W_{11}.

Example 3.9. *Consider $X_0(44)$ which is of genus 4. Since W_{11} is a bielliptic involution on $X_0(44)$ [18], it has six fixed points on $X_0(44)$. Applying Step I and Step II we have $D = -44$ and $\beta \equiv \pm 22, 66 (mod\ 176)$. First consider the case of $\beta = 20$, then we have decomposition as follows:*

$$\mathcal{Q}_{-44,44,22} = \mathcal{Q}^{\circ}{}_{-44,44,22} = \mathcal{Q}^{\circ}{}_{-44,44,22,1,1}.$$

We know that $\mathcal{Q}^{\circ}{}_{-44}/\Gamma(1) = \{[1,0,11], [3,2,4], [3,-2,4]\}$. Applying Step III we obtain by taking the inverse image of reduced forms under the map which is described in Proposition 2.2(2) the following forms:

$$\mathcal{Q}^{\circ}{}_{-44,44,22,1,1}/\Gamma(1) = \{[132,22,1], [44,22,3]\}.$$

Next, consider the case of $\beta = -22$, by the same way, we obtain the following forms:

$$\mathcal{Q}^{\circ}{}_{-44,44,-22,1,1}/\Gamma(1) = \{[132,-22,1], [44,-22,3]\}.$$

When $\beta = 66$, we have the following

$$\mathcal{Q}^{\circ}{}_{-44,44,66,1,1}/\Gamma(1) = \{[1100,66,1], [44,66,25]\}.$$

Moreover in Step IV, the corresponding elliptic elements are given as follows:

$$W_1 = \begin{pmatrix} 11 & -1 \\ 132 & -11 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 11 & -3 \\ 44 & -11 \end{pmatrix}, \quad W_3 = \begin{pmatrix} -11 & -1 \\ 132 & 11 \end{pmatrix},$$
$$W_4 = \begin{pmatrix} -11 & -3 \\ 44 & 11 \end{pmatrix}, \quad W_5 = \begin{pmatrix} 33 & -1 \\ 1100 & -33 \end{pmatrix}, \quad W_6 = \begin{pmatrix} 33 & -25 \\ 44 & -33 \end{pmatrix}.$$

Then Weierstrass points (fixed points) are:

$$\tau_1 = \frac{-1}{12} + \frac{\sqrt{-11}}{132}, \quad \tau_2 = \frac{-1}{4} + \frac{\sqrt{-11}}{44}, \quad \tau_3 = \frac{1}{12} + \frac{\sqrt{-11}}{132},$$
$$\tau_4 = \frac{1}{4} + \frac{\sqrt{-11}}{44}, \quad \tau_5 = \frac{-3}{100} + \frac{\sqrt{-11}}{1100}, \quad \tau_6 = \frac{-3}{4} + \frac{\sqrt{-11}}{44}.$$

In next example, we will use Algorithm 2.4 to find Weierstrass points on $X_0(38)$ fixed by W_{38}.

Example 3.10. *Consider $X_0(38)$ which is of genus 4. Since W_{38} is a bielliptic involution on $X_0(38)$ [18], it has six fixed points. Applying Step I and Step II, we have $(Q, \beta) = (152, 0)$ and*

$$\mathcal{Q}^{red}_{-152} = \{[1,0,38], [2,0,19], [3,2,13], [3,-2,13], [6,4,7], [6,-4,7]\}.$$

