Hamming Index for Some Classes of Graphs with Respect to Edge-Vertex Incidence Matrix

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Abstract
Hamming distance of a two bit strings u and v of length n is defined to be the number of positions of u and v with different digit. If G is a simple graph on n vertices and m edges and B is an edge–vertex incidence matrix of G, then every edge e of G can be labeled using a binary digit string of length n from the row of B which corresponds to the edge e. We discuss Hamming distance of two different edges of the graph G. Then, we present formulae for the sum of all Hamming distances between two different edges of G, particularly when G is a path, a cycle, and a wheel, and some composite graphs.

Keywords Simple graph, composite graph, incidence matrix, bit string, hamming distance, hamming index

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1. Introduction

Let $Z_2 = \{0, 1\}$. Notice that $\langle Z_2, +_2 \rangle$ is a group with binary operation addition modulo 2. A bit string x of length n can be thought as an element of the set $\overbrace{Z_2 \times Z_2 \times \cdots \times Z_2}^{n}$. Moreover, the set $\overbrace{Z_2 \times Z_2 \times \cdots \times Z_2}^{n}$ with binary operation defined by

$$\mathrm{x} \oplus_2 \mathrm{y} = (x_1 +_2 y_1, x_2 +_2 y_2, \ldots, x_n +_2 y_n)$$

is a group. For a bit string $\mathrm{x} \in Z_2 \times Z_2 \times \cdots \times Z_2$ the weight of x, denoted by $wt(\mathrm{x})$, is the number of 1 contained in x. For two bit strings $\mathrm{x}, \mathrm{y} \in Z_2 \times Z_2 \times \cdots \times Z_2$ the Hamming distance of x and y, denoted by $H_d(\mathrm{x}, \mathrm{y})$, is the number of positions in x and y with different digits. Hamming distance can also be defined by $H_d(\mathrm{x}, \mathrm{y}) = wt(\mathrm{x} \oplus_2 \mathrm{y})$ (Pless, 1998).

Let $G(V,E)$ be a simple graph on n vertices and m edges with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. If the edge $e = \{u, v\}$, we say that the edge e incident to vertices u and v. Two edges e_i and e_j are adjacent if they have one end vertex in common.

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An edge–vertex incidence matrix of a graph G is an $m \times n$ matrix $B = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} 1, & \text{if the edge } e_i \text{ incident to the vertex } v_j, \\ 0, & \text{if the edge } e_i \text{ not incident to the vertex } v_j. \end{cases}$$

Notice that every row of the edge–vertex incidence matrix of G is a bit string of length n and thus the edge–vertex incidence matrix contains m bit strings each of length n. For $i = 1, 2, \ldots, m$ we define $s(e_i) = B(i,:)$ where $B(i,:)$ is the ith row of B. For each two different edges e_i and e_j the Hamming distance between e_i and e_j is defined to be

$$H_d(s(e_i), s(e_j)) = H_d(B(i,:), B(j,:)) = wt(B(i,:) \oplus_2 B(j,:)).$$

The sum of Hamming distances of a graph G, denoted by $H(G)$, is defined as

$$\begin{aligned} H(G) &= \sum_{1 \le i<j \le m} H_d(s(e_i), s(e_j)) \\ &= \sum_{1 \le i<j \le m} H_d(B(i,:), B(j,:)) = \sum_{1 \le i<j \le m} wt(B(i,:) \oplus_2 B(j,:)). \end{aligned}$$

The sum of Hamming distances is also called the Hamming index.

The notion of Hamming distance and sum of Hamming distances of edges of simple graph G is initiated by Ramane *et al.* (2015). They discuss a general formula for Hamming index of a graph and apply this formula to get the Hamming index of classes of regular graph. Hamming distance and Hamming index of a graph can also be defined using the adjacency matrix of the graph (Ganagi and Ramane, 2016). In this paper, we discuss Hamming distance and sum of Hamming distances of simple graph with respect to its edge–vertex incidence matrix. We especially discuss a way of finding the Hamming index of a graph from the known Hamming index of its subgraph. We first present formulae for Hamming index of some graphs consist of a path or a cycle as its subgraph. We then present a formula for Hamming index of some composite graphs.

