The current issue and full text archive of this journal is available on Emerald Insight at:
https://www.emerald.com/insight/1319-5166.htm

Received 28 February 2019
Accepted 29 June 2019

On warped product bi-slant submanifolds of Kenmotsu manifolds

Siraj Uddin
Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Ion Mihai
Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania, and
Adela Mihai
Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Abstract
Chen (2001) initiated the study of CR-warped product submanifolds in Kaehler manifolds and established a general inequality between an intrinsic invariant (the warping function) and an extrinsic invariant (second fundamental form).

In this paper, we establish a relationship for the squared norm of the second fundamental form (an extrinsic invariant) of warped product bi-slant submanifolds of Kenmotsu manifolds in terms of the warping function (an intrinsic invariant) and bi-slant angles. The equality case is also considered. Some applications of derived inequality are given.

Keywords Warped products, Semi-slant, Pseudo-slant, Bi-slant submanifolds, Warped product bi-slant submanifolds, Kenmotsu manifolds

Paper type Original Article

1. Introduction

In [18], K. Kenmotsu studied one class of almost contact metric manifolds known as *Kenmotsu manifolds*. He proved that:

JEL Classification — 53C15, 53C40, 53C42, 53B25

Declaration of Competing Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The publisher wishes to inform readers that the article "On warped product bi-slant submanifolds of Kenmotsu manifolds" was originally published by the previous publisher of the *Arab Journal of Mathematical Sciences* and the pagination of this article has been subsequently changed. There has been no change to the content of the article. This change was necessary for the journal to transition from the previous publisher to the new one. The publisher sincerely apologises for any inconvenience caused. To access and cite this article, please use Uddin, S., Mihai, I., Mihai, A. (2019), "On warped product bi-slant submanifolds of Kenmotsu manifolds", *Arab Journal of Mathematical Sciences*, Vol. 27 No. 1, pp. 2-14. The original publication date for this paper was 05/07/2019.

Arab Journal of Mathematical Sciences
Vol. 27 No. 1, 2021
pp. 2-14
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1016/j.ajmsc.2019.06.001

1. Locally a Kenmotsu manifold is a warped product $I \times_f M$ of an interval I and a Kaehler manifold M, with warping function $f = ce^t$, where c is a nonzero constant.

2. A Kenmotsu manifold with constant sectional curvature is a space of constant curvature -1, and so it is locally a hyperbolic space.

A $(2m+1)$-dimensional manifold $\tilde{M}$ is said to be an *almost contact manifold* if it admits an endomorphism φ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η, which satisfy:

$$\varphi^2 = -I + \eta \otimes \xi, \varphi\xi = 0, \eta(\xi) = 0, \eta \circ \varphi = 0. \tag{1.1}$$

There exists a compatible metric g, which satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi), \tag{1.2}$$

for all vector fields X, Y on $\tilde{M}$ [6]. In addition, an almost contact metric manifold $\tilde{M}$ is said to be a *Kenmotsu manifold* [18] if the relation

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{1.3}$$

holds, where $\tilde{\nabla}$ is the Levi-Civita connection of g. From (1.3), for a Kenmotsu manifold $\tilde{M}$, we also have

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi. \tag{1.4}$$

As Kenmotsu manifolds are warped product manifolds, therefore it is interesting to investigate the geometry of its warped product submanifolds. The notion of warped submanifolds was first introduced by B.-Y. Chen as a CR-warped product submanifold of Kaehler manifolds in his series of articles [11,12]. He established a general sharp inequality between the main extrinsic invariant (the second fundamental form) and an intrinsic invariant (the warping function) of such submanifolds. Motivated by Chen's work many geometers studied warped product submanifolds for different spaces (see, e.g., [4,17,19–25,29,30] among many others. For the most up-to-date overview of this subject, see [13–15]).

On the other hand, J.L. Cabrerizro et al. studied in [7] bi-slant submanifolds of almost contact metric manifolds. In [28], the first author and B.-Y. Chen investigated warped product bi-slant submanifolds in Kaehler manifolds. They proved that there do not exist any warped product bi-slant submanifolds of Kaehler manifolds other than hemi-slant warped products and CR-warped products. The non-existence of warped product bi-slant submanifolds is proved in [2] for cosymplectic manifolds.

In this paper, we study warped product bi-slant submanifolds of a Kenmotsu manifold. The geometry of such submanifolds in Kenmotsu manifolds is quite different from Sasakian and cosymplectic case because in case of Kenmotsu manifolds such submanifolds exist while there is no proper warped product bi-slant submanifolds in Sasakain and cosymplectic as well. On their existence, we establish a generalized Chen type sharp inequality for the squared norm of the second fundamental form in terms of the warping function and bi-slant angles. The equality case is considered. Some applications are given in the last section.

2. Preliminaries

Let $\psi : M^n \to \tilde{M}^{(n+d)}$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+d)$-dimensional Riemannian manifold $\tilde{M}$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas are respectively given by [31]

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.2}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_V is the shape operator of M with respect to V. Moreover, $h : TM \times TM \to T^{\perp}M$ is the second fundamental form of M in $\tilde{M}$. Furthermore, A_V and h are related by

$$g(h(X, Y), V) = g(A_V X, Y), \tag{2.3}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$.

A submanifold M of a Riemannian manifold $\tilde{M}$ is said to be a *totally umbilical submanifold* if $h(X, Y) = g(X, Y)H$, for any $X, Y \in TM$, where $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ is the mean curvature vector of M. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$. Also, one denotes

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \quad h^r_{ij} = g(h(e_i, e_j), e_r), \tag{2.4}$$

with $i, j = 1, \ldots, n; r = n+1, \ldots, n+d$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{n+1}, \ldots, e_{n+d}\}$ an orthonormal basis of the normal space $T^{\perp}_p$, for any $p \in M$.

