Nonlinear Jordan centralizer of strictly upper triangular matrices

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Received 17 December 2018
Revised 24 August 2019
Accepted 25 August 2019

Abstract

Let $\mathcal{F}$ be a field of zero characteristic, let $N_n(\mathcal{F})$ denote the algebra of $n\times n$ strictly upper triangular matrices with entries in $\mathcal{F}$, and let $f: N_n(\mathcal{F})\to N_n(\mathcal{F})$ be a nonlinear Jordan centralizer of $N_n(\mathcal{F})$, that is, a map satisfying that $f(XY+YX)=Xf(Y)+f(Y)X$, for all $X, Y\in N_n(\mathcal{F})$. We prove that $f(X)=\lambda X+\eta(X)$ where $\lambda\in\mathcal{F}$ and η is a map from $N_n(\mathcal{F})$ into its center $\mathcal{Z}(N_n(\mathcal{F}))$ satisfying that $\eta(XY+YX)=0$ for every X, Y in $N_n(\mathcal{F})$.

Keywords Jordan centralizer, Strictly upper triangular matrices, Commuting map

Paper type Original Article

1. Introduction

Consider a ring R. An additive mapping $T: R\to R$ is called a left (respectively right) centralizer if $T(ab)=T(a)b$ (respectively $T(ab)=aT(b)$) for all $a, b\in R$. The map T is called a centralizer if it is a left and a right centralizer. The characterization of centralizers on algebras or rings has been a widely discussed subject in various areas of mathematics.

In [11] Zalar proved the following interesting result: if R is a 2-torsion free semiprime ring and T is an additive mapping such that $T(a^2)=T(a)a$ (or $T(a^2)=aT(a)$), then T is a centralizer. Vukman [10] considered additive maps satisfying similar conditions, namely $2T(a^2)=T(a)a+aT(a)$ for any $a\in R$, and showed that if R is a 2-torsion free semiprime ring then T is also a centralizer. Since then, the centralizers have been intensively investigated by many mathematicians (see, e.g., [2–5,7]).

Let R be a ring. An additive map $f: R\to R$, is called a Jordan centralizer of R if

$$\forall x, y\in R f(xy+yx)=xf(y)+f(y)x. \qquad (1)$$

The author would like to thank the referee for providing useful suggestions which served to improve this paper.

Declaration of Competing Interest: No author associated with this paper has disclosed any potential or pertinent conflicts which may be perceived to have impending conflict with this work. For full disclosure statements refer to https://doi.org/10.1016/j.ajmsc.2019.08.002.

The publisher wishes to inform readers that the article "Nonlinear Jordan centralizer of strictly upper triangular matrices" was originally published by the previous publisher of the *Arab Journal of Mathematical Sciences* and the pagination of this article has been subsequently changed. There has been no change to the content of the article. This change was necessary for the journal to transition from the previous publisher to the new one. The publisher sincerely apologises for any inconvenience caused. To access and cite this article, please use Hadj Ahmed, D. A. (2019), "Nonlinear Jordan centralizer of strictly upper triangular matrices", *Arab Journal of Mathematical Sciences*, Vol. 26 No. 1/2, pp. 197-201. The original publication date for this paper was 07/09/2019.

Arab Journal of Mathematical Sciences
Vol. 26 No. 1/2, 2020
pp. 197-201
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1016/j.ajmsc.2019.08.002

Recently, Ghomanjani and Bahmani [8] dealt with the structure of Lie centralizers of trivial extension algebras, whereas Fošner and Jing [6] studied Lie centralizers of triangular rings.

The inspiration of this paper comes from the articles [1,4,6] in which the authors deal with the Lie centralizer maps of triangular algebras and rings. In this note we will consider nonlinear Jordan centralizers on strictly upper triangular matrices over a field of zero characteristic.