2. Necessary background

Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. For each vertex v_i the degree of v_i, denoted by $\deg(v_i)$, is the number of edges in G that are incident to v_i. It is well known that $\sum_{i=1}^{n} \deg(v_i) = 2m$ (Korte and Vygen, 2006).

We note that by definition, the Hamming index of a graph can be calculated using the formula $H(G) = \sum_{1 \le i<j \le m} wt(B(i,:) \oplus_2 B(j,:))$. Employing this formula, the Hamming index of a certain graph can be determined using the following algorithm (Figure 1).

The following lemma, due to Ramane et al. (2015), presents a formula for Hamming distance between two edges of a simple graph. We present a simpler proof than in Ramane *et al.* (2015).

Lemma 2. Let G be a simple graph. Then

$$H_d(s(e_i), s(e_j)) = \begin{cases} 2, & \text{if } e_i \text{ adjacent to } e_j, \\ 4, & \text{if } e_i \text{ not adjacent to } e_j. \end{cases}$$

Proof. Since every edge of G is incident to exactly two vertices, then for each $i = 1, 2, \ldots, m$, we have $wt(s(e_i)) = 2$. Let $s(e_i) = x_1x_2x_3 \cdots x_n$ and $s(e_j) = y_1y_2y_3 \cdots y_n$. If edge e_i is adjacent to e_j, then there is a common vertex, say v_k, of edges e_i and e_j. This implies $x_k = y_k = 1$, and if $t \neq k$ then $x_t = y_t = 0$ or $x_t \neq y_t$. Therefore, if the edge e_i is adjacent to the edge e_j,

$$H_d(s(e_i), s(e_j)) = wt(s(e_i)) + w(s(e_j)) - 2 = 2.$$

If the edge e_i is not adjacent to the edge e_j, then there is no common vertex v_k of e_i and e_j. This implies for each $t = 1, 2, \ldots, n, x_t = y_t = 0$ or $x_t \neq y_t$. Therefore, if the edge e_i is not adjacent to the edge e_j, we have

$$H_d(s(e_i), s(e_j)) = wt(s(e_i)) + w(s(e_j)) = 4. \qquad \square$$

Theorem 3. Let G a simple graph on m edges. If there are *n(P)* adjacent pairs of edges in G, then $H(G) = 2m(m-1) - 2n(P)$.

Proof. By Lemma 2 we have $H(G) = 2n(P) + 4n(Q)$, where $n(Q)$ is the number of pairs of non-adjacent edges in G. Since the graph G contains m edges, then we have $n(Q) = m(m-1)/2 - n(P)$. Therefore, $H(G) = 2m(m-1) - 2n(P)$. $\square$

Proposition 4. Let P_n a path on $n \geq 3$ vertices. Then $H(P_n) = 2(n-2)^2$.

Proof. Notice that $E(P_n) = \{e_i = \{v_i, v_{i+1}\} : i = 1, 2, \ldots, n-1\}$. Therefore, there are $n(P) = n - 2$ pairs of adjacent edges e_i and e_{i+1}. Since P_n has $m = n - 1$ edges, then by Theorem 3 we have

$$H(P_n) = 2m(m-1) - 2n(P) = 2(n-1)(n-2) - 2(n-2) = 2(n-2)^2. \qquad \square$$

Proposition 5. Let C_n be a cycle on n vertices. Then $H(C_n) = 2n(n-2)$.

Proof. There are n pairs of adjacent edges e_i and e_{i+1} (we note that $e_{n+1} = e_1$), $i = 1, 2, \ldots, n$. Hence $n(P) = n$. By Theorem 3 we have

$$H(C_n) = 2m(m-1) - 2n(P) = 2n(n-1) - 2n = 2n(n-2). \qquad \square$$

Algorithm 1: Sum of Hamming Distances

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Input: The edge-vertex incidence matrix B
Output: Sum of Hamming distances H_B(G)
H_B(G) = 0;
for i = 1 to m – 1
    for j = i + 1 to m
        H_d(ij) = wt(B(i,:) ⊕2 B(j,:))
        H_B(G) = H_B(G) + H_d(ij)
    end
end
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Figure 1.
Algorithm for Sum of Hamming Distances