For a differentiable function f on a m-dimensional manifold $\tilde{M}$, the gradient $\overrightarrow{\nabla} f$ of f is defined as $g(\overrightarrow{\nabla} f, X) = X(f)$, for any X tangent to $\tilde{M}$.

For any vector field X tangent to M, we write

$$\varphi X = TX + FX, \tag{2.5}$$

where TX is the tangential component and FX is the normal component of φX. Thus, T is an endomorphism on the tangent bundle TM and F is a normal bundle valued 1-form of TM. A submanifold M is called *invariant* if F is identically zero, that is, $\varphi X \in TM$ for any $X \in TM$; while, M is *anti-invariant* if T is identically zero, that is, $\varphi X \in T^{\perp}M$, for any $X \in TM$.

Similarly, for any vector field V normal to M, we put

$$\varphi V = tV + fV, \tag{2.6}$$

where tV and fV are the tangential and normal components of φV, respectively.

Let M be a submanifold of an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$. Hence, if we denote $\mathfrak{D}$ the orthogonal distribution to ξ in TM, then $TM = \mathfrak{D} \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by ξ. For any nonzero vector X tangent to M at the point $p \in M$, such that X is not proportional to ξ_p, we denote by $\theta(X)$, the angle between φX and T_pM. In fact, since $\varphi\xi = 0$, $\theta(X)$ agrees with the angle between φX and $\mathfrak{D}_p$. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be slant [8], if for any non-zero vector X tangent to M at p such that X is not proportional to ξ_p, the angle $\theta(X)$ between φX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle\xi_p\rangle$. In this case $\mathfrak{D}$ is a slant distribution with slant angle θ.

A slant submanifold is said to be *proper slant*, if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$ [9,10]. We note that on a slant submanifold if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

A characterization of slant submanifolds was given in [8] as follows:

Theorem 1 *([8]). Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant* $\lambda \in [0, 1]$ *such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{2.7}$$

Furthermore, in such case, if θ is slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequences of (2.7)

$$g(TX, TY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.8}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.9}$$

for any X, Y tangent to M.

The following useful relation is obtained as a consequence of (2.7) in [26].

Theorem 2 ([26]). *Let M be a proper slant submanifold of an almost contact metric manifold $\tilde{M}$. Then*

$$(a) \quad tFX = \sin^2\theta(-X + \eta(X)\xi), \quad (b) \quad fFX = -FTX, \tag{2.10}$$

for any $X \in TM$.

Another characterization of slant submanifolds was given in [7]:

Theorem 3 ([7]). *Let $\mathfrak{D}$ be a distribution on M, orthogonal to ξ. Then, $\mathfrak{D}$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $(PT)^2X = -\lambda X$, for any $X \in \mathfrak{D}_p$ at $p \in M$, where P denotes the orthogonal projection on $\mathfrak{D}$. Furthermore, in this case, $\lambda = \cos^2\theta_{\mathfrak{D}}$.*

J.L. Cabrerizo et al. [7] defined bi-slant submanifolds as follows:

Definition 1. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *bi-slant* if there exists a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ on M such that:

(i) The tangent bundle TM admits the orthogonal direct decomposition: $TM = \mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus \langle\xi\rangle$.

(ii) Each $\mathfrak{D}_i$ $\forall i = 1, 2$ is a slant distribution with slant angle θ_i.

Given a bi-slant submanifold M, for any $X \in TM$, we put

$$X = P_1X + P_2X + \eta(X)\xi, \tag{2.11}$$

where P_iX denotes the component of X in $\mathfrak{D}_i$, for any $i = 1, 2$. In particular, if $X \in \mathfrak{D}_i$, then we obtain $X = P_iX$. If we define $T_i = P_i \circ T$, then we have

$$\varphi X = T_1X + T_2X + FX, \tag{2.12}$$

for any $X \in TM$. Given $i = 1, 2$, from Theorem 3, we get

$$T_i^2X = -\cos^2\theta_i X, \tag{2.13}$$

for any $X \in \mathfrak{D}_i$.

Non-trivial examples of bi-slant submanifolds are given in [7].

3. Warped product bi-slant submanifolds

Let $M_1 \times_f M_2$ be a warped product manifold of two Riemannian manifolds M_1 and M_2. Then from a result of [5], we have

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z \tag{3.1}$$

for any vector fields X, Z tangent to M_1, M_2, respectively.

Recently, B.-Y. Chen and the first author introduced the notion of warped product bi-slant submanifolds of Kaehler manifolds [28]. They proved the non-existence of proper warped product bi-slant submanifolds. Then, they introduced the notion of warped product pointwise bi-slant submanifolds and obtained several fundamental results [16]. In this section, we give some useful lemmas for warped product bi-slant submanifolds of Kenmotsu manifolds. First we define these submanifolds as follows:

A warped product $M_1 \times_f M_2$ of two slant submanifolds M_1 and M_2 with slant angles θ_1 and θ_2 of a Kenmotsu manifold $\tilde{M}$ is called a *warped product bi-slant submanifold*.

