Warped product pointwise semi-slant submanifolds of cosymplectic space forms and their applications

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Abstract
In this paper some characterizations for the existence of warped product pointwise semi-slant submanifolds of cosymplectic space forms are obtained. Moreover, a sharp estimate for the squared norm of the second fundamental form is investigated, the equality case is also discussed. By the application of derived inequality, we compute an expression for Dirichlet energy of the involved warping function. Finally, we also proved some classifications for these warped product submanifolds in terms of Ricci solitons and Ricci curvature. A non-trivial example of these warped product submanifolds is provided.

Keywords Semi-slant product, Pointwise slant submanifolds, Cosymplectic space form, Dirichlet energy, Ricci soliton, Ricci curvature

Paper type Original Article

1. Introduction
The study of warped product manifolds has been a favourite topic in the field of geometry due to its applications in Physics and relativistic theories [1]. Many basic solutions to Einstein field equations are given by warped products [1]. The concept of modelling of space–time near black holes uses the idea of warped product manifolds [2]. Schwartzschild space–time is an example of warped product $P \times_r S^2$ where the base $P = R \times R^+$ is a half plane $r > 0$ and fibre S^2 is the unit sphere. Under certain conditions, the Schwartzschild space–time becomes black hole. A cosmological model to model the universe as a space–time known as Robertson–Walker model is a warped product [3].

JEL Classification — 53C25, 53C40, 53C42, 53D15

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One of the important task in Physics and Engineering is to find the Dirichlet energy of smooth functions. Dirichlet energy is analogous to Kinetic energy. On a compact manifold M, the Dirichlet energy of any smooth function $\lambda : M \rightarrow R$ is given by

$$E(\lambda) = \frac{1}{2}\int_M \|\nabla\lambda\|^2 d\,V,$$

where $\nabla\lambda$ is the gradient of λ and dV is the volume element. It is obvious that $E(\lambda) \geq 0$ for any smooth function λ. We know that the manifolds of non-zero (constant) curvature cannot be represented as a product manifold. So, considering the fact that a Riemannian product of manifolds cannot has negative curvature, the idea of warped product of manifolds came into existence. To construct the class of manifolds of negative or non-positive curvature, R. L. Bishop and B. O'Neill [4] introduced this idea of warped product manifolds. Warped product manifolds (see definition in Section 2) are a generalized setting of product manifolds. Since warping functions of the warped product manifolds are positive valued smooth functions, our interest is to find the Dirichlet energy of these functions.

Some intrinsic properties of warped product manifolds were studied in [4]. Initial extrinsic studies of warped product manifolds in the almost complex setting were performed by B. Y. Chen [5,6] while obtaining some existence results for CR-submanifolds to be CR-warped product submanifolds in Kaehler manifolds. On the other hand, in the almost contact settings contact CR-warped product submanifolds were explored by Hasegawa et al. [7]. Many other geometers have also explored warped product manifolds in contact settings and various existence results have been obtained [8–11].

Warped product pointwise semi-slant submanifold is another generalized class of warped product semi-slant submanifolds and contact CR-warped product submanifolds. In [12], Park studied the warped product pointwise semi-slant warped product submanifolds. After that, Ali and Ozel [13] extended this study in the setting of cosymplectic manifolds and they obtained some optimal inequalities related to the second fundamental form and warping function. Warped product pointwise semi-slant submanifolds for almost contact and almost complex manifolds were explored in (see [14–16]).

On the other hand the Gradient Ricci soliton are extensively investigated in the theory of relativity, physics as well as in the differential geometry. The classification results related to Ricci Soliton and Gradient Ricci solitons with the warped product structure have been established in [17–20]. Moreover, the Ricci curvature has a significant nature in Riemannian geometry, for example Ricci flat is a solution of Einstein field equation on a Riemannian manifold in which cosmological constant vanishes. More clearly, in the theory of general relativity the Ricci tensor is correlated with Einstein's field equation to study the material contents of universe. So, in comparison with Riemannian curvature, the Ricci curvature is more significant in the theory of relativity and physics.

In the present article, we study warped product pointwise semi-slant submanifolds of cosymplectic space forms and obtain some interesting inequalities for warped product pointwise semi-slant submanifolds. We estimate the squared norm of the second fundamental form in terms of warping function and slant function. The equality case is discussed accordingly. We explored some applications of equality case of the derived inequality. More precisely, we calculate Dirichlet energy of the warping functions by using our derived inequality. Finally, we obtain the classification of the warped product pointwise semi-slant submanifolds admitting the gradient Ricci soliton, in terms of Ricci curvature and second fundamental form, some existence results are also established.

The paper is organized as follows: Section 2 is devoted to basic definitions, formulae and preliminary results which are required for the subsequent study of the paper. In Section 3, we explore the existence of warped product pointwise semi-slant submanifolds in cosymplectic space forms and prove our main results. A non-trivial example is given for the warped product pointwise semi-slant submanifolds in a cosymplectic manifold. Using the derived inequality, formulae for Dirichlet energy of warping function is obtained in Section 4. Conclusions are presented in Section 5. Bibliography is given at the end of the paper.

2. Preliminaries

A $(2n+1)$-dimensional C^∞-manifold $\bar{M}$ is said to have an *almost contact structure* if on $\bar{M}$ there exist a tensor field Ψ of type $(1,1)$, a vector field η and a 1-form ζ satisfying the following properties [21]

$$\Psi^2 = -I + \zeta\eta\otimes\eta, \quad \Psi\eta = 0, \quad \zeta\circ\Psi = 0, \quad \zeta(\eta) = 1. \tag{2.1}$$

The manifold $\bar{M}$ with the structure (Ψ,η,ζ) is called *almost contact metric manifold*. There exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$, satisfying the following

$$\zeta(Y) = g(Y,\eta), \quad g(\Psi Y,\Psi V) = g(Y,V) - \zeta(Y)\zeta(V), \tag{2.2}$$

for all $Y, V \in T\bar{M}$ where $T\bar{M}$ is the tangent bundle of $\bar{M}$.

An almost contact metric structure (Ψ,η,ζ,g) is said to be *cosymplectic manifold* if it satisfies the following tensorial equation [21]

$$(\bar{\nabla}_Y\Psi)V = 0, \tag{2.3}$$

for any $Y, V \in T\bar{M}$, where $\bar{\nabla}$ denotes the Riemannian connection of the metric g. Moreover, for a cosymplectic manifold

$$\bar{\nabla}_Y\eta = 0. \tag{2.4}$$

A cosymplectic manifold $\bar{M}$ is said to be a *cosymplectic space form* [21] if it has constant Ψ-holomorphic sectional curvature c and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of cosymplectic space form $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(Y_1,Y_2)U = \frac{c}{4}\{&g(Y_2,V)Y_1 - g(Y_1,V)Y_2 + g(Y_1,\Psi V)\Psi Y_2, \\ &- g(Y_2, \Psi V)\Psi Y_1 + 2g(Y_1,\Psi Y_2)\Psi V + \zeta(Y_1)\zeta(V)Y_2 \\ &- \zeta(Y_2)\zeta(V)Y_1 + g(Y_1,V)\zeta(Y_2)\eta - g(Y_2,V)\zeta(Y_1)\eta\}\end{aligned} \tag{2.5}$$

for any vector fields Y_1, Y_2, V on $\bar{M}$.