3. Main results

Let G be a graph and F be a subgraph of G. We discuss the Hamming index of the graph G in term of the Hamming index of its subgraph F. We note that

$$\begin{aligned} H(G) &= \sum_{e_i,e_j\in E(G)} H_d(s(e_i),s(e_j)) \\ &= \sum_{e_i,e_j\in E(F)} H_d(s(e_i),s(e_j)) + \sum_{e_i\in E(F),e_j\notin E(F)} H_d(s(e_i),s(e_j)) + \sum_{e_i,e_j\notin E(F)} H_d(s(e_i),s(e_j)) \\ &= H(F) + \sum_{e_i\in E(F),e_j\notin E(F)} H_d(s(e_i),s(e_j)) + \sum_{e_i,e_j\notin E(F)} H_d(s(e_i),s(e_j)). \end{aligned}$$

A fan $F_{1,n}$ is a graph on $n+1$ vertices and $2n-1$ edges, where its vertex set is $V(F_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$ and its edge set is

$$E(F_{1,n}) = \{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, n-1\} \cup \{\{v_i, v_{n+1}\} : i = 1, 2, \ldots, n\}.$$

Theorem 6. For a fan $F_{1,n}$, we have $H(F_{1,n}) = 7n^2 - 17n + 12$.fo

Proof. Notice that a fan $F_{1,n}$ contains the path P_n as its subgraph where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, n-1\}$. From Proposition 4, we have $H(P_n) = 2(n-2)^2 = 2n^2 - 8n + 8$.

There are $n(n-1)$ pairs of edges e_i and e_j for which $e_i \in P_n$ and $e_j \notin P_n$. Among them there are $2(n-1)$ adjacent edges. This implies

$$\sum_{e_i\in P_n,e_j\notin P_n} H_d(s(e_i),s(e_j)) = 2(2(n-1)) + 4(n^2-3n+2) = 4n^2 - 8n + 4.$$

There are $n(n-1)/2$ pairs of edges e_i and e_j for which $e_i \notin P_n$ and $e_j \notin P_n$. All of them are adjacent pairs. This implies

$$\sum_{e_i\notin P_n,e_j\notin P_n} H_d(s(e_i),s(e_j)) = 2(n(n-1)/2) = n^2 - n.$$

We now conclude that

$$\begin{aligned} H(F_{1,n}) &= H(P_n) + \sum_{e_i\in P_n,e_j\notin P_n} H_d(s(e_i),s(e_j)) + \sum_{e_i,e_j\notin P_n} H_d(s(e_i),s(e_j)) \\ &= (2n^2-8n+8) + (4n^2-8n+4) + (n^2-n) = 7n^2 - 17n + 12. \end{aligned}$$

An n-wheel W_n is a graph on $n+1$ vertices and $2n$ edges with vertex set $V(W_n) = \{v_1, v_2, \ldots, v_{n+1}\}$ and edge set $E(W_n) = \{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, n-1\} \cup \{\{v_n, v_1\}\} \cup \{\{v_i, v_{i+n}\} : i = 1, 2, \ldots, n\}$. □

Theorem 7. If $n \geq 2$, then $H(W_n) = 7n^2 - 9n$.

Proof. We note that the cycle C_n is a subgraph of W_n. Proposition 5 guarantees that $H(C_n) = 2n^2 - 4n$. There are n^2 pairs of edges e_i and e_j for which $e_i \in C_n$ and $e_j \notin C_n$. Among them there are $2n$ pairs of adjacent edges. This implies

$$\sum_{e_i \in C_n, e_j \notin C_n} H_d(s(e_i), s(e_j)) = 2(2n) + 4(n^2 - 2n) = 4n^2 - 4n.$$

There are $n(n-1)/2$ pairs of edges e_i and e_j for which $e_i, e_j \notin W_n$. All of them are adjacent pairs. This implies

$$\sum_{e_i, e_j \notin W_n} H_d(s(e_i), s(e_j)) = 2(n(n-1)/2) = n^2 - n.$$

We now conclude that

$$\begin{aligned} H(W_n) &= H(C_n) + \sum_{e_i \in C_n, e_j \notin C_n} H_d(s(e_i), s(e_j)) + \sum_{e_i, e_j \notin C_n} H_d(s(e_i), s(e_j)) \\ &= (2n^2 - 4n) + (4n^2 - 4n + (n^2 - n) = 7n^2 - 9n. \end{aligned}$$