A warped product bi-slant submanifold $M = M_1 \times_f M_2$ is called *proper* if both M_1 and M_2 are proper slant submanifolds with slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$ of $\tilde{M}$. A warped product bi-slant submanifold $M_1 \times_f M_2$ is a contact CR-warped product if $\theta_1 = 0, \theta_2 = \frac{\pi}{2}$ or $\theta_2 = 0, \theta_1 = \frac{\pi}{2}$; such submanifolds were discussed in [3,27]. Also, it is a warped product pseudo-slant submanifold if $\theta_1 = \theta$ and $\theta_2 = \frac{\pi}{2}$ [1] or $\theta_2 = \theta$ and $\theta_1 = \frac{\pi}{2}$ [23]. The warped product bi-slant submanifolds with slant angles $\theta_1 = 0, \theta_2 = \theta$ or $\theta_2 = 0, \theta_1 = \theta$ were discussed in [25,30].

Let $M = M_1 \times_f M_2$ be a warped product bi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M, where M_1 and M_2 are proper slant submanifolds of $\tilde{M}$. Then, we distinguish 2 cases:

(i) ξ is tangent to M_1;

(ii) ξ is tangent to M_2.

From (1.4), (2.1) and (3.1), the second case is trivial i.e., there does not exist any proper warped product bi-slant submanifold of a Kenmotsu manifold when the structure vector field is tangent to the fiber.

Now, we start with the case (i). Throughout this paper, we assume that the tangent spaces of M_1 and M_2, respectively are $\mathfrak{D}_1$ and $\mathfrak{D}_2$. From now on, we use the following conventions: X_1, Y_1 are vector fields in $\mathfrak{D}_1$ and X_2, Y_2 are vector fields in $\mathfrak{D}_2$.

Lemma 1. *Let $M = M_1 \times_f M_2$ be a warped product bi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \mathfrak{D}_1$. Then, we have*

(i) $\xi(\ln f) = 1$;

(ii) $g(h(X_1, Y_1), FX_2) = g(h(X_1, X_2), FY_1)$,

for any $X_1, Y_1 \in \mathfrak{D}_1$ and $X_2 \in \mathfrak{D}_2$.

Proof. First part is trivial and can be obtained by using (1.4), (2.1) and (3.1). For the second part, we have

$$g(\tilde{\nabla}_{X_1} Y_1, X_2) = -g(\nabla_{X_1} X_2, Y_1) = -X_1(\ln f)\, g(Y_1, X_2) = 0, \tag{3.2}$$

for any $X_1, Y_1 \in \mathfrak{D}_1$ and $X_2 \in \mathfrak{D}_2$. Also, from (1.2) and the fact that ξ is tangent to M_1, we have

$$g(\tilde{\nabla}_{X_1} Y_1, X_2) = g(\varphi\tilde{\nabla}_{X_1} Y_1, \varphi X_2) = g(\tilde{\nabla}_{X_1}\varphi Y_1, \varphi X_2) - g\big((\tilde{\nabla}_{X_1}\varphi) Y_1, \varphi X_2\big). \tag{3.3}$$

Form (3.2) and (3.3), we obtain

$$\begin{aligned} g((\tilde{\nabla}_{X_1}\varphi) Y_1, \varphi X_2) &= g(\tilde{\nabla}_{X_1}\varphi Y_1, \varphi X_2) = g(\tilde{\nabla}_{X_1} T_1 Y_1, T_2 X_2) \\ &\quad + g(\tilde{\nabla}_{X_1} T_1 Y_1, FX_2) + g(\tilde{\nabla}_{X_1} FY_1, \varphi X_2). \end{aligned}$$

Then using the covariant derivative property of Riemannian connection and (1.2), (1.3), (2.1) and (3.1), we derive

$$\eta(Y_1)\, g(T_1X_1, T_2X_2) = -X_1(\ln f)\, g(T_1Y_1, T_2X_2) + g(h(X_1, T_1Y_1), FX_2) + g((\tilde{\nabla}_{X_1}\varphi)FY_1, X_2) - g(\tilde{\nabla}_{X_1}\varphi FY_1, X_2).$$

By the orthogonality of vector fields, the left hand side and the first term in the right hand side vanish identically. Then using (1.3), (2.6) and (2.10), we find

$$0 = g(h(X_1, T_1Y_1), FX_2) + \sin^2\theta_1\, g\,(\tilde{\nabla}_{X_1}Y_1, X_2) - \sin^2\theta_1\, g\,(\tilde{\nabla}_{X_1}\xi, X_2) + g(\tilde{\nabla}_{X_1}FT_1Y_1, X_2).$$

Using (1.4), (2.2), (2.3), (3.1) and the orthogonality of vector fields, we derive

$$g(h(X_1, X_2), FT_1Y_1) = g(h(X_1, T_1Y_1), FX_2). \tag{3.4}$$

Interchanging Y_1 by T_1Y_1 in (3.4), we obtain

$$\cos^2\theta_1\, g(h(X_1, X_2), FY_1) = \cos^2\theta_1 g(h(X_1, Y_1), FX_2) - \cos^2\theta_1\eta(Y_1)\, g\,(h(X_1, \xi), X_2). \tag{3.5}$$

Since for a submanifold M of a Kenmotsu manifold $\tilde{M}$, $h(X, \xi) = 0, \forall X \in TM$, then the second part of the lemma follows from above relation. Hence, the proof is complete. ∎

Lemma 2. *Let $M = M_1 \times_f M_2$ be a warped product bi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \mathfrak{D}_1$, where M_1 and M_2 are proper slant submanifolds of $\tilde{M}$ with slant angles θ_1 and θ_2, respectively. Then, we have*

$$g(h(X_2, Y_2), FT_1X_1) - g(h(X_2, T_1X_1), FY_2) = T_1X_1(\ln f)\, g(X_2, T_2Y_2) - \cos^2\theta_1(X_1(\ln f) - \eta(X_1))\, g(X_2, Y_2) \tag{3.6}$$

for any $X_1 \in \mathfrak{D}_1$ and $X_2, Y_2 \in \mathfrak{D}_2$.

