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So how to make group decisions? Arrow's impossibility theorem 70 years after

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Abstract

Purpose – In 1951, Kenneth Arrow proved that it is not possible to have a group decision-making procedure that satisfies reasonable requirements like fairness. From the theoretical viewpoint, this is a great result – well-deserving the Nobel Prize that was awarded to Professor Arrow. However, from the practical viewpoint, the question remains – so how should we make group decisions? A usual way to solve this problem is to provide some reasonable heuristic ideas, but the problem is that different seemingly reasonable idea often lead to different group decision – this is known, e.g. for different voting schemes.

Design/methodology/approach – In this paper we analyze this problem from the viewpoint of decision theory, the basic theory underlying all our activities – including economic ones.

Findings – We show how from the first-principles decision theory, we can extract explicit recommendations for group decision making.

Originality/value – Most of the resulting recommendations have been proposed earlier. The main novelty of this paper is that it provides a unified coherent narrative that leads from the fundamental first principles to practical recommendations.

Keywords Group decisions, Arrow's impossibility theorem, Utility

Paper type Research paper

1. Arrow's impossibility theorem: a brief reminder and the resulting challenge

1.1 What is Arrow's impossibility theorem

In 1951, Kenneth Arrow proved his famous result – for which he later received a Nobel Prize – that no voting system is perfect; see Arrow (1951). To be more precise, Arrow tried to find a voting system – i.e. an algorithm that transforms individual preferences into a resulting group preference – that would satisfy the following three seemingly natural conditions:

(1) if all voters prefer a to b, then the group should also prefer a to b;

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(2) if for some alternatives *a* and *b*, the voters' preference of *a* or *b* remains unchanged, then the group preference between *a* and *b* should also remain unchanged – even if voters' preferences about other pairs of alternatives change; and

(3) there is no dictator – i.e. there is no voter whose preferences are always reproduced as group preferences.

To his surprise – and to the surprise of the whole scientific community – instead of finding such "perfect" voting procedure, Arrow prove a theorem that such a perfect system is not possible. In other words, he proved that it is not possible to have a voting system that satisfies all three requirements.

1.2 Resulting challenge

There are many impossibility results in sciences – or, to be more precise, there are many theories that imply that some processes are not possible. Probably historically the first such principle was the energy conservation law, according to which we cannot get energy out of nothing, the perpetuum mobile – a machine that continues working without any fuel – is just not possible. Next was the second law of thermodynamics, which states, crudely speaking, that energy cannot pass from a colder to a hotter object: if we place a kettle on a cold stove, it will not boil. Twentieth century physics added to these examples:

(1) according to special relativity theory, we cannot travel with a speed faster than the speed of light;

(2) according to quantum physics, we cannot precisely measure both coordinates and momentum of a particle: if we measure coordinates, momentum changes, and if we measure momentum, the particle's coordinates change.

Engineers live within these limitations, these limitations help them avoid pursuing impossible tasks and concentrate on working devices. In all such cases, an impossibility result is a challenge but not a catastrophe.

But group decision making is a different issue. We *have* to make group decisions, we cannot just stop making them – and we want decisions to be fair, to satisfy some version of Arrow's assumptions. So what should we do?

A usual way to solve this problem is to provide some reasonable heuristic ideas, but the problem is that different seemingly reasonable idea often lead to different group decisions – this is known, e.g. for different voting schemes.

1.3 What we do in this paper

In this paper, we provide a brief overview of how, in spite of Arrow's result, we can make reasonable group decisions.

To be more precise, we analyze this problem from the viewpoint of decision theory, the basic theory underlying all our activities – including economic ones – and describe the corresponding recommendations. Most of the resulting recommendations have been proposed earlier. The main objective of this paper is to provide a unified coherent narrative that leads from the fundamental first principles to practical recommendations.