N	$g_0(N)$	W_Q	$g_0^Q(N)$	$v(Q)$	Weierstrass points
$1-21$	≤ 1		*None*		
22	2	W_2, W_{22}	1	2	*None*
		W_{11}	0	6	$-\frac{1}{2}+\frac{\sqrt{-11}}{22}, \pm\frac{1}{6}+\frac{\sqrt{-11}}{66}, \pm\frac{1}{4}+\frac{\sqrt{-11}}{44}, -\frac{1}{12}+\frac{\sqrt{-11}}{132}$
23	2	W_{23}	0	6	$\frac{\sqrt{-23}}{23}, \frac{1}{2}+\frac{\sqrt{-23}}{46}, \pm\frac{1}{3}+\frac{\sqrt{-23}}{69}, \pm\frac{1}{4}+\frac{\sqrt{-23}}{92}$
$24-25$	≤ 1		*None*		
26	2	W_2, W_{13}	1	2	*None*
		W_{26}	0	6	$\frac{\sqrt{-26}}{26}, \pm\frac{1}{3}+\frac{\sqrt{-26}}{78}, \pm\frac{2}{5}+\frac{\sqrt{-26}}{130}, -\frac{5}{31}+\frac{\sqrt{-26}}{806}$
27	1		*none*		
28	2	W_4, W_{28}	1	2	*None*
		W_7	0	6	$\pm\frac{1}{4}+\frac{\sqrt{-7}}{28}, \pm\frac{1}{8}+\frac{\sqrt{-7}}{56}, \pm\frac{3}{8}+\frac{\sqrt{-7}}{56}$
29	2	W_{29}	0	6	$\frac{\sqrt{-29}}{29}, \frac{1}{2}+\frac{\sqrt{-29}}{58}, \pm\frac{1}{3}+\frac{\sqrt{-29}}{87}, \pm\frac{1}{5}+\frac{\sqrt{-29}}{145}$
30	3	W_5, W_6, W_{30}	1	4	*None*
		W_2, W_3, W_{10}	2	0	*None*
		W_{15}	0	8	$-\frac{1}{2}+\frac{\sqrt{-15}}{30}, \pm\frac{1}{4}+\frac{\sqrt{-15}}{60}, \pm\frac{1}{6}+\frac{\sqrt{-15}}{120}, -\frac{3}{8}+\frac{\sqrt{-15}}{120}, -\frac{1}{16}+\frac{\sqrt{-15}}{240}, -\frac{3}{34}+\frac{\sqrt{-15}}{510}$
31	2	W_{31}	0	6	$\frac{\sqrt{-31}}{31}, \frac{1}{2}+\frac{\sqrt{-31}}{62}, \pm\frac{1}{4}+\frac{\sqrt{-31}}{124}, \pm\frac{2}{5}+\frac{\sqrt{-31}}{155}$
32	1		*None*		
33	3	W_3	2	0	*None*
		W_{11}	0	8	$\pm\frac{1}{3}+\frac{\sqrt{-11}}{33}, \pm\frac{1}{6}+\frac{\sqrt{-11}}{66}, \pm\frac{2}{9}+\frac{\sqrt{-11}}{99}, \pm\frac{1}{12}+\frac{\sqrt{-11}}{132}$
		W_{33}	1	4	*None*
34	3	W_2, W_{17}	1	4	*None*
		W_{34}			
35	3	W_5	1	4	*None*
		W_7	2	0	*None*
		W_{35}	0	8	$\frac{\sqrt{-35}}{35}, \pm\frac{1}{4}+\frac{\sqrt{-35}}{140}, -\frac{1}{6}+\frac{\sqrt{-35}}{210}, \pm\frac{1}{12}+\frac{\sqrt{-35}}{420}, -\frac{5}{12}+\frac{\sqrt{-35}}{420}, -\frac{1}{18}+\frac{\sqrt{-35}}{630}$
36	1		*none*		
37	2	W_{37}	1	2	*none*
38	4	W_2	2	2	*none*
		W_{19}	1	6	$\pm\frac{1}{2}+\frac{\sqrt{-76}}{38}, \pm\frac{1}{4}+\frac{\sqrt{-76}}{152}, \pm\frac{1}{10}+\frac{\sqrt{-76}}{380}$
		W_{38}	1	6	$\frac{\sqrt{-38}}{38}, \frac{2}{39}+\frac{\sqrt{-38}}{741}, \frac{2}{51}+\frac{\sqrt{-38}}{1938}, \frac{20}{273}+\frac{\sqrt{-38}}{546}, \frac{128}{2737}+\frac{\sqrt{-38}}{2737}, \frac{88}{1297}+\frac{\sqrt{-38}}{1297}$
39	3	W_3	1	4	*none*
		W_{13}	2	0	*none*
		W_{39}	0	8	$\frac{\sqrt{-39}}{39}, \pm\frac{1}{4}+\frac{\sqrt{-39}}{156}, \pm\frac{1}{8}+\frac{\sqrt{-39}}{312}, -\frac{3}{8}+\frac{\sqrt{-39}}{312}, -\frac{5}{16}+\frac{\sqrt{-39}}{624}, -\frac{1}{20}+\frac{\sqrt{-39}}{780}$
40	3	W_5, W_8	2	0	*none*
		W_{40}	1	4	*none*
		$\begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$	0	8	$\pm\frac{1}{4}+\frac{\sqrt{-5}}{20}, \pm\frac{1}{12}+\frac{\sqrt{-5}}{60}, \pm\frac{5}{12}+\frac{\sqrt{-5}}{60}, \pm\frac{5}{56}+\frac{\sqrt{-5}}{180}$
41	3	W_{41}	0	8	$\frac{\sqrt{-41}}{41}, \frac{1}{2}+\frac{\sqrt{-41}}{82}, \pm\frac{1}{3}+\frac{\sqrt{-41}}{123}, \pm\frac{2}{5}+\frac{\sqrt{-41}}{205}, \pm\frac{1}{6}+\frac{\sqrt{-41}}{246}$
42	5	W_2, W_7	3	0	*none*
		W_3, W_6	2	4	*none*
		W_{21}, W_{42}			
		W_{14}	1	8	$\pm\frac{1}{3}+\frac{\sqrt{-14}}{42}, \pm\frac{1}{15}+\frac{\sqrt{-14}}{210}, \frac{2}{57}+\frac{\sqrt{-14}}{798}, -\frac{4}{75}+\frac{\sqrt{-14}}{1050}, -\frac{4}{225}+\frac{\sqrt{-14}}{3150}, -\frac{2}{3}+\frac{\sqrt{-14}}{42}$
43	3	W_{43}	1	4	*none*
44	4	W_4	2	2	*none*
		W_{11}	1	6	$\pm\frac{1}{12}+\frac{\sqrt{-11}}{132}, \pm\frac{1}{4}+\frac{\sqrt{-11}}{44}, -\frac{3}{4}+\frac{\sqrt{-11}}{44}, -\frac{3}{100}+\frac{\sqrt{-11}}{1100}$
		W_{44}	1	6	$\frac{\sqrt{-44}}{44}, \frac{52}{687}+\frac{\sqrt{-44}}{687}, \frac{2}{59}+\frac{\sqrt{-44}}{2596}, \frac{1}{15}+\frac{\sqrt{-44}}{660}, \frac{4}{141}+\frac{\sqrt{-44}}{6204}, \frac{1}{9}+\frac{\sqrt{-44}}{396}$
45	3	W_5, W_9, W_{45}	1	4	*none*
46	5	W_2	3	0	*none*
		W_{23}	0	12	$-\frac{1}{2}+\frac{\sqrt{-23}}{46}, \pm\frac{1}{4}+\frac{\sqrt{-23}}{92}, \pm\frac{1}{6}+\frac{\sqrt{-23}}{138}, \pm\frac{1}{8}+\frac{\sqrt{-23}}{184}, \pm\frac{3}{8}+\frac{\sqrt{-23}}{184}, \pm\frac{1}{12}+\frac{\sqrt{-23}}{276}, -\frac{1}{24}+\frac{\sqrt{-23}}{552}$
		W_{46}	2	4	*none*
47	4	W_{47}	0	10	$\frac{\sqrt{-47}}{47}, \frac{1}{2}+\frac{\sqrt{-47}}{94}, \pm\frac{1}{3}+\frac{\sqrt{-47}}{141}, \pm\frac{1}{4}+\frac{\sqrt{-47}}{188}, \pm\frac{1}{6}+\frac{\sqrt{-47}}{282}, \pm\frac{2}{7}+\frac{\sqrt{-47}}{329}$
48	3	W_3, W_{16}	2	0	*none*
		W_{48}	1	4	*none*
		$\begin{pmatrix} -6 & -1 \\ -48 & 6 \end{pmatrix}$	0	8	$\pm\frac{1}{8}+\frac{\sqrt{-3}}{24}, \pm\frac{3}{8}+\frac{\sqrt{-3}}{24}, \pm\frac{3}{56}+\frac{\sqrt{-3}}{168}, \pm\frac{17}{56}+\frac{\sqrt{-3}}{168}$
49	1		*none*		
50	2	W_2, W_{25}	1	2	*none*
		W_{50}	0	6	$\frac{\sqrt{-2}}{10}, \pm\frac{3}{11}+\frac{\sqrt{-2}}{110}, \pm\frac{1}{17}+\frac{\sqrt{-2}}{170}, -\frac{7}{43}+\frac{\sqrt{-2}}{430}$