Proof. For any $X_1 \in \mathfrak{D}_1$ and $X_2, Y_2 \in \mathfrak{D}_2$, we have

$$g(\tilde{\nabla}_{X_2}X_1, Y_2) = X_1(\ln f)\, g(X_2, Y_2). \tag{3.7}$$

On the other hand, we also have

$$g(\tilde{\nabla}_{X_2}X_1, Y_2) = g(\varphi\tilde{\nabla}_{X_2}X_1, \varphi Y_2) = g(\tilde{\nabla}_{X_2}\varphi X_1, \varphi Y_2) - g\big((\tilde{\nabla}_{X_2}\varphi)X_1, \varphi Y_2\big).$$

Using (1.3) and (2.5), we arrive at

$$g(\tilde{\nabla}_{X_2}X_1, Y_2) = g(\tilde{\nabla}_{X_2}T_1X_1, T_2Y_2) + g(\tilde{\nabla}_{X_2}T_1X_1, FY_2) + g(\tilde{\nabla}_{X_2}FX_1, \varphi Y_2) + \eta(X_1)\, g\,(\varphi X_2, \varphi Y_2).$$

From (1.2), (2.1) and (3.1), the above equation takes the form

$$g(\tilde{\nabla}_{X_2}X_1, Y_2) = T_1X_1(\ln f)g(X_2, T_2Y_2) + g(h(X_2, T_1X_1), FY_2) - g(\tilde{\nabla}_{X_2}\varphi FX_1, Y_2) + g((\tilde{\nabla}_{X_2}\varphi)FX_1, Y_2) + \eta(X_1)\, g\,(X_2, Y_2).$$

Again, using (1.3) and (2.6), we derive

$$g(\tilde{\nabla}_{X_2}X_1, Y_2) = T_1X_1(\ln f)\, g\,(X_2, T_2Y_2) + g(h(X_2, T_1X_1), FY_2) - g(\tilde{\nabla}_{X_2}t\, FX_1, Y_2) - g(\tilde{\nabla}_{X_2}f\, FX_1, Y_2) + \eta(X_1)\, g\,(X_2, Y_2).$$

Then, from (2.10) and (3.1), we obtain

$$\begin{aligned} g(\tilde{\nabla}_{X_2} X_1, Y_2) &= T_1X_1(\ln f)\, g(X_2, T_2Y_2) + g(h(X_2, T_1X_1), FY_2) \\ &+ \sin^2\theta_1 X_1(\ln f)\, g(X_2, Y_2) \\ &- \sin^2\theta_1 \eta(X_1)\, g(X_2, Y_2) + g(\tilde{\nabla}_{X_2} FT_1X_1, Y_2) + \eta(X_1)\, g(X_2, Y_2) \\ &= T_1X_1(\ln f)\, g(X_2, T_2Y_2) + g(h(X_2, T_1X_1), FY_2) \\ &+ \sin^2\theta_1 X_1(\ln f)\, g(X_2, Y_2) \\ &+ \cos^2\theta_1 \eta(X_1)\, g(X_2, Y_2) - g(h(X_2, Y_2), FT_1X_1). \end{aligned} \tag{3.8}$$

Thus, the result follows from (3.7) and (3.8), which proves the lemma completely. ■

The following useful relations are easily derived by interchanging X_1 by T_1X_1, X_2 by T_2X_2 and Y_2 by T_2Y_2 in Lemma 2.

$$\begin{aligned} g(h(X_2, Y_2), FX_1) - g(h(X_1, X_2), FY_2) &= T_1X_1(\ln f)\, g(X_2, Y_2) \\ &- (X_1(\ln f) - \eta(X_1))\, g(T_2X_2, Y_2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} g(h(T_2X_2, Y_2), FT_1X_1) - g(h(T_2X_2, T_1X_1), FY_2) &= \cos^2\theta_2 T_1X_1(\ln f)\, g(X_2, Y_2) \\ &- \cos^2\theta_1 (X_1(\ln f) \\ &- \eta(X_1))\, g(T_2X_2, Y_2), \end{aligned} \tag{3.10}$$

$$\begin{aligned} g(h(X_2, T_2Y_2), FT_1X_1) - g(h(X_2, T_1X_1), FT_2Y_2) &= -\cos^2\theta_2 T_1X_1(\ln f)\, g(X_2, Y_2) \\ &+ \cos^2\theta_1 (X_1(\ln f) \\ &- \eta(X_1))\, g(T_2X_2, Y_2), \end{aligned} \tag{3.11}$$

$$\begin{aligned} g(h(T_2X_2, T_2Y_2), FT_1X_1) - g(h(T_2X_2, T_1X_1), FT_2Y_2) &= -\cos^2\theta_2 T_1X_1(\ln f) \\ &\quad g(T_2X_2, Y_2) \\ &- \cos^2\theta_1 \cos^2\theta_2 (X_1(\ln f) \\ &- \eta(X_1))\, g(X_2, Y_2), \end{aligned} \tag{3.12}$$

$$\begin{aligned} g(h(T_2X_2, Y_2), FX_1) - g(h(T_2X_2, X_1), FY_2) &= T_1X_1(\ln f)\, g(T_2X_2, Y_2) \\ &+ \cos^2\theta_2 (X_1(\ln f) \\ &- \eta(X_1))\, g(X_2, Y_2), \end{aligned} \tag{3.13}$$

$$\begin{aligned} g(h(X_2, T_2Y_2), FX_1) - g(h(X_2, X_1), FT_2Y_2) &= T_1X_1(\ln f)\, g(X_2, T_2Y_2) \\ &- \cos^2\theta_2 (X_1(\ln f) \\ &- \eta(X_1))\, g(X_2, Y_2) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} g(h(T_2X_2, T_2Y_2), FX_1) - g(h(T_2X_2, X_1), FT_2Y_2) &= \cos^2\theta_2 T_1X_1(\ln f)\, g(X_2, Y_2) \\ &- \cos^2\theta_2 (X_1(\ln f) - \eta(X_1)) \\ &\quad g(T_2X_2, Y_2). \end{aligned} \tag{3.15}$$