2. Natural idea: use more information about the individual preferences

2.1 Need to use additional information about individual preferences

In Arrow's theorem, all we know about each individual preference is which of the alternatives this individual prefers. For example, in the case of actual elections, an individual prefers Candidate 1, next best is Candidate 2, and finally, Candidate 3.

In practice, however, we have more information. For example, in many elections, we have two reasonable candidates and one extremist candidate. In this case, Candidate 2 may be slightly worse than Candidate 1, but electing Candidate 3 would be, in the view of many voters, a disaster. In this case, the difference between Candidates 1 and 2 is much smaller than the difference between each of these candidates and the awful Candidate 3.

On the other hand, it may happen that we have one reasonable candidate and two extreme candidates – an extreme leftist and an extreme rightist. In this case, for most voters, both Candidate 2 and Candidate 3 will be disasters, so the difference between Candidate 1 and any of the two extreme candidates is much larger than the difference between the two extreme folks.

In both cases, we have the exact same order of candidates: Candidate 1 is the best, Candidate 2 is the second best and Candidate 3 is the worst. Since Arrow's theorem assumes that this order is all we know, this theorem treats these two situations equally. But, of course, from the common sense viewpoint, these are completely different situations, in which we should make different decisions.

A natural idea is thus not to restrict ourselves to order between preferences, but to extract additional information from the individuals: not only *who* they prefer, but also *to what extent* they prefer the corresponding alternative.

2.2 How do we describe this additional information

How can we describe the strength of different preferences in precise terms? This is a question to which decision theory has a reasonable solution – a reasonable way to describe each individual's degree of preference of each alternative by a number; see, e.g. Fishburn (1969), Luce and Raiffa (1989), Raiffa (1997), Nguyen *et al.* (2009) and Kreinovich (2014). Let us describe this solution.

In general, to assign a numerical value to each situation (in our case, to each alternative), we need to have a continuous scale. For physical quantities, such scales are provided by the corresponding measuring instrument: a clock measures time, a ruler measures length, etc. It turns out that such a continuous scale is possible for comparing alternatives as well. Indeed, suppose that we want to select between several alternatives $A_1, \ldots, A_n$; later other alternatives may be added. We can also consider "ideal" alternatives:

(1) a very bad alternative A_- which is worse than any alternative A_i that we will actually encounter, and

(2) a very good alternative A_+ which is better than anything that we will actually encounter.

For example:

(1) we can imagine a candidate who is so bad that even the worst of folks on the ballot are angels in comparison, and

(2) we can also imagine a perfect candidate who will solve all our problems and make everyone prosper.

In economic decision making – e.g. in investing money – such "ideal" alternatives are easy to describe; for example:

(1) A_- means that we lose all the money, and

(2) A_+ means that our investment will grow 100 times in a year.

Now, for each number p from the interval [0, 1], we can consider a "mixed" alternative – we will denote it by $M(p)$ – in which someone flips several coins or performs any other random process, and then, depending on the result of this process:

(1) assigns us the very good alternative A_+ with probability p, and

(2) assigns us the very bad alternative A_- with the remaining probability $1 - p$.

Since the value p continuously changes from 0 to 1, the mixed alternatives $M(p)$ form a continuous scale.

Whatever alternatives people face, they always make some choice. So to find out how much an individual values an alternative A, let this individual compare it with mixed alternatives $M(p)$ corresponding to different values p.

(1) For some values p, the alternative A will be better; we will denote this by

$$M(p) < A.$$

(2) For some other values p, the mixed alternative $M(p)$ will be better:

$$A < M(p).$$

(3) Finally, for some values p, the individual may claim that A and $M(p)$ are equally good; we will denote this by

$$A \sim M(p).$$

For $p = 0$, the mixed alternative means that we get the very bad alternative A_- with probability 1. By our selection of the very bad alternative A_-, this alternative is much worse than any alternative A that we will actually encounter, so $M(0) = A_- < A$.