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g. The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connections ∇ and $\nabla^\perp$ on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + \sigma(Y_1,Y_2), \tag{2.6}$$

$$\bar{\nabla}_{Y_1}\xi = -A_\xi Y_1 + \nabla^\perp_{Y_1}\xi, \tag{2.7}$$

for each $Y_1, Y_2 \in TM$ and $\xi \in T^\perp M$, where σ and A_ξ are the second fundamental form and the shape operator respectively for the immersion of M into $\bar{M}$, they verify the relation

$$g(\sigma(Y_1,Y_2),\xi) = g(A_\xi Y_1, Y_2), \tag{2.8}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced metric on M.

For a submanifold M of a Riemannian manifold $\bar{M}$, the equation of Codazzi is given by

$$(\bar{R}(Y_1, Y_2)V)^{\perp} = \nabla^{\perp}_{Y_1}\sigma(Y_1, V) - \nabla^{\perp}_{Y_2}\sigma(Y_1, V) + \sigma(\nabla_{Y_2}Y_1, V) - \sigma(\nabla_{Y_1}Y_2, V) + \sigma(Y_1, \nabla_{Y_2}V) - \sigma(Y_2, \nabla_{Y_1}V)$$

where $(\bar{R}(Y_1, Y_2)V)^{\perp}$ denotes the normal component of the curvature tensor $\bar{R}(Y_1, Y_2)V$.

If PY and FY denote the tangential and normal component of ΨY respectively for any $Y \in TM$, we can write

$$\Psi Y = PY + FY. \tag{2.9}$$

Similarly, for any $\xi \in T^{\perp}M$, we write

$$\Psi\xi = t\xi + f\xi, \tag{2.10}$$

where $t\xi$ and $f\xi$ are the tangential and normal components of $\Psi\xi$ respectively. Thus P (resp. f) is 1–1 tensor field on TM (resp. $T^{\perp}M$) and t (resp. F) is a tangential (resp. normal) valued 1-form on $T^{\perp}M$ (resp. TM). The covariant derivatives of the tensor fields Ψ, P and F are defined as

$$(\bar{\nabla}_{Y_1}\Psi)Y_2 = \nabla_{Y_1}\Psi Y_2 - \Psi\nabla_{Y_1}Y_2 \tag{2.11}$$

$$(\bar{\nabla}_{Y_1}P)Y_2 = \nabla_{Y_1}PY_2 - P\nabla_{Y_1}Y_2, \tag{2.12}$$

$$(\bar{\nabla}_{Y_1}F)Y_2 = \nabla^{\perp}_{Y_1}FY_2 - F\nabla_{Y_1}Y_2. \tag{2.13}$$

From Eqs. (2.3), (2.6), (2.7), (2.9) and (2.10), we have

$$(\bar{\nabla}_{Y_1}P)Y_2 = A_{FY_2}Y_1 + t\sigma(Y_1, Y_2) - g(Y_1, Y_2)\eta \tag{2.14}$$

$$(\bar{\nabla}_{Y_1}F)Y_2 = f\sigma(Y_1, Y_2) - \sigma(Y_1, PY_2). \tag{2.15}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{m}\sum_{s=1}^{m}\sigma(e_s, e_s),$$

where m is the dimension of M and $\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame of vector fields on M. The squared norm of the second fundamental form σ is defined as

$$\|\sigma\|^2 = \sum_{r,s=1}^{m} g(\sigma(e_r, e_s), \sigma(e_r, e_s)). \tag{2.16}$$

A submanifold M of $\bar{M}$ is said to be a *totally geodesic submanifold* if $\sigma(Y_1, Y_2) = 0$ and *totally umbilical submanifold* if $\sigma(Y_1, Y_2) = g(Y_1, Y_2)H$, for each $Y_1, Y_2 \in TM$.

Definition. *([22])*. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be *slant submanifold* if for any $y \in M$ and $Y_1 \in T_yM - \langle\eta\rangle$, the angle between Y_1 and ΨY_1 is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of M in $\bar{M}$. If $\theta = 0$, the submanifold is *invariant submanifold* and if $\theta = \pi/2$ then it is *anti-invariant submanifold*. If $\theta \neq 0, \pi/2$, it is *proper slant submanifold*.

For slant submanifolds of the contact metric manifolds J. L. Cabrerizo et al. [22] proved the following lemma.

Lemma 2.1. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\eta \in TM$, then M is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(I - \zeta \otimes \eta), \tag{2.17}$$

where $\lambda = -\cos^2\theta$.

The notion of pointwise slant submanifolds was introduced by F. Etayo [23] as a natural generalization of the slant submanifolds in the setting of almost Hermitian manifolds. Later, B. Y. Chen and O. J. Garay [24] investigated pointwise slant submanifolds for almost Hermitian manifolds and obtained some fundamental results. A step forward, K. S. Park [12] extended the concept of pointwise slant submanifolds in almost contact metric manifolds. Recently, Uddin and Al-Khalidi [25] modified the definition of pointwise slant submanifolds for almost contact metric manifolds. More precisely, a submanifold M of an almost contact metric manifold $\bar{M}$ is said to be pointwise slant submanifold if for any $X \in T_xM$ such that η is tangential to M, the angle $\theta(X)$ between ΨX and $T_xM - \{0\}$ is independent of the choice of non zero vector field $X \in T_pM - \{0\}$. In this case θ is treated as the function on M, which is called the slant function of the point wise slant function. Now, we have the following characterizing theorem

Theorem 2.2. *([25]). Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\eta \in TM$. Then, M is pointwise slant if and only if*

$$P^2 = \cos^2\theta(-I + \zeta \otimes \eta) \tag{2.18}$$

for some real valued function θ on TM

Thus, one has the following consequences of the above formula.

$$g(PY_1, PY_2) = \cos^2\theta[g(Y_1, Y_2) - \zeta(Y_1)\zeta(Y_2)], \tag{2.19}$$

$$g(FY_1, FY_2) = \sin^2\theta[g(Y_1, Y_2) - \zeta(Y_1)\zeta(Y_2)] \tag{2.20}$$

for all $Y_1, Y_2 \in TM$.

Pointwise semi-slant submanifolds were defined and studied by Park [12] as a natural generalization of contact CR-submanifolds in terms of slant function. Now, we have the following definition

Definition. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a pointwise semi-slant submanifold if there exist two orthogonal complementary distributions D and D^θ on M such that

(i) The tangent bundle TM can be written as $TM = D \oplus D^\theta \langle \eta \rangle$,

(ii) The distribution D is invariant,

(iii) The distribution D^θ is pointwise slant with a slant function θ.

As a generalization of the product manifolds, one can consider the warped product of manifolds which are defined as follows.

Let (C_1, g_{C_1}) and (C_2, g_{C_2}) be the two Riemannian manifolds with g_{C_1} and g_{C_2} as their Riemannian metrics resp. and Φ be a positive differentiable function on C_1. Let $\pi_1 : C_1 \times C_2 \to C_1$, $\pi_2 : C_1 \times C_2 \to C_2$ are the projection maps given by $\pi_{C_1}(c_1, c_2) = c_1$ and $\pi_{C_2}(c_1, c_2) = c_2$ for every $(c_1, c_2) \in C_1 \times C_2$. The warped product $M = C_1 \times_\Phi C_2$ [4] is the manifold $C_1 \times C_2$ equipped with the Riemannian structure such that

$$g(Y_1, Y_2) = g_1(\pi_{1*}Y_1, \pi_{1*}Y_2) + (\Phi \circ \pi_1)^2 g_2(\pi_{2*}Y_1, \pi_{2*}Y_2),$$

for all $Y_1, Y_2 \in TM$, where π_{i*} denotes the tangent map corresponding to π_i for each i. The function Φ is called the *warping function* of the warped product manifold. If the warping function is constant then the warped product manifold M is said to be *trivial warped product*.