Throughout this article, $\mathcal{F}$ is a field of zero characteristic. Let $M_n(\mathcal{F})$ and $N_n(\mathcal{F})$ denote the algebra of all $n\times n$ matrices and the algebra of all $n\times n$ strictly upper triangular matrices over $\mathcal{F}$, respectively. We use $diag(a_1, a_2, \ldots, a_n)$ to represent a diagonal matrix with diagonal $(a_1, a_2, \ldots, a_n)$ where $a_i\in\mathcal{F}$. The set of all $n\times n$ diagonal matrices over $\mathcal{F}$ is denoted by $D_n(\mathcal{F})$. Let I_n be the identity in $M_n(\mathcal{F})$, $J=\sum_{i=1}^{n-1}E_{i,i+1}$ and $\{E_{ij}:1\le i,j\le n\}$ the canonical basis of $M_n(\mathcal{F})$, where E_{ij} is the matrix with 1 in the (i,j) position and zeros elsewhere. By $C_{N_n(\mathcal{F})}(X)$ we will denote the centralizer of the element X in the ring $N_n(\mathcal{F})$.

The notation $f:N_n(\mathcal{F})\to N_n(\mathcal{F})$ means a nonlinear map satisfying $\forall X, Y\in N_n(\mathcal{F})$: $f(XY+YX)=Xf(Y)+f(Y)X$.

Notice that it is easy to check that the $\mathcal{Z}(N_n(\mathcal{F}))=\mathcal{F}E_{1n}$.

The main result in this paper is the following:

Theorem 1. *Let $\mathcal{F}$ be a field of zero characteristic. If $f:N_n(\mathcal{F})\to N_n(\mathcal{F})$ is a nonlinear Jordan centralizer then there exists $\lambda\in\mathcal{F}$ and a map $\eta:N_n(\mathcal{F})\to\mathcal{Z}(N_n(\mathcal{F}))$ satisfying $\eta(XY+YX)=0$ for every X, Y in $N_n(\mathcal{F})$ such that $f(X)=\lambda X+\eta(X)$ for all X in $N_n(\mathcal{F})$.*

2. Proof of the main result

Let us start with some basic properties of Lie centralizers.

Lemma 2. *Let f be a nonlinear Jordan centralizer of $N_n(\mathcal{F})$. Then*

(1) $f(0)=0$,

(2) *For every $X, Y\in N_n(\mathcal{F})$, we have $f(XY+YX)=Yf(X)+f(X)Y$.*

Proof. To prove (1) it suffices to notice that

$$f(0)=0f(0)+f(0)0=0.$$

(2) Observe that if $f(XY+YX)=Yf(X)+f(X)Y$, Interchanging X and Y in the above identity, we have $f(XY+YX)=Yf(X)+f(X)Y$. ■

Lemma 3. *Let f be a nonlinear Jordan centralizer of $N_n(\mathcal{F})$. Then*

(1) $f(\sum_{i=1}^{n-1}a_iE_{i,i+1})=\sum_{i=1}^{n-1}b_iE_{i,i+1}$,

(2) There exists $\lambda\in\mathcal{F}$ such that $f(J)=\lambda J$.

Proof. Let $D=\sum_{i=1}^{n}\alpha_iE_{i,i}\in D_n(\mathcal{F})$, As $\mathcal{F}$ is infinite, we can find a set $\{\alpha_i\in\mathcal{F}/1\le i\le n\}$ whose elements satisfy conditions: $\alpha_i+\alpha_{i+1}=1$ for $1\le i\le n-1$ and $\alpha_i+\alpha_j\ne 1$ for $j\ne i+1$.

(1) Consider $A\in M_n(\mathcal{F})$. It is well known that $DA+AD=A$ if and only if $A=\sum_{i=1}^{n}a_iE_{i,i+1}$. Hence, if $A=\sum_{i=1}^{n-1}a_iE_{i,i+1}$,, we have $A=DA+AD$. Thus $f(A)=f(DA+AD)=Df(A)+f(A)D$. Therefore $f(A)=\sum_{i=1}^{n-1}b_iE_{i,i+1}$.

(2) As in (1), let $N=\sum_{i=1}^{n-1}(-1)^iE_{i,i+1}\in N_n(\mathcal{F})$, consider $A=\sum_{i=1}^{n-1}a_iE_{i,i+1}$. for some $a_i\in\mathcal{F}$. Then $NA+AN=0$ if and only if $A=aJ$ for some $a\in\mathcal{F}$.

Indeed, $f(J)=\sum_{i=1}^{n-1}a_iE_{i,i+1}$. by (1). Thus, $0=f(0)=f(NA+AN)=Nf(A)+f(A)N$. Hence, there exists $\lambda\in\mathcal{F}$ such that $f(J)=\lambda J$. ■

We will need the following lemma.