For $p = 1$, we get the very good alternative A_+ with probability 1. By our selection of the very good alternative A_+, this alternative is much better than any alternative A that we will actually encounter, so $A < A_+ = M(1)$.

As the probability p of a very good alternative A_+ increases, the mixed alternative becomes better and better: if $p < p'$, then $M(p) < M(p')$. Thus:

(1) if we had $A < M(p)$ and $p < p'$, then $A < M(p) < M(p')$ and hence, $A < M(p')$;

(2) similarly, if we had $M(p') < A$ and $p < p'$, then $M(p) < M(p') < A$ and thus, $M(p) < A$.

One can show that in this case, there exists the "threshold" value $u(A)$ – which is equal to $\sup\{p : M(p) < A\} = \inf\{p : A < M(p)\}$ – for which:

(1) for all $p < u(A)$, we have $M(p) < A$, and

(2) for all $p > u(A)$, we have $A < M(p)$.

This threshold value is known as the *utility* of the alternative A.

In particular, for any small $\varepsilon > 0$, we have

$$M(u(A) - \varepsilon) < A < M(u(A) + \varepsilon).$$

When the value ε is very small – e.g. equal to 0.001 or even to 0.0001 – no one will notice the corresponding difference in probabilities: 0.5 or 0.501? So, from the practical viewpoint, alternatives $M(u(A) - \varepsilon)$ and $M(u(A) + \varepsilon)$ are indistinguishable. Thus, the intermediate alternatives A and $M(u(A))$ are also practically indistinguishable. We will therefore say that the original alternative A is *equivalent* to the mixed alternative $M(u(A))$ and denote this by

$$A \equiv M(u(A)).$$

This way, to each alternative A, we assign a number $u(A)$ so that $A < B$ if and only if $u(A) < u(B)$. In other words, we get the desired continuous numerical scale for describing preferences.

2.3 Utility is not uniquely determined

The numerical value of utility depends on our choice is A_- and A_+. If we select a different pair (A'_-, A'_+), we will, in general, get different numerical utility values $u'(A)$. What is the relation between $u(A)$ and $u'(A)$?

Let us consider the case when $A_- < A'_- < A'_+ < A_+$. In this case, in the original scale – based on the original selection of A_- and A_+ – each of the new alternatives A'_- and A'_+ has a certain utility $u(A'_-)$ and $u(A'_+)$. Here, by definition of utility:

(1) A'_- is equivalent to a mixed alternative $M(u(A'_-))$ in which we get A_+ with probability $u(A'_-)$ and A_- with probability $1-u(A'_-)$; and

(2) A'_+ is equivalent to a mixed alternative $M(u(A'_+))$ in which we get A_+ with probability $u(A'_+)$ and A_- with probability $1-u(A'_+)$.

For each alternative A, the fact that it has utility $u'(A)$ in the new scale means that A is equivalent to a mixed alternative $M'(u'(A))$ in which:

(1) we get A'_+ with probability $u'(A)$, and

(2) we get A'_- with probability $1 - u'(A)$.

Since each A'_- and A'_+ is, itself, equivalent to a mixed event, thus A is equivalent to the following two-stage mixed alternative:

(1) first, we select A'_+ with probability $u'(A)$ and A'_- with the remaining probability $1 - u'(A)$;

(2) then, depending on which of the alternatives $A'_\pm$ we selected on the first stage, we select A_+ with probability $u(A'_\pm)$ and A_- with the remaining probability $1-u(A'_\pm)$.

In this two-stage scheme, we get either A_+ or A_-, and the probability of selecting A_+ is equal to

$$p = u'(A)\cdot u(A'_+) + (1 - u'(A))\cdot u(A'_-) = u(A'_-) + u'(A)\cdot(u(A'_+) - u(A'_-)).$$

The alternative A is thus equivalent to a mixed event $M(p)$ in which we get A_+ with probability p and A_- with the remaining probability. Thus, by definition of utility, the above value p is the utility $u(A)$ of the alternative A in the original scale, i.e.

$$u(A) = u(A'_-) + u'(A)\cdot(u(A'_+) - u(A'_-)).$$

So, utilities $u(A)$ and $u'(A)$ corresponding to different pairs (A_-, A_+) can be obtained from each other by a linear transformation.