Let Y_1 be a vector field on C_1 and Y_2 be a vector field on C_2, then from Lemma 7.3 of [3], we have

$$\nabla_{Y_1} Y_2 = \nabla_{Y_2} \left(\frac{Y_1 \Phi}{\Phi} \right) Y_2 \tag{2.21}$$

where ∇ is the Levi-Civita connection on M. For a warped product $M = C_1 \times_{\Phi} C_2$ it is easy to observe that

$$\nabla_{Y_1} Y_2 = \nabla_{Y_2} Y_1 = (Y_1 ln\, \Phi) Y_2 \tag{2.22}$$

for $Y_1 \in TC_1$ and $Y_2 \in TC_2$.

$\nabla\Phi$ is the gradient of Φ and is defined as

$$g(\nabla\Phi, Y) = Y\Phi, \tag{2.23}$$

for all $Y \in TM$.

Let M be an m-dimensional Riemannian manifold with the Riemannian metric g and let $\{e_1, e_2, \ldots, e_m\}$ be an orthogonal basis of TM. Then as a result of (2.23), we get

$$\|\nabla\Phi\|^2 = \sum_{i=1}^{m} (e_i(\Phi))^2. \tag{2.24}$$

The Laplacian of Φ is defined by

$$\Delta\Phi = \sum_{i=1}^{m} \left\{ \left(\nabla_{e_i} e_i \right) \Phi - e_i e_i \Phi \right\}. \tag{2.25}$$

The Hessian tensor for a differentiable function Φ is symmetric covariant tensor of rank 2 and is defined as

$$\Delta\Phi = -traceH^{\Phi} \tag{2.26}$$

or we can also write

$$Hess^{\Phi} = -\Delta\Phi \tag{2.27}$$

Now, we state the Hopf's Lemma.

Hopf's Lemma [26]. If M is an m-dimensional connected compact Riemannian manifold. If Φ is a differentiable function on M s. t. $\Delta\Phi \geq 0$ everywhere on M (or $\Delta\Phi \leq 0$ everywhere on M), then Φ is a constant function.

For a compact orientable Riemannian manifold M with or without boundary and as a consequences of the integration theory of manifolds, we have [27]

$$\int_M \Delta\Phi dV = 0, \tag{2.28}$$

where Φ is a function on M and dV is the volume element of M.

The Ricci soliton idea was given by Hamilton [28]. It is regarded as the natural generality of Einstein metrics and they are the self similar solution of the Ricci flow $\frac{\partial}{\partial t} g(t) = -2Ric(t)$. If there exists a smooth vector field Y such that the Ricci tensor meets the following condition

$$Ric + \frac{1}{2}\mathcal{L}_Y g = \alpha, \tag{2.29}$$

for any constant α, where $\mathcal{L}_Y$ is the Lie derivative, then the Riemannian metric g on a complete Riemannian manifold $\bar{M}$ is named as Ricci Soliton. If $\alpha < 0,\ \alpha = 0$ and $\alpha > 0$ then the Ricci soliton is called expanding, steady and Shrinking respectively. If we specify $Y = \nabla\Phi$ for a smooth function Φ defined on $\bar{M}$, then g admits gradient Ricci soliton with the potential function Φ. For this case (2.29) takes the form

$$Ric + \nabla^2\Phi = \alpha g. \tag{2.30}$$

Since the Laplacian Δ and the gradient ∇^2 are related as $\Delta = \nabla^2$. Thus, in terms of Hessian (2.29) can be expressed as

$$Ric = \alpha g + Hess^{\Phi}. \tag{2.31}$$

Note 2.1. If the potential function Φ is constant on a gradient Ricci soliton, then $(M, g, \nabla\Phi, \lambda)$ is an Einstein manifold.

3. Warped product pointwise semi-slant submanifolds

In [12], K. S. Park, proved the existence of the warped product pointwise semi-slant warped product submanifolds of the type $N_T \times_\Phi N_\theta$ of cosymplectic manifolds and achieved the following lemma

Lemma 3.1. *Let $M = N_T \times_\Phi N_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\bar{M}$ such that $\eta \in TN_T$, where N_T and N_θ are invariant and pointwise slant submanifolds of $\bar{M}$, respectively. Then*

$$g(\sigma(Y, W), FPZ) = -\cos^2\theta Y \ln\Phi\, g(W, Z) - \Phi Y \ln\Phi g(W, PZ), \tag{3.1}$$

and

$$g(\sigma(\Psi Y, W), FZ) = Y \ln\Phi g(W, Z) - \Psi Y \ln\Phi g(W, TZ), \tag{3.2}$$

for any $Y \in TN_T$ and $Z, W \in TN_\theta$.

Now let $M = N_T \times_\Phi N_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\bar{M}$ and we consider the vector field η tangent to N_T. If D is invariant distribution and D^θ is proper point wise slant distribution with the slant function θ, then the tangent bundle TM and $T^\perp M$ are decomposed (resp.) as follows

$$TM = D \oplus D^\theta \oplus \langle\eta\rangle,$$

$$T^\perp M = FD^\theta \oplus \mu,$$

where μ is the orthogonal complementary distribution of FD^θ in $T^\perp M$. It is easy to see that μ is an invariant subbundle of $T^\perp M$ with respect to Ψ.

In view of the above direct decomposition, the second fundamental form σ can be written as

$$\sigma(U_1, U_2) = \sigma_{FD^\theta}(U_1, U_2) + \sigma_\mu(U_1, U_2), \tag{3.3}$$

for $U_1, U_2 \in TM$, where $\sigma_{FD^\theta}(U_1, U_2)$ and $\sigma_\mu(U_1, U_2)$ are the components of $\sigma(U_1, U_2)$ in the normal sub-bundles FD^θ and μ respectively. Moreover if $\{V_1, V_2, \ldots, V_q\}$ be a local orthonormal frame of vector fields of D^θ, then

$$\sigma_{FD^\theta}(U_1, U_2) = \sum_{r=1}^{q} \sigma^r(U_1, U_2) FV_r \tag{3.4}$$

where

$$\sigma^r(U_1, U_2) = \csc^2\theta g(\sigma(U_1, U_2), FV_r) \tag{3.5}$$

To ensure the existence, we construct an example of a warped product pointwise semi-slant submanifold of the type $M = N_T \times_\Phi N_\theta$ in cosymplectic manifold with η tangent to N_T.

Example. Let $\bar{M} = C^5 \times R$ be a Riemannian product of Euclidean space C^5 with line R such that the structure vector field $\eta = \frac{\partial}{\partial t}$, 1-form $\zeta = dt$ and metric $g = g_1 + dt^2$, where g_1 is the metric on Euclidean space. Then $(\bar{M}, \Psi, \eta, \zeta, g)$ is a cosymplectic manifold. Let $\phi : M^5 \to \bar{M}^{11}$ be a point wise semi-slant submanifold such that $0 < u, v < 1$ is defined as

$$e_1 = u\tanh x,\quad e_2 = v\tanh x,\quad e_3 = u\tanh y,\quad e_4 = v\tanh y,\quad e_5 = u\,\mathrm{sech}\,x,$$

$$e_6 = v\mathrm{sech}\ x,\quad e_7 = u\mathrm{sech}\ y,\quad e_8 = v\mathrm{sech}\,y,\quad e_9 = x,\quad e_{10} = y,\quad e_{11} = 0.$$