4. An inequality for warped product bi-slant submanifolds

Let $M = M_1 \times_f M_2$ be a warped product bi-slant submanifolds of a Kenmotsu manifold $\tilde{M}$; we decompose the normal bundle of M as follows

$$T^{\perp}M = F\mathfrak{D}_1 \oplus F\mathfrak{D}_2 \oplus \mu, \quad \mu \perp F\mathfrak{D}_1 \oplus F\mathfrak{D}_2, \tag{4.1}$$

where μ is a φ-invariant normal subbundle of $T^{\perp}M$.

A warped product bi-slant submanifold $M = M_1 \times_f M_2$ of a Kenmotsu manifold $\tilde{M}$ is said to be *mixed totally geodesic*, if $h(X, Z) = 0$, for any $X \in \mathfrak{D}_1$ and $Z \in \mathfrak{D}_2$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the tangent bundles of M_1 and M_2, respectively.

Now, we set the following frame fields for an n-dimensional warped product bi-slant submanifold $M = M_1 \times_f M_2$ of a $(2m+1)$-dimensional Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_1, where M_1 and M_2 are proper slant submanifolds of $\tilde{M}$ with slant angles θ_1 and θ_2 respectively. Let us consider the dimensions $\dim(M_1) = 2p+1$ and $\dim M_2 = 2q$, i.e., $n = 2p + 1 + 2q$. Then the orthonormal frames of the corresponding tangent spaces $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively are given by $\{e_1, \ldots, e_p, e_{p+1} = \sec\theta_1 T_1 e_1, \ldots, e_{2p} = \sec\theta_1 T_1 e_p, e_{2p+1} = \xi\}$ and $\{e_{2p+2} = \bar{e}_1, \ldots, e_{2p+1+q} = \bar{e}_q, e_{2p+q+2} = \bar{e}_{q+1} = \sec\theta_2 T_2 \bar{e}_1, \ldots, e_n = \bar{e}_{2q} = \sec\theta_2 T_2 \bar{e}_q\}$. Thus, the orthonormal frame fields of the normal subbundles of $F\mathfrak{D}_1, F\mathfrak{D}_2$ and μ, respectively are $\{e_{n+1} = \tilde{e}_1 = \csc\theta_1 F e_1, \ldots, e_{n+p} = \tilde{e}_p = \csc\theta_1 F e_p, e_{n+p+1} = \tilde{e}_{p+1} = \csc\theta_1 \sec\theta_1 F T_1 e_1, \ldots, e_{n+2p} = \tilde{e}_{2p} = \csc\theta_1 \sec\theta_1 F T_1 e_p\}$, $\{e_{n+2p+1} = \tilde{e}_{2p+1} = \hat{e}_1 = \csc\theta_2 F\bar{e}_1, \ldots, e_{n+2p+q} = \tilde{e}_{2p+q} = \hat{e}_q = \csc\theta_2 F\bar{e}_q, e_{n+2p+q+1} = \tilde{e}_{2p+q+1} = \hat{e}_{q+1} = \csc\theta_2 \sec\theta_2 F T_2 \bar{e}_1, \ldots, e_{n+2p+2q} = \tilde{e}_{2p+2q} = \hat{e}_{2q} = \csc\theta_2 \sec\theta_2 F T_2 \bar{e}_q\}$ and $\{e_{2n} = \tilde{e}_n, \ldots, e_{2m+1} = \tilde{e}_{2(m-n+1)}\}$.

Now, using the results of Section 3 and the above frame fields, we give following main result of this paper.

Theorem 4. *Let $M = M_1 \times_f M_2$ be a mixed totally geodesic warped product bi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_1, where M_1 and M_2 are proper slant submanifolds of $\tilde{M}$ with slant angles θ_1 and θ_2, respectively. Then*

(i) The second fundamental form h of M satisfies the following inequality

$$\|h\|^2 \geq 2q \csc^2\theta_1 \left(\cos^2\theta_1 + \cos^2\theta_2\right)\left(\|\vec{\nabla}(\ln f)\|^2 - 1\right), \tag{4.2}$$

where $2q = \dim M_2$ and $\vec{\nabla}(\ln f)$ is the gradient of $\ln f$ along M_1.

(ii) If equality sign in (i) holds identically, then:

(a) M_1 is a totally geodesic submanifold of $\tilde{M}$;

(b) M_2 is a totally umbilical submanifold of $\tilde{M}$.