Lemma 4 *(Lemma 2.1, [9]). Suppose that $\mathcal{F}$ is an arbitrary field. If $G, H \in UT_n(\mathcal{F})$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, then G and H are conjugated in $UT_n(\mathcal{F})$.*

Here $UT_n(\mathcal{F})$ is the multiplicative group of $n \times n$ upper triangular matrices with only 1's in the main diagonal. From the lemma above we obtain the following corollary.

Corollary 5. *Let $\mathcal{F}$ be a field. For every $A = \sum_{1 \leq i < j \leq n} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, there exists $B \in T_n(\mathcal{F})$ such that $B^{-1}AB = J$ and $T_n(\mathcal{F})$ is the ring of upper triangular matrices.*

Proof. Let A be a matrix in $N_n(\mathcal{F})$ of the mentioned form. Then $I_n + A$ is a unitriangular matrix. Let us notice first that there exists $B_1 \in D_n(\mathcal{F})$ such that $(B_1^{-1}AB_1)_{i,i+1} = 1$ for all $i \in \mathbb{N}$. We can construct $B_1 \in D_n(\mathcal{F})$ recursively by:

$$(B_1)_{11} = 1, \ (B_1)_{i+1,i+1} = (B_1)_{ii} \cdot (A_{i,i+1})^{-1} \text{for } i \geq 1.$$

Consider the matrix $I_n + B_1^{-1}AB \in UT_n(\mathcal{F})$. The unitriangular matrices $I_n + J$ and $I_n + B_1^{-1}AB$ fulfill the condition in Lemma 4. Hence, there exists $B_2 \in UT_n(\mathcal{F})$ such that $I_n + J = B_2^{-1}(I_n + B_1^{-1}AB_1)B_2$. Then $J = B_2^{-1}(B_1^{-1}AB_1)B_2$. Taking $B = B_1B_2 \in T_n(\mathcal{F})$, we get $J = B^{-1}AB$ as wanted. ■

Lemma 6. *Let $A = \sum_{i<j} a_{ij} E_{ij}$ be a matrix in $N_n(\mathcal{F})$ with $a_{i,i+1} \neq 0$ for every $i = 1, \ldots, n-1$. Then there exists $\lambda_A \in \mathcal{F}$ such that $f(A) = \lambda_A A$.*

Proof. Since $A = \sum_{1 \leq i < j \leq n} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $T \in T_n(\mathcal{F})$ such that $TAT^{-1} = J$ by the previous corollary. Define $h : N_n(\mathcal{F}) \to N_n(\mathcal{F})$ by $h(X) = Tf(T^{-1}XT)T^{-1}$. Then h is a nonlinear Jordan centralizer map. Indeed, $\forall X, Y \in N_n(\mathcal{F})$, we have:

$$\begin{aligned}
h(XY + YX) &= Tf\big(T^{-1}(XY + YX)T\big)T^{-1} \\
&= Tf\big(T^{-1}(XY + YX)T\big)T^{-1} \\
&= Tf\big(T^{-1}XTT^{-1}YT + T^{-1}YTT^{-1}XT\big)T^{-1} \\
&= Tf\big((T^{-1}XT)(T^{-1}YT) + (T^{-1}YT)(T^{-1}XT)\big)T^{-1} \\
&= T\big[(T^{-1}XT)f(T^{-1}YT) + f(T^{-1}YT)(T^{-1}XT)\big]T^{-1} \\
&= XTf\big(T^{-1}YT\big)T^{-1} + Tf\big(T^{-1}YT\big)T^{-1}X \\
&= Xh(Y) + h(Y)X
\end{aligned}$$

Hence, $h(J) = \lambda_A J$ by lemme 2.2. Then

$$Tf(A)T^{-1} = Tf\big(T^{-1}(TAT^{-1})T\big)T^{-1} = h(J) = \lambda_A J = \lambda_A TAT^{-1}.$$

Multiplying the left and right sides by T^{-1} and T respectively yields $f(A) = \lambda_A A$. ■

Now we wish to extend Lemma 2.3 to all elements of $N_n(\mathcal{F})$. In order to do this, let us introduce the following set:

$$\mathcal{S} = \big\{B = (b_{ij}) \in N_n(\mathcal{F}) : b_{i,i+1} \neq 0 \, \forall i = 1, \ldots, n-1\big\}.$$

This set has an important property that is established below.