The tangent space TM is spanned by the vector fields X_1, X_2, X_3, X_4, X_5 such that

$$X_1 = \mathrm{sech}\,y\frac{\partial}{\partial e_8} + \mathrm{sech}\ x\frac{\partial}{\partial e_6} + \tanh y\frac{\partial}{\partial e_4} + \tanh x\frac{\partial}{\partial e_2},$$

$$X_2 = \mathrm{sech}\ y\frac{\partial}{\partial e_7} + \mathrm{sech}\,x\frac{\partial}{\partial e_5} + \tanh y\frac{\partial}{\partial e_3} + \tan x\frac{\partial}{\partial e_1},$$

$$X_3 = \eta = \frac{\partial}{\partial t},$$

$$X_4 = \frac{\partial}{\partial e_9} + v\tanh x\frac{\partial}{\partial e_6} + u\tan x\frac{\partial}{\partial e_5} - v\,\mathrm{sech}\ x\frac{\partial}{\partial e_2} - u\,\mathrm{sech}\ x\frac{\partial}{\partial e_1},$$

$$X_5 = \frac{\partial}{\partial e_{10}} + v\tanh y\frac{\partial}{\partial e_7} + u\tan y\frac{\partial}{\partial e_8} - v\,\mathrm{sech}\,y\frac{\partial}{\partial e_4} - u\mathrm{sech}\,y\frac{\partial}{\partial e_3}.$$

Then $D^\theta = \mathrm{span}\{X_4, X_5\}$ is a pointwise slant distribution with slant function $\cos^{-1}(\frac{1}{u^2+v^2+1})$ and invariant distribution $D = \mathrm{span}\{X_1, X_2, X_3\}$. Thus $M^5 = N_T \times_\Phi N_\theta$ is a non-trivial warped product pointwise semi-slant submanifold of M^{11} with the warping function $\Phi = \sqrt{(u^2 + v^2 + 1)}$.

In this section, for convention, we denote by $Y, X \in TN_T$ and $V, U \in TN_\theta$ as the vector fields of respective tangent bundles of N_T and N_θ. At first, some initial results to be proved.

Lemma 3.2. *Let $N_T \times_\Phi N_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then*

(i) $\eta ln\,\Phi = 0$,

(ii) $g(\sigma(\Psi Y, V), FV) = Yln\,\Phi\|V\|^2$,

(iii) $g(\sigma(\Psi Y, V), \Psi\sigma(Y, V)) = \|\sigma_\mu(Y, V)\|^2 + \cos^2\theta(Yln\,\Phi)^2\|V\|^2$,

for all $Y \in TN_T$ and $V \in TN_\theta$, where σ_μ is the μ component of the second fundamental form σ.

Proof. From (2.4), (2.6), and (2.22), it is easy to see that $\eta ln\,\Phi = 0$. Moreover, part (ii) is a particular case of (3.2). To prove part (iii), on making use of (2.6) and (2.3), we get

$$\sigma(\Psi Y, V) = \Psi\sigma(Y, V) + \Psi\nabla_V Y - \nabla_V\Psi Y.$$

Now using (2.22), the above equation can be written as

$$\sigma(\Psi Y, V) = \Psi\sigma(Y, V) + Y\ln\Phi\Psi V - \Psi Y\ln\Phi V.$$

Comparing the normal parts

$$\sigma(\Psi Y, V) = \Psi\sigma_\mu(Y, V) + Y\ln\Phi FV,$$

taking inner product with $\Psi\sigma(Y, V)$, we get

$$g(\sigma(\Psi Y,\ V), \Psi\sigma(Y, V)) = \|\sigma_\mu(Y, V)\|^2 + Y\ln\Phi g(\Psi\sigma(Y, V), FV). \tag{3.6}$$

Calculating the last term of above equation by using (2.6), (2.3), and (2.20) as follows

$$g(\Psi\sigma(Y, V), FV) = g(\sigma(\Psi Y, V), FV) - \sin^2\theta Y\ln\Phi\|V\|^2.$$

Utilizing part (ii), we get

$$g(\Psi\sigma(Y, V), FV) = \cos^2\theta Y\ln\Phi\|V\|^2,$$

using in (3.6), we get the required result.

Lemma 3.3. *Let $M = N_T \times_\Phi N_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\bar{M}$. Then*

$$g(\sigma(Y, PV), FV) = -g(\sigma(Y, V), FPV) = -\cos^2\theta Y\ln\Phi\|V\|^2$$

for all $Y \in TN_T$, $V \in TN_\theta$.

Proof. From the part (ii) of Lemma 3.2, we may obtain

$$g(\sigma(PY, V), FU) + g(\sigma(PY, U), FV) = 2Y\ln\Phi g(V, U)$$

for any $Y \in TN_T$ and $V, U \in TN_\theta$. Replacing U by $PV \in D^\theta$ and using the fact that V and PV are orthogonal, we have

$$g(\sigma(Y, PV), FV) = -g(\sigma(Y, V), FPV). \tag{3.7}$$

By using (2.12), (2.14), and (2.22), we have

$$PY\ln\Phi V - Y\ln\Phi PV = t\sigma(Y, V).$$

Now taking inner product with $U \in TN_\theta$ in the above equation, we have

$$PY\ln\Phi g(V, U) - Y\ln\Phi g(PV, U) = -g(\sigma(Y, V), FU).$$

Interchanging V and U and subtracting the resultant from the above equation, we get

$$-g(\sigma(Y, V), FU) + g(\sigma(Y, U), FV) = 2Y\ln\Phi g(V, PU).$$

In particular, replacing U by $PV \in D^\theta$, the above equation yields

$$g(\sigma(Y, V), FPV) - g(\sigma(Y, PV), FV) = -2\cos^2\theta Y\ln\Phi\|V\|^2. \tag{3.8}$$

Using (3.7), we obtain

$$g(\sigma(Y, PV), FV) = -g(\sigma(Y, V), FPV) = -\cos^2\theta Y\ln\Phi\|V\|^2. \quad \square \tag{3.9}$$

Lemma 3.4. *On a warped product pointwise semi-slant submanifold $M = N_T \times_{\Phi} N_\theta$ of a cosymplectic manifold $\bar{M}$, we have*

$$\sum_{i=1}^{p}\Bigg[\sum_{j,k=1}^{2q} g\big(\sigma(\Psi e_i, e^k), Fe^j\big) g\big(\sigma(e_i, Pe^k), Fe^j\big) - g\big(\sigma(e_i, e^k), Fe^j\big) g\big(\sigma(\Phi e_i, Pe^k), Fe^j\big)\Bigg] = -4q\cos^2\theta \|\nabla ln\,\Phi\|^2,$$

where $\{e_0 = \eta, e_1, e_2, \ldots, e_p, \Psi e_1, \Psi e_2, \ldots, \Psi e_p\}$ and $\{e^1, e^2, \ldots, e^q, e^{q+1} = \sec\theta Pe^1, e^{q+2} = \sec\theta Pe^2, \ldots, e^{2q} = \sec\theta Pe^q\}$ are the frames of the orthonormal vector fields on TN_T and TN_θ respectively.