Table 1.
Weierstrass points on $X_0(N)$ for $N \leq 50$ by W_Q

we find quadratic forms $[a, b, c]$ *which is* $SL_2(\mathbb{Z})$*-equivalent with* $\mathcal{Q}_{-152}^{red}$ *and* $gcd(a, 38) = 1$ *as follows (respectively):*

$$[a, b, c] = \{[1, 0, 38], [21, -38, 19], [3, 2, 13], [3, -2, 13], [7, 10, 9], [7, 18, 17]\}.$$

Applying Step III we have:

$$[1, 0, 38] = [1, 0, 38] \circ \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad [741, -76, 2] = [21, -38, 19] \circ \begin{pmatrix} 19 & -1 \\ 1 & 1 \end{pmatrix},$$
$$[1938, -152, 3] = [3, 2, 13] \circ \begin{pmatrix} 25 & -1 \\ 1 & 1 \end{pmatrix}, \quad [546, -80, 3] = [3, -2, 13] \circ \begin{pmatrix} 13 & -1 \\ 1 & 1 \end{pmatrix},$$
$$[2737, -256, 6] = [7, 10, 9] \circ \begin{pmatrix} 21 & -1 \\ 1 & 1 \end{pmatrix}, \quad [1297, -176, 6] = [7, 18, 17] \circ \begin{pmatrix} 15 & -1 \\ 1 & 1 \end{pmatrix}.$$

From Step IV, the Weierstrass points are (respectively):

$$\tau_1 = \frac{\sqrt{-38}}{38}, \qquad \tau_2 = \frac{2}{39} + \frac{\sqrt{-38}}{741}, \qquad \tau_3 = \frac{2}{51} + \frac{\sqrt{-38}}{1938},$$
$$\tau_4 = \frac{20}{273} + \frac{\sqrt{-38}}{546}, \qquad \tau_5 = \frac{128}{2737} + \frac{\sqrt{-38}}{2737}, \qquad \tau_6 = \frac{88}{1297} + \frac{\sqrt{-38}}{1297}.$$

We list in Table 1 Weierstrass points on $X_0(N)$ for $N \leq 50$ fixed by the Atkin–Lehner involutions. We have used Maple and Wolfram Mathematica for the numerical computations:

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