Lemma 7. *Let $\mathcal{F}$ be a field. Every element of $N_n(\mathcal{F})$ can be written as a sum of at most two elements of $\mathcal{S}$.*

Proof. If $a_{i,i+1} \neq 0$ for all $i = 1, \ldots, n-1$, then A belongs to $\mathcal{S}$, so there is nothing to prove. If A is not in $\mathcal{S}$, then we can define B_1 and B_2 as follows:

$$(B_1)_{ij} = \begin{cases} a_{i,i+1} - b_i & \text{if } j = i+1 \\ a_{ij} & \text{if } j > i+1, \end{cases} \qquad (B_2)_{ij} = \begin{cases} b_i & \text{if} \quad j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

where b_i is an element in $\mathcal{F}$ different from $a_{i,i+1}$. It is easy to see that B_1, B_2 are in $\mathcal{S}$, and $A = B_1 + B_2$, so we wanted. ■

Lemma 8. *Let $\mathcal{F}$ be a field. For arbitrary elements A, B of $N_n(\mathcal{F})$, there exists $\lambda_{A,B} \in \mathcal{F}$ such that*

$$f(A+B) = f(A) + f(B) + \lambda_{A,B} E_{1n}.$$

Proof. For any A, B, X of $N_n(\mathcal{F})$, we have

$$\begin{aligned} f((A+B)X + X(A+B)) &= Xf(A+B) + f(A+B)X \\ &= Xf(A+B) + f(A+B)X \\ &= Af(X) + f(X)A + Bf(X) + f(X)B \\ &= f(AX + XA) + f(BX + XB) \\ &= Xf(A) + f(A)X + Xf(B) + f(B)X \end{aligned}$$

hence

$$X(f(A) + f(B) - f(A+B)) = (f(A+B) - f(B) - f(A))\,X$$

which implies that $(f(A+B) - f(A) - f(B))^2 \in \mathcal{Z}(N_n(\mathcal{F}))$. Thus, there exists $\lambda_{A,B} \in \mathcal{F}$ such that $f(A+B) = f(A) + f(B) + \lambda_{A,B} E_{1n}$. ■

Now we can prove the main theorem.

Proof of Theorem 1. For every $X \in N_n(\mathcal{F})$ there exists a A, $B \in \mathcal{S}$ such that $X = A + B$.

First take A, $B \in \mathcal{S}$ such that $AB + BA \neq 0$. Then, by Lemma 2.3, $f(A) = \lambda_A A$, $f(B) = \lambda_B B$ for some λ_A, $\lambda_B \in \mathcal{F}$. Since f is nonlinear Jordan centralizer map, the following holds:

$$f(AB + BA) = Af(B) + f(B)A = Bf(A) + f(A)B$$

we must have $\lambda_A = \lambda_B$.

Consider now A and B from $\mathcal{S}$ such that $AB + BA = 0$. Then there exists $C \in \mathcal{S}$ such that the pairs C and A, C and B, C are $AC + CA \neq 0$ and $BC + CB \neq 0$, so we have $\lambda_A = \lambda_C$ and $\lambda_B = \lambda_C$.

Thus, there exists $\lambda \in \mathcal{F}$, $\eta : N_n(\mathcal{F}) \to \mathcal{Z}(N_n(\mathcal{F}))$ nonlinear Jordan centralizer map such that $f(X) = \lambda X + \eta(X)$ for all $X \in N_n(\mathcal{F})$.

we have

$$\begin{aligned} f(XY + YX) &= \lambda(XY + YX) + \eta(XY + YX) \\ &= Xf(Y) + f(Y)X \\ &= X(\lambda Y + \eta(Y)) + (\lambda Y + \eta(Y))X \\ &= \lambda(XY + YX) + X\eta(Y) + \eta(Y)X \end{aligned}$$

we obtain that $\eta(XY + YX) = X\eta(Y) + \eta(Y)X$ for all $X, Y \in N_n(\mathcal{F})$.

Now we use Lemma 2.5 we get $f(X) = \lambda X + \eta(X)$ for all $X \in N_n(\mathcal{F})$, where $\eta : N_n(\mathcal{F}) \to \mathcal{Z}(N_n(\mathcal{F}))$ is a nonlinear Jordan centralizer map and $\eta(X) = 0$ for all $X \in \mathcal{S}$. □

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