Proof. First, we expand the left hand term in the following way

$$\sum_{i=1}^{p}\Bigg[\sum_{j,k=1}^{2q} g\big(\sigma(\Psi e_i, e^k), Fe^j\big) g\big(\sigma(e_i, Pe^k), Fe^j\big)\Bigg]$$

$$= \sum_{i=1}^{p}\Bigg[\sum_{j=1}^{2q} g\big(\sigma(\Psi e_i, e^j), Fe^j\big) g\big(\sigma(e_i, Pe^j), Fe^j\big) + \sum_{j\neq k=1}^{2q} g\big(\sigma(\Psi e_i, e^k), Fe^j\big) g\big(\sigma(e_i, Pe^k), Fe^j\big)\Bigg]$$

$$= \sum_{i=1}^{p}\Bigg[\sum_{j=1}^{2q} g\big(\sigma(\Psi e_i, e^j), Fe^j\big) g\big(\sigma(e_i, Pe^j), Fe^j\big) + \sum_{j=1}^{q} g\big(\sigma(\Psi e_i, e^j), Fe^{j+q}\big) g\big(\sigma(e_i, Pe^j), Fe^{j+q}\big) + \sum_{j=1}^{q} g\big(\sigma(\Psi e_i, e^{j+q}), Fe^j\big) g\big(\sigma(e_i, Pe^{j+q}), Fe^j\big)\Bigg]$$

$$= \sum_{i=1}^{p}\Bigg[\sum_{j=1}^{2q} g\big(\sigma(\Psi e_i, e^j), Fe^j\big) g\big(\sigma(e_i, Pe^j), Fe^j\big) + \sec^2\theta \sum_{j=1}^{q} g\big(\sigma(\Psi e_i, e^j), FPe^j\big) g\big(\sigma(e_i, Pe^j), FPe^j\big) - \sum_{j=1}^{q} g\big(\sigma(\Psi e_i, Pe^j), Fe^j\big) g\big(\sigma(e_i, e^j), Fe^j\big)\Bigg].$$

Using part (ii) of Lemmas 3.2, 3.3 and utilizing (2.24), we get

$$\sum_{i=1}^{p}\Bigg[\sum_{j,k=1}^{2q} g\big(\sigma(\Psi e_i, e^k), Fe^j\big) g\big(\sigma(e_i, Pe^k), Fe^j\big)\Bigg]$$

$$= \sum_{i=1}^{p}\Big[-2q\cos^2\theta (e_i ln\,\Phi)^2 - 2q\cos^2\theta (\Psi\, e_i ln\,\Phi)^2\Big]$$

$$= -2q\cos^2\theta \|\nabla ln\,\Phi\|^2.$$

Replacing e_i by Ψe_i in above equation, we get

$$\sum_{i=1}^{p}\left[\sum_{j,k=1}^{2q} g\left(\sigma(e_i,e^k),Fe^j\right)g\left(\sigma(\Psi e_i,Pe^k),Fe^j\right)\right] = 2q\cos^2\theta\|\nabla ln\,\Phi\|^2.$$

Subtracting the above two findings, we get the required result. □

Now, we prove the following characterization.

Theorem 3.5. *Let $M = N_T \times_\Phi N_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic space form $\bar{M}(c)$ such that N_T is a compact submanifold. Then M is a Riemannian product submanifold if the following inequalities hold*

$$\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu\left(e_i,e^j\right)\right\|^2 \le cpq\sin^2\theta - 2q\left(\cos^2\theta + 2\cos^2\theta\right)\|\nabla ln\,\Phi\|^2$$

and

$$\sum_{i=1}^{p}\sum_{j=1}^{2q} g\left(\sigma_\mu(\Psi e_i,e^j),\sigma_\mu\left(e_i,Pe^j\right)\right)\ge 0,$$

where σ_μ denotes the component of σ in μ, and $(2p+1)$ and $2q$ are the dimensions of N_T, and N_θ respectively.

Proof. For any unit vector fields $Y \in TN_T$ and $V \in TN_\theta$, using (2.1), (2.5), and (2.20), we have

$$\bar{R}(Y,\ \Psi Y,\ V,\ FV) = -\frac{c}{2}\sin^2\theta\|Y\|^2\|V\|^2 \tag{3.10}$$

On the other hand by Codazzi equation

$$\begin{aligned}\bar{R}(Y,\Psi Y,V,FV) = g(.(\nabla^\perp_Y\sigma(\Psi Y,V),FV) - g\left(\nabla^\perp_{\Psi Y}\sigma(Y,V),FV\right)\\ +g(\sigma(Y,\nabla_{\Psi Y}V),FV) - g(\sigma(\Psi Y,\nabla_Y V),FV)\\ -g(\sigma(\nabla_Y\Psi Y,V),FV) + g(\sigma(\nabla_{\Psi Y}Y,V),FV).\end{aligned} \tag{3.11}$$

Now, we compute the values of the terms involved in (3.11). First, we have

$$g\left(\nabla^\perp_Y\sigma(\Psi Y,V),FV\right) = Yg(\sigma(\Psi Y,V),FV) - g\left(\sigma(\Psi Y,V),\nabla^\perp_Y FV\right).$$

Using the part (ii) of Lemma 3.2 in the above equation, we get

$$\begin{aligned}g\left(\nabla^\perp_Y\sigma(\Psi Y,V),FV\right) = Y^2\,ln\,\Phi\|V\|^2 + 2(Y ln\,\Phi)^2\|V\|^2\\ -g\left(\sigma(\Psi Y,V),\nabla^\perp_Y FV\right).\end{aligned} \tag{3.12}$$

We calculate the last term of (3.12) using (2.9) as follows

$$g\left(\sigma(\Psi Y,V),\nabla^\perp_Y FV\right) = g(\sigma(\Psi Y,V),\bar{\nabla}_Y(\Psi V - PV)).$$

By the use of (2.6) and (2.11) the above equation takes the form

$$\begin{aligned}g\left(\sigma(\Psi Y,V),\nabla^\perp_Y FV\right) = g(\sigma(\Psi Y,V),(\bar{\nabla}_Y\Psi)V + \Psi\bar{\nabla}_Y V)\\ -g(\sigma(\Psi Y,V),\sigma(Y,PV)).\end{aligned}$$

Using (2.3), (2.6), (2.22), part (ii), and (iii) of Lemma 3.2, we obtain

$$\begin{aligned}g\left(\sigma(\Psi Y,V),\nabla^\perp_Y FV\right) = (Y\,ln\,\Phi)^2\left(1+\cos^2\theta\right)\|V\|^2 + \|\sigma_\mu(Y,V)\|^2\\ -g(\sigma(\Psi Y,V),\sigma(Y,PV)).\end{aligned} \tag{3.13}$$

Utilizing (3.13) in (3.12), we get

$$g\left(\nabla^{\perp}_{Y}\sigma(\Psi Y, V), FV\right) = Y^2 ln\, \Phi \|V\|^2 + (Y\, ln\, \Phi)^2 \sin^2\theta \|V\|^2 \\ -\|\sigma_\mu(Y, V)\|^2 + g(\sigma(\Psi Y, V), \sigma(Y, PV)). \tag{3.14}$$

Similarly, we can write

$$g\left(\nabla^{\perp}_{\Psi Y}\sigma(Y, V), FV\right) = -(\Psi Y)^2 ln\, \Phi \|V\|^2 - (\Psi Y\, ln\, \Phi)^2 \sin^2\theta \|V\|^2 \\ +\|\sigma_\mu(\Psi Y, V)\|^2 + g(\sigma(Y, V), \sigma(\Psi Y, PV)). \tag{3.15}$$

From the part (ii) of Lemma 3.2, we have

$$g(A_{FV}V, \Psi Y) = Y\, ln\, \Phi \|V\|^2.$$

Replacing Y by $\nabla_Y Y$ (using the totally geodesicness of N_T, $\nabla_Y Y \in TN_T$) in the above equation, we have

$$g(A_{FV}V, \Psi \nabla_Y Y) = \nabla_Y Y\, ln\, \Phi \|V\|^2.$$

By using (2.6), the above equation takes the form

$$g(A_{FV}V, \Psi \bar{\nabla}_Y Y - \Psi\sigma(Y, Y)) = \nabla_Y Y\, ln\, \Phi \|V\|^2.$$