Proof. From (2.4), we have

$$\|h\|^2 = \sum_{i,\,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,\,j=1}^{n} g(h(e_i, e_j), e_r)^2.$$

Splitting the above expression for the tangent bundles of M_1 and M_2, we derive

$$\begin{aligned}\|h\|^2 = &\sum_{r=1}^{2m+1} \sum_{i,\,j=1}^{2p+1} g(h(e_i, e_j), e_r)^2 + 2\sum_{r=1}^{2m+1} \sum_{i=1}^{2p+1} \sum_{j=1}^{2q} g(h(e_i, \bar{e}_j), e_r)^2 \\ &+ \sum_{r=1}^{2m+1} \sum_{i,\,j=1}^{2q} g(h(\bar{e}_i, \bar{e}_j), e_r)^2.\end{aligned} \tag{4.3}$$

Since M is mixed totally geodesic, then the second term in the right hand side of (4.3) is identically zero. Thus, we find

$$\begin{aligned}\|h\|^2 = &\sum_{r=1}^{2p}\sum_{i,\,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 + \sum_{r=1}^{2q}\sum_{i,\,j=1}^{2p+1} g(h(e_i,e_j),\widehat{e}_r)^2 \\ &+ \sum_{r=n}^{2(m-n+1)}\sum_{i,\,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 \\ &+ \sum_{r=1}^{2p}\sum_{i,\,j=1}^{2q} g(h(\overline{e}_i,\overline{e}_j),\tilde{e}_r)^2 + \sum_{r=1}^{2q}\sum_{i,\,j=1}^{2q} g(h(\overline{e}_i,\overline{e}_j),\widehat{e}_r)^2 \\ &+ \sum_{r=n}^{2(m-n+1)}\sum_{i,\,j=1}^{2q} g(h(\overline{e}_i,\overline{e}_j),\tilde{e}_r)^2.\end{aligned} \tag{4.4}$$

The third and sixth terms have μ-components and we could not find any relation for warped products in terms of μ-components, therefore we shall leave these two terms. Also, we could not find any relations for $g(h(e_i,e_j),\tilde{e}_r), i,j = 1,\ldots,2p+1, r = 1,\ldots,2p$ and $g(h(\overline{e}_i,\overline{e}_j),\widehat{e}_r), i,j,r = 1,\ldots,2q$. Therefore, we also leave the first and fifth terms. By using Lemma 1(ii) for a mixed totally geodesic warped product the second term in the right hand side is also zero. Thus, the evaluated term is only fourth term which can be expressed by using the constructed frame fields as follows

$$\begin{aligned}\|h\|^2 \geq &\csc^2\theta_1 \sec^2\theta_2 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(\overline{e}_i, T_2\overline{e}_j), Fe_r)^2 \\ &+ \csc^2\theta_1 \sec^2\theta_2 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(T_2\overline{e}_i, \overline{e}_j), Fe_r)^2 \\ &+ \csc^2\theta_1 \sec^2\theta_1 \sec^2\theta_2 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(\overline{e}_i, T_2\overline{e}_j), FT_1e_r)^2 \\ &+ \csc^2\theta_1 \sec^2\theta_1 \sec^2\theta_2 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(T_2\overline{e}_i, \overline{e}_j), FT_1e_r)^2 \\ &+ \csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(\overline{e}_i,\overline{e}_j), Fe_r)^2 \\ &+ \csc^2\theta_1 \sec^2\theta_1 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(\overline{e}_i,\overline{e}_j), FT_1e_r)^2 \\ &+ \csc^2\theta_1 \sec^4\theta_2 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(T_2\overline{e}_i, T_2\overline{e}_j), Fe_r)^2 \\ &+ \csc^2\theta_1 \sec^2\theta_1 \sec^4\theta_2 \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g(h(T_2\overline{e}_i, T_2\overline{e}_j), FT_1e_r)^2.\end{aligned} \tag{4.5}$$

Using Lemma 2 and the relations (3.9)–(3.15) for a mixed totally geodesic warped product submanifold, we derive

$$\begin{aligned}
\|h\|^2 \geq & 2q\csc^2\theta_1\cos^2\theta_2\sum_{r=1}^{p}(e_r(\ln f)-\eta(e_r))^2 \\
& + 2q\csc^2\theta_1\sec^2\theta_1\cos^2\theta_2\sum_{r=1}^{p}(T_1e_r(\ln f))^2 \\
& + 2q\csc^2\theta_1\sum_{r=1}^{p}(T_1e_r(\ln f))^2 + 2q\csc^2\theta_1\cos^2\theta_1\sum_{r=1}^{p}(e_r(\ln f)-\eta(e_r))^2 \\
& = 2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\sum_{r=1}^{p}(e_r(\ln f)-\eta(e_r))^2 \\
& + 2q\csc^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)\sum_{r=1}^{p}(T_1e_r(\ln f))^2.
\end{aligned} \tag{4.6}$$

Since $\eta(e_r)=0, r=1,\ldots,2p$, then above expression will be

$$\begin{aligned}
\|h\|^2 \geq & 2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\sum_{r=1}^{p}(e_r(\ln f))^2 \\
& + 2q\csc^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)\sum_{r=1}^{2p+1}(T_1e_r(\ln f))^2 \\
& - 2q\csc^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)\sum_{r=p+1}^{2p}g(e_r, T_1\overrightarrow{\nabla}(\ln f))^2 \\
& - 2q\csc^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)(T_1e_{2p+1}(\ln f))^2.
\end{aligned} \tag{4.7}$$

Since $T_1e_{2p+1}=T_1\xi=0$, (4.7) can be written as

$$\begin{aligned}
\|h\|^2 \geq & 2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\sum_{r=1}^{p}(e_r(\ln f))^2 \\
& + 2q\csc^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)\|T_1\overrightarrow{\nabla}(\ln f)\|^2 \\
& - 2q\csc^2\theta_1\sec^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)\sum_{r=1}^{p}g(T_1e_r, T_1\overrightarrow{\nabla}(\ln f))^2.
\end{aligned} \tag{4.8}$$

Now, we compute the second term as follows

$$\|T_1\overrightarrow{\nabla}(\ln f)\|^2 = g(T_1\overrightarrow{\nabla}(\ln f), T_1\overrightarrow{\nabla}(\ln f)).$$