An n-Sheep Steering Wheel, denoted by SSW_n, is a graph on $2n+1$ vertices and $3n$ edges such that its vertex set is $V(SSW_n) = V(W_n) \cup \{v_{n+2}, v_{n+3}, \ldots, v_{2n+1}\}$ and its edge set is $E(SSW_n) = E(W_n) \cup \{\{v_i, v_{n+1+i}\} : i = 1, 2, \ldots, n\}$. □

Theorem 8. If $n \geq 3$, then $H(SSW_n) = 17n(n-1)$.

Proof. Notice that an n-wheel W_n is a subgraph of SSW_n. If the pair of edges e_i and e_j are both in W_n, then by Theorem 7

$$\sum_{e_i, e_j \in E(W_n)} H_d(s(e_i), s(e_j)) = 7n^2 - 9n.$$

Let the pair of edges e_i and e_j are such that e_i is in W_n and e_j is not in W_n. There are $3n$ adjacent pairs of such edges. There are $n(2n-3)$ non-adjacent pairs of such edges. Therefore

$$\sum_{e_i \in E(W_n), e_j \notin E(W_n)} H_d(s(e_i), s(e_j)) = 2(3n) + 4(n(2n-3)) = 8n^2 - 6n.$$

There are $n(n-1)/2$ pairs of non-adjacent edgese_i and e_j such that both of them are not in W_n. There are no pairs of adjacent edges e_i and e_j such that both of them are not in W_n. Therefore

$$\sum_{e_i, e_j \notin E(W_n)} H_d(s(e_i), s(e_j)) = 2n(n-1).$$

We now conclude that

$$\begin{aligned} H(SSW_n) = \sum_{e_i, e_j \in E(SSW_n)} H_d(s(e_i), s(e_j)) &= 7n^2 - 9n + 8n^2 - 6n + 2n^2 - 2n \\ &= 17n(n-1). \end{aligned}$$

An (*n*,*m*)-Jahangir graph, denoted by $J_{n,m}$, is a graph on $nm+1$ vertices and $m(n+1)$ edges such that its vertex set is $V(J_{n,m}) = V(C_{mn}) \cup \{v_{mn+1}\}$ and its edge set is

$$E(J_{n,m}) = E(C_{mn}) \cup \{\{v_{(j-1)n+1}, v_{mn+1}\} : j = 1, 2, \ldots, m\}. \qquad \square$$

Theorem 9. If $J_{n,m}$ is a (*n*,*m*)-Jahangir graph, then $H(J_{n,m}) = (2nm + 4m)(nm - 2) + m(m + 3)$.

Proof. Notice that the cycle C_{nm} is a subgraph of the Jahangir graph $J_{n,m}$. If the pair of edges e_i and e_j are both in C_{nm}, then by Proposition 5 we have

$$\sum_{e_i, e_j \in C_{nm}} H_d(s(e_i), s(e_j)) = H_B(C_{nm}) = 2nm(nm - 2).$$

If the pair of edges e_i and e_j is such that e_i is in C_{nm} and e_j is not in C_{nm}, then there are $2m$ pairs of edges that are adjacent and there are $m(nm-2)$ pairs of edges which are not adjacent. Therefore

$$\sum_{v_i \in V(C_{nm}), v_j \notin V(C_{nm})} H_d(s(e_i), s(e_j) = 2(2m) + 4m(nm - 2).$$

If both edges e_i and e_j are not in C_{nm}, then there are $m(m-1)/2$ pairs of edges which are adjacent and there are no pairs of edges which are not adjacent. Therefore

$$\sum_{v_i, v_j \notin V(C_{nm})} H_d(s(e_i), s(e_j)) = m(m - 1).$$

We now conclude that

$$\begin{aligned} H_B(J_{n,m}) &= 2nm(nm - 2) + 4m + 4m(nm - 2) + m(m - 1) \\ &= (2nm + 4m)(nm - 2) + m(m + 3). \end{aligned}$$

We now discuss the hamming index of two classes of composite graphs. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are two graphs. The graph $G_1 \cup G_2$ is a graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup V(G_2)$. $\square$

Theorem 10. Let G_1 be a graph on m_1 edges and G_2 be a graph on m_2 edges. If $V(G_1) \cap V(G_2) = \varnothing$, then $H(G_1 \cup G_2) = H(G_1) + H(G_2) + 4m_1m_2$.