By using the fact that the first factor N_T is totally geodesic in M, it can be easily verified that $\sigma(Y_1, Y)_2 \in \mu$, for all $Y_1, Y_2 \in TN_T$. Using this and (2.11) in the above equation, we get

$$g(\sigma(\nabla_Y \Psi Y, V), FV) = \nabla_Y Y\, ln\, \Phi \|V\|^2. \tag{3.16}$$

Similarly, we can write

$$g(\sigma(\nabla_{\Psi Y} Y, V), FV) = -\nabla_{\Psi Y} \Psi Y\, ln\, \Phi \|V\|^2. \tag{3.17}$$

By use of (2.22) and the part (ii) of Lemma 3.2, we have

$$g(\sigma(\Psi Y, \nabla_Y V), FV) = (Y\, ln\, \Phi)^2 \|V\|^2 \tag{3.18}$$

and

$$g(\sigma(Y, \nabla_{\Psi Y} V), FV) = -(\Psi Y ln\, \Phi)^2 \|V\|^2. \tag{3.19}$$

Substituting values from (3.10), (3.14), (3.15), (3.16), (3.17), (3.18), and (3.19) in (3.11), we obtain

$$-\frac{c}{2}\sin^2\theta \|Y\|^2 \|V\|^2 = Y^2 ln\, \Phi \|V\|^2 + (\Psi Y)^2 ln\, \Phi \|V\|^2 \\ -(Y\, ln\, \Phi)^2 \cos^2\theta \|Y\|^2 - (\Psi Y\, ln\, \Phi)^2 \cos^2\theta \|V\|^2 \\ -\|\sigma_\mu(Y, V)\|^2 - \|\sigma_\mu(\Psi Y, V)\|^2 \\ -\nabla_Y Y\, ln\, \Phi \|V\|^2 - \nabla_{\Psi Y} \Psi Y\, ln\, \Phi \|V\|^2 \\ +g(\sigma(\Psi Y, V), \sigma(Y, PV)) - g(\sigma(Y, V), \sigma(\Psi Y, PV)).$$

Let $\{e_0 = \eta, e_1, e_2, \ldots, e_p, e_{p+1} = \Psi e_1, e_{p+2} = \Psi e_2, \ldots, e_{2p} = \Psi e_p\}$ be the orthonormal frame on TN_T and $\{e^1, e^2, \ldots, e^q, e^{q+1} = \sec\theta Pe^1, e^{q+2} = \sec\theta Pe^2, \ldots, e^{2q} = \sec\theta Pe^q\}$ be an orthonormal frame on TN_θ. Now, using the decomposition (3.3), formula (3.5), (2.6) and

(2.3), the above equation takes the form

$$
\begin{aligned}
-\frac{c}{2}\sin^2\theta\|Y\|^2\|V\|^2 = {} & Y^2 ln\,\Phi\|V\|^2 + (\Psi Y)^2 ln\,\Phi\|V\|^2 \\
& -(Y\,ln\,\Phi)^2\cos^2\theta\|V\|^2 - (\Psi\,Y\,ln\,\Phi)^2\cos^2\theta\|V\|^2 \\
& -\|\sigma_\mu(Y,V)\|^2 - \|\sigma_\mu(\Psi Y,V)\|^2 \\
& -\nabla_Y Y ln\,\Phi\|V\|^2 - \nabla_{\Psi Y}\Psi\,Y ln\,\Phi\|V\|^2 \\
& +\csc^2\theta\sum_{j=1}^{2q}[g(\sigma(\Psi Y,V),FV_j)g(\sigma(X,PV),FV_j) \\
& -g(\sigma(\Psi Y,V),FV_j)g(\sigma(\Psi Y,PV),FV_j)]\|V_j\|^2 \\
& +2g(\sigma_\mu(\Psi Y,V),\sigma_\mu(Y,PV)).
\end{aligned}
$$

Now summing the above equation over $i = 1,2,\ldots,p$ and $j = 1,2,\ldots 2q$, using (2.24), (2.25) and part (iii) of Lemma 3.2, one can get

$$
\begin{aligned}
2q\Delta(ln\,\Phi) = {} & cpq\sin^2\theta - 2q\cos^2\theta\|\nabla ln\,\Phi\|^2 - \sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu(e_i,e^j)\right\|^2 \\
& -4q\cot^2\theta\|\nabla ln\,\Phi\|^2 + 2\sum_{i=1}^{p}\sum_{j=1}^{2q} g(\sigma_\mu(\Psi e_i,e^j),\sigma_\mu(e_i,Pe^j)).
\end{aligned} \tag{3.20}
$$

From (3.20) if

$$
\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu(e_i,e^j)\right\|^2 \le cpq\,\sin^2\theta - 2q(\cos^2\theta + 2\cot^2\theta)\,\|\nabla ln\,\Phi\|^2
$$

and

$$
\sum_{i=1}^{p}\sum_{j=1}^{2q} g(\sigma_\mu(\Psi e_i,e^j),\sigma_\mu(e_i,Pe^j)) \ge 0,
$$

then $\Delta ln\,\Phi \ge 0$, so by the Hopf's Lemma, $ln\,\Phi$ is constant that mean Φ is constant, which proves the theorem. □

In the next theorem, we obtain the squared norm of the second fundamental form in terms of the warping function and slant function.

Theorem 3.6. *Let $\bar{M}(c)$ be a $(2n+1)$-dimensional cosymplectic space form and $M = N_T \times_\Phi N_\theta$ be an m-dimensional warped product pointwise semi-slant submanifold such that N_T is a $(2p+1)$-dimensional invariant submanifold and N_θ be a $2q$-dimensional proper pointwise slant submanifold of $\bar{M}(c)$. If*

$$
\sum_{i=1}^{p}\sum_{j=1}^{2q} g(\sigma(\Psi e_i,e^j),\sigma(e_i,Pe^j)) \ge 0,
$$

then

(i) *The squared norm of the second fundamental form* σ *satisfies*

$$\|\sigma\|^2 \geq cpq\sin^2\theta + 2q\sin^2\theta\|\nabla ln\,\Phi\|^2 - 2q\Delta(ln\,\Phi). \tag{3.21}$$

(ii) *The equality sign of* (3.21) *holds identically if and only if*

(a) N_T *is totally geodesic invariant submanifold of* $\bar{M}(c)$*. Hence* N_T *is a cosymplectic space form.*

(b) N_θ *is totally umbilical submanifolds of* $\bar{M}(c)$*.*

(c) $\sum_{i=1}^{p}\sum_{j=1}^{2q} g(\sigma(\Psi e_i, e^j),\ \sigma(e_i, Pe^j)) = 0$.

Proof. From (3.20), we have

$$\sum_{i=1}^{2p}\sum_{j=1}^{2q}\|\sigma_\mu(e_i, e^j)\|^2 \geq cpq\sin^2\theta - 2q(\cos^2\theta + 2\cot^2\theta)\|\nabla ln\,\Phi\|^2 - 2q\Delta(ln\,\Phi). \tag{3.22}$$

For the orthonormal frames $\{e_0 = \eta, e_1, e_2, \ldots, e_p, e_{p+1} = \Psi e_1, e_{p+2} = \Psi e_2, \ldots, e_{2p} = \Psi e_p\}$ and $\{e^1, e^2, \ldots, e^q, \sec\theta Pe^1, \sec\theta Pe^2, \ldots, \sec\theta Pe^q\}$, in view of the formulae (3.4), (3.5), part (ii) of Lemma 3.2, we get

$$\sum_{i=0}^{2p}\sum_{j=1}^{2q}\|\sigma_{FD_\theta}(e_i, e^j)\|^2 = \sum_{i=0}^{2p}\sum_{j,k=1}^{2q}\csc^2\theta g(\sigma(e_i, e^j), Fe^k)^2$$
$$= \csc^2\theta\sum_{i=0}^{2p}\left[\sum_{j=1}^{2q} g(\sigma(e_i, e^j), Fe^j)^2 + \sum_{j\neq k=1}^{2q} g(\sigma(e_i, e^j), Fe^k)^2\right]$$
$$= \csc^2\theta\sum_{i=0}^{2p}\left[2q(e_i ln\,\Phi)^2 + \sec^2\theta\sum_{j=1}^{q}\left\{g(\sigma(e_i, e^j), FPe^j)^2 + g(\sigma(e_i, Pe^j), Fe^j)^2\right\}\right].$$

Further using Lemma 3.3 and (2.24), the above equation reduced to

$$\sum_{i=0}^{2p}\sum_{j=1}^{2q}\|\sigma_{FD_\theta}(e_i, e^j)\|^2 = 2q\csc^2\theta\|\nabla ln\,\Phi\|^2 + 2q\cot^2\theta\|\nabla ln\,\Phi\|^2. \tag{3.23}$$

From (3.22), (3.23) we get the required inequality.