In other words, utility – just like time or temperature – is defined modulo a linear transformation.

2.4 So how to make group decisions: analysis of the problem

For each alternative A, we know its utility $u_i(A)$ for each individual. So, each alternative is characterized by a tuple of real numbers $y = (y_1, y_2, \ldots)$, where $y_i = u_i(A)$ is the utility of this alternative for the i-th individual.

Based on these tuples, we need to decide which alternative is better for the group. In other words, for every two alternatives A and B, we need to decide, for the group as a whole:

(1) whether B is better (we will denote this by $A < B$),

(2) whether A is better ($B < A$),

(3) or whether, to the group, A and B are equivalent; we will denote this by

$$A \sim B.$$

In other words, we need to define a transitive and reflexive (i.e. $A \leq A$) relation $\leq$ on the set of all possible alternatives – i.e. in effect, on the set of all tuples y. Such relations are known as *pre-orders*. We are interested in *non-trivial* pre-orders – i.e. pre-orders for which $y \nsim y'$ for some y and y'.

It is also reasonable to require that the pre-order be *continuous* in the following sense: if $y^{(k)} \leq y'$ for all k and $y^{(k)} \to y$, then we should have $y \leq y'$. Indeed, the limit $y^{(k)} \to y$ means that for large k, the tuple $y^{(k)}$ is very close to y – and thus, practically indistinguishable from y. Since $y^{(k)}$ and y are indistinguishable, from $y^{(k)} \leq y'$, we conclude that $y \leq y'$.

Finally, we need to take into account that utilities are not uniquely defined: they are defined modulo linear transformations. Some of this non-uniqueness can be eliminated if we take into account that in group decision making, there is always an option not to do anything – not to build an airport if we cannot agree on its location, not to elect anyone if we cannot agree, etc. This do-nothing alternative is usually worse than anything on the ballot, so we can take it as the very bad alternative A_{-}. For this alternative A_{-}, its utility is equal to 0, for all other alternatives, it should be positive. Once we fix the 0 point, the only remaining linear transformations are re-scalings, when we replace each individual utility value $u_i(A)$ with a new value $u'_i(A) = c_i \cdot u_i(A)$ for some $c_i > 0$.

It is reasonable to require that the resulting relation between tuples should not change if we simply re-scale these values – just like fundamental physical formulas should not change if we simply change the unit of time. In other words, the relation $\leq$ should be *invariant* with respect to such changes, i.e. for all possible positive values y_i and c_i, we should have:

$$(y_1, y_2, \ldots) \leq (y'_1, y'_2, \ldots) \Leftrightarrow (c_1 \cdot y_1, c_2 \cdot y_2, \ldots) \leq (c_1 \cdot y'_1, c_2 \cdot y'_2, \ldots).$$

It turns out – see, e.g. Jaimes *et al.* (2012) – that each such invariant continuous non-trivial pre-order has the form

$$(y_1, y_2, \ldots) \leq (y'_1, y'_2, \ldots) \Leftrightarrow \prod_i y_i^{\alpha_i} \leq \prod_i (y'_i)^{\alpha_i},$$

for some values $\alpha_i > 0$.

If we additionally require that all individual are treated equally, i.e. that $\alpha_1 = \alpha_2 = \ldots$, then we conclude that

$$(y_1, y_2, \ldots) \leq (y'_1, y'_2, \ldots) \Leftrightarrow \left(\prod_i y_i\right)^{\alpha} \leq \left(\prod_i y'_i\right)^{\alpha},$$

i.e. equivalently, that

$$(y_1, y_2, \ldots) \leq (y'_1, y'_2, \ldots) \Leftrightarrow \prod_i y_i \leq \prod_i y'_i.$$

Thus, we need to select an alternative for which the product of utilities is the largest. This selection was first proposed by yet another Nobel Prize winner John Nash – see Nash (1953) – and is known as *Nash's bargaining solution*.