Using (2.8), we get

$$\|T_1\overrightarrow{\nabla}(\ln f)\|^2 = \cos^2\theta_1\left(g(\overrightarrow{\nabla}(\ln f), \overrightarrow{\nabla}(\ln f)) - (\eta(\overrightarrow{\nabla}\ln f))^2\right).$$

Using the gradient definition and Lemma 1(i), we get

$$\|T_1\overrightarrow{\nabla}(\ln f)\|^2 = \cos^2\theta_1\left(\|\overrightarrow{\nabla}(\ln f)\|^2 - 1\right). \tag{4.9}$$

Then from (4.8) and (4.9), we derive

$$\begin{aligned}\|h\|^2 \geq\ & 2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\sum_{r=1}^{p}\left(e_r(\ln f)\right)^2\\ &+2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\left(\|\overrightarrow{\nabla}(\ln f)\|^2-1\right)\\ &-2q\csc^2\theta_1\left(1+\cos^2\theta_2\sec^2\theta_1\right)\csc^2\theta_1\sum_{r=1}^{p}g(e_r,\overrightarrow{\nabla}(\ln f))^2\\ &=2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\sum_{r=1}^{p}\left(e_r(\ln f)\right)^2\\ &+2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\left(\|\overrightarrow{\nabla}(\ln f)\|^2-1\right)\\ &-2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\sum_{r=1}^{p}\left(e_r(\ln f)\right)^2\\ &=2q\csc^2\theta_1\left(\cos^2\theta_1+\cos^2\theta_2\right)\left(\|\overrightarrow{\nabla}(\ln f)\|^2-1\right),\end{aligned}$$

which is inequality (4.2). For the equality case, since M is mixed totally geodesic, then

$$h(\mathfrak{D}_1,\mathfrak{D}_2)=\{0\}. \tag{4.10}$$

From the leaving third and sixth terms in the right hand side of (4.4), we have

$$h(X,Y)\perp\mu, \tag{4.11}$$

for any $X,Y\in TM$. Also, from the leaving first term of (4.4), we find

$$h(\mathfrak{D}_1,\mathfrak{D}_1)\perp F\mathfrak{D}_1. \tag{4.12}$$

Then from (4.11) and (4.12), we conclude that

$$h(\mathfrak{D}_1,\mathfrak{D}_1)\subset F\mathfrak{D}_2. \tag{4.13}$$

But for a mixed totally geodesic submanifold, from Lemma 1(ii), we get

$$h(\mathfrak{D}_1,\mathfrak{D}_1)\perp F\mathfrak{D}_2. \tag{4.14}$$

Thus, from (4.13) and (4.14), we find

$$h(\mathfrak{D}_1,\mathfrak{D}_1)=\{0\}. \tag{4.15}$$

Since M_1 is totally geodesic in M [5,11], then using this fact with (4.10) and (4.15) we conclude that M_1 is a totally geodesic submanifold of $\tilde{M}$, which is the first relation of inequality. Similarly, from the leaving fifth term in the right hand side of (4.4), we get

$$h(\mathfrak{D}_2,\mathfrak{D}_2)\perp F\mathfrak{D}_2. \tag{4.16}$$

Thus, (4.11) and (4.16) yield

$$h(\mathfrak{D}_2,\mathfrak{D}_2)\subset F\mathfrak{D}_1. \tag{4.17}$$

Since M_2 is totally umbilical in M [5,11], using this fact with (4.10) and (4.17), we conclude that M_2 is a totally umbilical submanifold of $\tilde{M}$, which proves the statement (b). Hence, the proof is complete. ■

5. Some applications of derived inequality

In this section, we give some applications of the derived inequality (4.2).

1. In Theorem 4, if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, then the warped product bi-slant $M_1 \times_f M_2$ takes the form $M_\perp \times_f M_T$ i.e., M is a contact CR-warped product studied in [27]. In this case, the inequality (4.2) will be $\|h\|^2 \geq 2q(\|\overrightarrow{\nabla}(\ln f)\|^2 - 1)$, which is the main Theorem 3.4 of [27].

2. If we consider $\theta_1 = \theta$ and $\theta_2 = 0$ in a warped product bi-slant submanifold $M = M_1 \times_f M_2$, then M is a warped product semi-slant submanifold of the form $M_\theta \times_f M_T$ studied in [30]. Then, the inequality (4.2) change into $\|h\|^2 \geq 2q(\csc^2\theta + \cot^2\theta)(\|\overrightarrow{\nabla}(\ln f)\|^2 - 1)$, which is Theorem 4.2 of [30]. Thus, Theorem 4.2 of [30] is a special case of Theorem 4.

3. Also, if we consider $\theta_1 = \theta$ and $\theta_2 = \frac{\pi}{2}$ in a warped product bi-slant submanifold $M = M_1 \times_f M_2$ of a Kenmotsu manifold $\tilde{M}$, then M turns into a warped product pseudo-slant submanifold of the form $M = M_\theta \times_f M_\perp$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. In this case, if we put $\theta_1 = \theta$ and $\theta_2 = \frac{\pi}{2}$ in Theorem 4, then inequality (4.2) will be $\|h\|^2 \geq 2q\cot^2\theta(\|\overrightarrow{\nabla}(\ln f)\|^2 - 1)$, which the inequality (5.1) of [1]. Hence, Theorem 5.1 of [1] is a special case of Theorem 4.