To prove the part (ii), let σ' be the second fundamental form for the immersion of N_θ in M. Then for any $U, V \in TN_\theta$ and $Y \in TN_T$, using Gauss formula, we have

$$g(\sigma'(U, V), V) = g(\nabla_U V, Y) = -Yln\,\Phi g(U, V).$$

Using (2.23), we have

$$g(\sigma'(U, V), Y) = -g(U, V)g(\nabla ln\,\Phi, Y),$$

or

$$\sigma'(U, V) = -g(U, V)\nabla ln\,\Phi. \tag{3.24}$$

If the equality sign of (3.21) holds identically, then we obtain

$$\sigma(D,D)=0, \qquad \sigma(D^{\theta},D^{\theta})=0, \tag{3.25}$$

$$g(\sigma_{\mu}(\Psi D,D^{\theta}), \quad \sigma_{\mu}(D,PD^{\theta}))=0. \tag{3.26}$$

The first condition of (3.25) implies that N_T is totally geodesic submanifold in M. On the other hand it is easy to see that $g(\sigma(Y_1,\Psi Y_2),FV)=0$, for all $Y_1,Y_2 \in TN_T$, $V \in TN_{\theta}$. It follows that N_T is totally geodesic in $\bar{M}(c)$ and hence is a cosymplectic space form. Moreover, the second condition of (3.25) together with (3.24) implies that N_{θ} is a totally umbilical submanifold.

This proves the theorem. □

4. Some applications

Theorem 3.6 motivates us to obtain formulae to calculate the Dirichlet energy involving warping function Φ. We denote by $E(\Phi)$ the Dirichlet energy of a function Φ. For a compact orientable warped product pointwise semi-slant submanifold $M=N_T\times_{\Phi}N_{\theta}$ in a cosymplectic space form $\bar{M}(c)$, we compute the Dirichlet energy of the warping function Φ in the following theorem.

Theorem 4.1. *Let $M=N_T\times_{\Phi}N_{\theta}$ be a compact orientable warped product pointwise semi-slant submanifold of a cosymplectic space form $\bar{M}(c)$, such that N_T be a $(2p+1)$-dimensional invariant submanifold tangent to the structure vector field η and N_{θ} be a $2q$-dimensional pointwise slant submanifold of $\bar{M}(c)$. Then for each $x\in N_{\theta}$, the Dirichlet energies of the warping function satisfy the following*

$$E(\ln\Phi)=\frac{1}{4q\sin^2\theta}\int_{N_T\times\{x\}}\|\sigma\|^2dV+\frac{cp}{4}Vol(N_T)$$

if and only if

(i) *N_T is totally geodesic invariant submanifold of $\bar{M}(c)$ and is a cosymplectic space form,*

(ii) *N_{θ} is totally umbilical submanifolds of $\bar{M}(c)$,*

(iii) $\sum_{i=1}^{p}\sum_{j=1}^{2q}g(\sigma(\Psi e_i,e^j),\sigma(e_i,Pe^j))=0,$

Proof. On integrating the equality case of the inequality (3.21) and using the definition of Dirichlet energy and (2.28), we get the required result. □

If $\theta=\pi/2$, then the compact orientable warped product pointwise semi-slant submanifolds become contact CR-warped product submanifolds. The following can be deduced from Theorem 4.1.

Corollary 4.2. *Let $M=N_T\times_{\Phi}N_{\perp}$ be a compact orientable contact CR-warped product submanifold of a cosymplectic space form $\bar{M}(c)$, such that N_T be a $(2p+1)$-dimensional invariant submanifold tangent to the structure vector field η and $N_{\perp}$ be an $2q$-dimensional anti-invariant submanifold of $\bar{M}(c)$. Then for each $x\in N_{\perp}$, we have*

$$E(\ln\Phi)=\frac{1}{4q}\int_{N_T\times\{x\}}\|\sigma\|^2dV+\frac{cp}{4}Vol(N_T)$$

if and only if

(i) N_T is totally geodesic invariant submanifold of $\bar{M}(c)$ and is a cosymplectic space form,

(ii) *$N_\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}(c)$.*

If the equality sign of (3.21) holds, then

$$\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu(e_i,e^j)\right\|^2 = cpq\sin^2\theta - 2q(\csc^2\theta + 2\cot^2\theta)\|\nabla ln\,\Phi\|^2 - 2q\Delta ln\,\Phi. \tag{4.1}$$

Since, the Laplacian of a smooth function Φ is the trace of the Hessian of the function. In terms of Hessian, (4.1) can be written as follows

$$\begin{aligned}\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu(e_i,e^j)\right\|^2 &= cpq\sin^2\theta - 2q(\csc^2\theta + 2\cot^2\theta)\|\nabla ln\,\Phi\|^2\\ &+ 2q\sum_{i=1}^{p}\left[Hess^{ln\,\Phi}(e_i,e_i) + Hess^{ln\,\Phi}(\Psi e_i,\Psi e_i)\right]\end{aligned} \tag{4.2}$$

Now, we have the following classification theorem for the warped product pointwise semi-slant submanifolds admitting the gradient Ricci soliton satisfying the equality case of (3.21).

Theorem 4.3. *Let $\bar{M}(c)$ be a $2n+1$-dimensional cosymplectic space form and $M = N_T \times_\Phi N_\theta$ be a warped product pointwise semi-slant submanifold admitting a shrinking gradient Ricci soliton. If*

$$\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu(e_i,e^j)\right\|^2 + 4pq = cpq\sin^2\theta + 2q\sum_{i=1}^{2p}Ric(e_i,e_i), \tag{4.3}$$

then one of the following is true

(i) The slant function $\theta = \pi/2$ i.e., M is a contact CR-warped product submanifold,

(ii) The warping function Φ is constant i.e., M is trivial Riemannian product pointwise semi-slant submanifold.