Proof. If pairs of edges e_i and e_j are both G_1, then

$$\sum_{e_i, e_j \in E(G_1)} H_d(s(e_i), s(e_j)) = H(G_1).$$

If pairs of edges e_i and e_j are both on G_2, then

$$\sum_{e_i, e_j \in E(G_2)} H_d(s(e_i), s(e_j)) = H(G_2).$$

If e_i is on G_1 and e_j is on G_2, then e_i and e_j are not adjacent. Since there are m_1m_2 pairs of such edges, by Lemma 3 we have

$$\sum_{e_i\in E(G_1),e_j\in E(G_2)} H_d(s(e_i),s(e_j)) = 4m_1m_2.$$

Therefore, $H(G_1 \cup G_2) = H(G_1) + H(G_2) + 4m_1m_2$.

Let G be a graph on n vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ and m edges $E(G)$. Define the thorn graph G^+ of G to be a graph on $2n$ vertices and $m+n$ edges such that

$$V(G^+) = V(G) \cup \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$$

and

$$E(G^+) = E(G) \cup \{\{v_i, v_{i+n}\} : i = 1, 2, \ldots, n\}. \qquad \square$$

Theorem 11. Let G be a graph on n vertices and m edges, then $H(G^+) = H(G) + 4mn + 2n^2 - 2n - 4m$.

Proof. If the pair of edges e_i and e_j are both lie on $E(G)$, then by definition

$$\sum_{e_i,e_j\in E(G)} H_d(s(e_i),s(e_j)) = H(G).$$

There are $n(n-1)/2$ pairs of edges e_i and e_j where both of them are not on $E(G)$. Since e_i and e_j are not adjacent, then

$$\sum_{e_i,e_j\notin E(G)} H_d(s(e_i),s(e_j)) = 2n(n-1).$$

We next consider the pairs of edges e_i and e_j where $e_i \in E(G)$ and $e_j \notin E(G)$. If the edge e_j is incident to some vertex v_k then e_j is adjacent to $\deg(v_k)$ edges in $E(G)$ and e_j is not adjacent to $(m - \deg(v_k))$ edges in $E(G)$. This implies there are $\sum_{k=1}^{n}\deg(v_k) = 2m$ adjacent pairs of edges and there are $\sum_{k=1}^{n}(m - \deg(v_k)) = nm - 2m$ pairs of non-adjacent edges. This implies

$$\sum_{e_i\in E(G),e_j\notin E(G)} H_d(s(e_i),s(e_j)) = 2(2m) + 4(nm - 2m) = 4nm - 4m.$$

Thus $H(G^+) = H(G) + 4nm + 2n^2 - 2n - 4m$.

For $n \geq 3$, an n-Sun, denoted by S_n, is a graph on $2n$ vertices and $2n$ edges with $V(S_n) = \{v_1, v_2, \ldots, v_{2n}\}$ and $E(S_n) = \{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, n, v_{n+1} = v_1\} \cup \{\{v_i, v_{i+n}\} : i = 1, 2, \ldots, n\}$. $\square$

Corollary 12. If $n \geq 3$, then $H(S_n) = 2n(4n-5)$.

Proof. Notice that $S_n = C_n^+$. Proposition 5 guarantees that $H(C_n) = 2n(n-2)$. Since a cycle C_n has $m = n$ edges, by Theorem 11 we have

$$\begin{aligned} H(S_n) &= H(C_n) + 4mn + 2n^2 - 2n - 4m \\ &= 2n(n-2) + 4n^2 + 2n^2 - 2n - 4n = 2n(4n-5). \end{aligned}$$ □

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