4. Similarly, if we assume that $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \theta$ in a warped product bi-slant submanifold $M = M_1 \times_f M_2$ of a Kenmotsu manifold $\tilde{M}$, then M is a warped product pseudo-slant submanifold $M = M_\perp \times_f M_\theta$ such that $M_\perp$ is an anti-invariant submanifold and M_θ is a proper slant submanifold of $\tilde{M}$. Thus inequality (4.2) takes the form $\|h\|^2 \geq 2q\cos^2\theta(\|\overrightarrow{\nabla}(\ln f)\|^2 - 1)$, which the inequality (4.1) of [23]. Hence, Theorem 4.1 of [23] is a special case of Theorem 4.

References

[1] F.R. Al-Solamy, M.F. Naghi, S. Uddin, Geometry of warped product pseudo-slant submanifolds of Kenmotsu manifolds, Quaest. Math. 42 (3) (2019) 373–389, http://dx.doi.org/10.2989/16073606.2018.1452800.

[2] L.S. Alqahtani, M.S. Stankovic, S. Uddin, Warped product bi-slant submanifolds of cosymplectic manifolds, Filomat 31 (16) (2017) 5065–5071, http://dx.doi.org/10.2298/FIL1716065A.

[3] K. Arslan, R. Ezentas, I. Mihai, C. Murathan, Contact CR-warped product submanifolds in Kenmotsu space forms, J. Korean Math. Soc. 42 (2005) 1101–1110.

[4] M. Atceken, Warped product semi-slant submanifolds in Kenmotsu manifolds, Turk. J. Math. 36 (2012) 319–330.

[5] R.L. Bishop, B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145 (1969) 1–49.

[6] D.E. Blair, Contact Manifolds in Riemannian Geometry, in Lecture Notes in Mathematics, vol. 509, Springer-Verlag, New York, 1976.

[7] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez, M. Fernandez, Semi-slant submanifolds of a Sasakian manifold, Geom. Dedicata 78 (2) (1999) 183–199.

[8] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez, M. Fernandez, Slant submanifolds in Sasakian manifolds, Glasg. Math. J. 42 (2000) 125–138.

[9] B.-Y. Chen, Geometry of Slant Submanifolds, Katholieke Universiteit Leuven, 1990.

[10] B.-Y. Chen, Slant immersions, Bull. Aust. Math. Soc. 41 (1990) 135–147.

[11] B.-Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds, Monatsh. Math. 133 (2001) 177–195.

[12] B.-Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds II, Monatsh. Math. 134 (2001) 103–119.

[13] B.-Y. Chen, Pseudo-Riemannian Geometry δ-Invariants and Applications, World Scientific, Hackensack, NJ, 2011.

[14] B.-Y. Chen, Geometry of warped product submanifolds: a survey, J. Adv. Math. Stud. 6 (2) (2013) 1–43.

[15] B.-Y. Chen, Differential Geometry of Warped Product Manifolds and Submanifolds, World Scientific, Hackensack, NJ, 2017.

[16] B.-Y. Chen, S. Uddin, Warped product pointwise bi-slant submanifolds of Kaehler manifolds, Publ. Math. Debr. 92 (1–2) (2018) 183–199.

[17] I. Hasegawa, I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, Geom. Dedicata 102 (2003) 143–150.

[18] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J. 24 (1972) 93–103.

[19] K. Matsumoto, I. Mihai, Warped product submanifolds in Sasakian space forms, SUT J. Math. 38 (2) (2002) 135–144.

[20] A. Mihai, Warped product submanifolds in complex space forms, Acta Sci. Math. Szeged 70 (2004) 419–427.

[21] A. Mihai, Warped product submanifolds in quaternion space forms, Rev. Roum. Math. Pures Appl. 50 (2005) 283–291.

[22] C. Murathan, K. Arslan, R. Ezentas, I. Mihai, Warped product submanifolds in Kenmotsu space forms, Taiwanese J. Math. 10 (6) (2006) 1431–1441.

[23] M.F. Naghi, S. Uddin, F.R. Al-Solamy, Warped product submanifolds of Kenmotsu manifolds with slant fiber, Filomat 32 (6) (2018) 2115–2126.

[24] B. Sahin, Warped product pointwise semi-slant submanifolds of Kaehler manifolds, Port. Math. 70 (2013) 252–268.

[25] S. Uddin, Geometry of warped product semi-slant submanifolds of Kenmotsu manifolds, Bull. Math. Sci. 8 (2018) 435–451, http://dx.doi.org/10.1007/s13373-017-0106-9.

[26] S. Uddin, F.R. Al-Solamy, Warped product pseudo-slant immersions in Sasakian manifolds, Publ. Math. Debr. 91 (3–4) (2017) 331–348, http://dx.doi.org/10.5486/PMD.2017.7640.

[27] S. Uddin, A. Alghanemi, M.F. Naghi, F.R. Al-Solamy, Another class of warped product CR-submanifolds in Kenmotsu manifolds, J. Math. Comput. Sci. 17 (2017) 148–157.

[28] S. Uddin, B.-Y. Chen, F.R. Al-Solamy, Warped product bi-slant immersions in Kaehler manifolds, Mediterr. J. Math. 14 (2) (2017) http://dx.doi.org/10.1007/s00009-017-0896-8, Art. 95, p. 11.

[29] S. Uddin, V.A. Khan, K.A. Khan, Warped product submanifolds of a Kenmotsu manifold, Turk. J. Math. 36 (2012) 319–330.

[30] S. Uddin, M.F. Naghi, F.R. Al-Solamy, Another class of warped product submanifolds of Kenmotsu manifolds, RACSAM 112 (4) (2018) 1141–1155, http://dx.doi.org/10.1007/s13398-017-0415-6.

[31] K. Yano, M. Kon, Structures on Manifolds, in: Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.

Corresponding author
Siraj Uddin can be contacted at: siraj.ch@gmail.com