Proof. Suppose that warped product pointwise semi-slant submanifold $M = N_T \times_\Phi N_\theta$ satisfies the basic equation of the Ricci soliton, such that the potential function $\tau = ln\,\Phi$, then

$$Ric(X,Y) = \lambda g(X,Y) + Hess^\tau(X,Y), \tag{4.4}$$

for all $X, Y \in TN_T$. Considering that $\{e_1, e_2, \ldots, e_p, e_{p+1} = \Psi e_1, \ldots, e_{2p} = \Psi e_p\}$ be an orthonormal frame of the vector fields on TN_T. Now, taking summation over $i = 1, 2, \ldots, p$ for $X = Y$ in (4.4), we have

$$\sum_{i=1}^{p}Ric(e_i,e_i) = \lambda p + +\sum_{i=1}^{p}Hess^\tau(e_i,e_i). \tag{4.5}$$

Replacing e_i by Ψe_i in above equation, we get

$$\sum_{i=1}^{p}Ric(\Psi e_i,\Psi e_i) = \lambda p + +\sum_{i=1}^{p}Hess^\tau(\Psi e_i,\Psi e_i). \tag{4.6}$$

From (4.5) and (4.6), we have

$$\sum_{i=1}^{2p} Ric(e_i, e_i) = 2\lambda p + + \sum_{i=1}^{p} (Hess^{\tau}(e_i, e_i) + Hess^{\tau}(\Psi e_i, \Psi e_i)). \tag{4.7}$$

By the assumption that the equality case of (3.21) holds, then by (4.2)

$$\frac{1}{2q} \sum_{i=1}^{2p} \sum_{j=1}^{2q} \left\| \sigma_{\mu}(e_i, e^j) \right\|^2 = \frac{cp}{2} \sin^2\theta - (\cos^2\theta + 2\cot^2\theta) \|\nabla ln\,\Phi\|^2 + \sum_{i=1}^{2p} Ric(e_i, e_i) - 2\lambda p, \tag{4.8}$$

or

$$\sum_{i=1}^{2p} \sum_{j=1}^{2q} \left\| \sigma_{\mu}(e_i, e^j) \right\|^2 = cpq \sin^2\theta - 2q(\cos^2\theta + 2\cot^2\theta) \|\nabla ln\,\Phi\|^2 + 2q \sum_{i=1}^{2p} Ric(e_i, e_i) - 4pq\lambda$$

By the assumption (4.3), we get

$$(\cos^2\theta + 2\cot^2\theta) \|\nabla ln\,\Phi\|^2 = 0.$$

From the last equation it is evident that $\theta = \pi/2$ or the warping function is constant, which proves the theorem.

If the submanifold $M = N_T \times_{\Phi} N_{\theta}$ admits the steady gradient Ricci soliton, then from last theorem, it is easy to conclude the following

Theorem 4.4. *Let $\bar{M}(c)$ be a $2n+1$-dimensional cosymplectic space form and $M = N_T \times_{\Phi} N_{\theta}$ be a warped product pointwise semi-slant submanifold admitting a steady gradient Ricci soliton. If*

$$\sum_{i=1}^{2p} \sum_{j=1}^{2q} \left\| \sigma_{\mu}(e_i, e^j) \right\|^2 = cpq \sin^2\theta - 2q \sum_{i=1}^{2p} Ric(e_i, e_i)\,, \tag{4.9}$$

then one of the following is true

(i) *The slant function $\theta = \pi/2$ i.e., M is a contact CR-warped product submanifold,*

(ii) *The warping function Φ is constant i.e., M is trivial Riemannian product pointwise semi-slant submanifold.*

In terms of Ricci curvature, we have the following classification

Theorem 4.5. *Let $\bar{M}(c)$ be a $2n+1$-dimensional cosymplectic space form and $M = N_T \times_{\Phi} N_{\theta}$ be a warped product pointwise semi-slant submanifold with the equality case of (3.21) holds. If the following holds*

$$2q \int_M Ric(\nabla ln\,\Phi, -) dV = cpq \sin^2\theta - \sum_{i=1}^{2p} \sum_{j=1}^{2q} \left\| \sigma_{\mu}(e_i, e^j) \right\|^2, \tag{4.10}$$

then one of the following statement is true

(i) *The slant function* $\theta = \pi/2$ *i.e., M is a contact CR-warped product submanifold,*

(ii) *The warping function* Φ *is constant i.e., M is trivial Riemannian product pointwise semi-slant submanifold.*

Proof. For a connection ∇ on a smooth manifold M, there exists a tensor R of type (1, 3) called the curvature tensor of the connection ∇ defined by

$$-\nabla_U \nabla_V W + \nabla_V \nabla_U W - \nabla_{[U,V]} W = R(U, V)W \tag{4.11}$$

for all $U, V, W \in TM$.

For a warping function $\tau = ln\,\Phi$, from (4.11), we have

$$\nabla^2 d(\tau)(V, U, W) - \nabla^2 d(\tau)(U, V, W) = d(\tau)R(U, V)W. \tag{4.12}$$

By the smoothness property of Φ on N_T and $\nabla^2_{UV} = \nabla_U \nabla_V - \nabla_{\nabla_U V}$, then the curvature tensor $R(U, V)W$ behaves like a derivative. Since $d\tau$ is closed, then it is easy to see that $\nabla^2 d(\tau)(U, V, W) = \nabla^2 d(\tau)(V, U, W)$, for any vector fields $U, V, W \in TN_T$. Now, for a local orthonormal frame $\{e_1, e_2, \ldots, e_{2p}\}$ on N_T and for a fixed point $t \in N_T$ such that $\nabla_{ei}(e_j)(t) = 0$, for $1 \leq i, j \leq 2p + 1$. If we specify $\nabla_{ei}(U)(t) = 0$, for any $U \in TN_T$ and taking trace with respect to V and W in the following equation

$$\nabla^2 d(\tau)(V, U, W) = \nabla^2 d(\tau)(U, V, W),$$

and utilizing (4.12), we have

$$\sum_{i=1}^{p} \left(\nabla^2 d(\tau)\right)(e_i, e_j, U) = -d(\Delta(\tau))(U) + Ric(\nabla, U). \tag{4.13}$$

Further solving left hand side, the above equation takes the form

$$div(Hess^{\tau})(U) + d(\Delta(\tau))(U) = Ric(\nabla\tau, U), \tag{4.14}$$

or

$$div(Hess^{\tau}) + d(\Delta(\tau)) = Ric(\nabla\tau, -). \tag{4.15}$$

As $M = N_T \times_{\Phi} N_\theta$ is a compact orientable warped product submanifold, then on integrating

$$\Delta(\tau) + \int_M div(Hess^{\tau})dV = \int_M Ric(\nabla\tau, -)dV,$$

where dV is the volume element.

Since $\Delta\Phi = -div(\nabla\Phi)$ [28] and $\int_M div(U)dV = 0$ for any $U \in TN_T$. So, it is easy to conclude that $\int_M div(Hess^{\tau})dV = 0$. Then

$$\Delta(\tau) = \int_M Ric(\nabla\tau, -)dV. \tag{4.16}$$

Utilizing above equation in (4.1), we have

$$\begin{aligned} \sum_{i=1}^{2p} \sum_{j=1}^{2q} \left\|\sigma_\mu(e_i, e^j)\right\|^2 = cpq\sin^2\theta - 2q(\csc^2\theta + 2\cot^2\theta)\|\nabla ln\,\Phi\|^2 \\ - 2q\int_M Ric(\nabla ln\,\Phi, -)dV. \end{aligned} \tag{4.17}$$

By the assumption (4.10), we get

$$(\cos^2\theta + 2\cot^2\theta)\|\nabla ln\,\Phi\|^2 = 0.$$

From the above equation it is evident that $\theta = \pi/2$ or the warping function Φ is constant, which proves the theorem.

5. Conclusion

In this paper, by using Hopf's Lemma, we obtained the characterizing inequalities for the existence of warped product pointwise semi-slant submanifolds of cosymplectic space forms. Moreover, we also worked out an estimation for the squared norm of the second fundamental form in terms of the warping function and slant function. To strengthen our results, we provided a non-trivial example of a warped product pointwise semi-slant submanifold in a cosymplectic manifold. Moreover, some applications in the form of the Dirichlet energy of the warping function are derived. The results obtained may be helpful in further studies on the Dirichlet energy of smooth functions.

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