Let us summarize the resulting recommendation.

2.5 How to make group decisions: recommendation

(1) First, we take the no-decision situation as A_- and select some very good ideal alternative – much better than all available ones – as A_+.

(2) Then, for each individual i and for each alternative A, we ask the individual to compare the alternative A with mixed alternatives $M(p)$ in which:

- we get A_+ with probability p, and
- we get A_- with the remaining probability $1 - p$.

We increase the value p from 0 to 1 and find the value p for which A is equivalent to $M(p)$. This value p is the utility $u_i(A)$ of the alternative A for the i-th individual.

(3) Finally, we select the alternative A for which the product of utilities is the largest.

2.6 How does this fit with Arrow's conditions

Out of the three Arrow's conditions, the first and the third one are clearly satisfied. However, the second one is not – and, as will show, that this condition is not satisfied make sense. Indeed, we can imagine a situation when initially:

(1) half of the people strongly preferred A to B, while

(2) another half weakly preferred B.

In this case, from the common sense viewpoint, it makes sense to select A.

But suppose then that some new information changes everyone's opinion about A to the worse: e.g. a minor scandal is revealed about the candidate A. As a result:

(1) those who strongly preferred A continue to prefer A but only weakly, while

(2) those who weakly preferred B now prefer it strongly.

In this case, it is reasonable for the group to start selecting B – although, if we ignore which preferences are strong and which are weak (as we do in the original Arrow's formulation), the two situations look identical.

3. But what is we only know preferences – as in the original Arrow's setting?

In the previous section, we considered the case when we thoroughly analyze the opinions of each individual. This is not realistic to expect in national elections, when millions vote. In such situations, at best, we have individual preferences between candidates. How can we then make a decision?

Suppose that all we know for each individual is the order between the candidates. In terms of the utilities, this means that the only information that we have about the utilities u_j of different alternatives j is that $0 < u_1 < \ldots < u_n < 1$. We do not know the exact values u_j, so all the tuples $(u_1, \ldots, u_n)$ satisfying the above inequality are possible. We do not have any reason to believe that some of these tuples are more probable and some are less probable. So, it makes sense to require that all such tuples are equally probable, i.e. that we have a uniform distribution on the set of all the tuples that satisfy the above inequality.

To make a utility-based decision, we need to select *one* such tuple for each individual. It is reasonable to select a tuple $\bar{u}_i$ for which the mean square difference from the actual tuple is the smallest possible:

$$\int \sum_{j=1}^{n} (\bar{u}_j - u_j)^2 \, du_1 \ldots du_n \to \min.$$

It is known that, in general, this minimum is attained when each $\bar{u}_i$ is equal to the mean value of the corresponding random variable, and that this mean value is equal to $u_j = j/(n+1)$; see, e.g. David and Nagaraja (2003), Arnold *et al.* (2008), Ahsanullah *et al.* (2013), Kosheleva *et al.* (2016).

In this case, maximizing the product of utilities means selecting an alternative A for which maximizing the product of its ranks $j = r_i(A)$ is the largest – or, equivalently, for which the sum $\sum_i \log(r_i(A))$ of the logarithms of the ranks is the largest.

Similarly, we can handle situations in which, in addition to (or instead of) ranking, we have other information about the alternatives. For example, if each voter only ranks the top three candidates, this is equivalent to the case $n = 4$ and thus, leads to assigning:

(1) the mean utility value 4/5 to the top candidate,

(2) the value 3/5 to the second best,

(3) the value 2/5 to the third best and

(4) the value 1/5 to all other candidates.

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