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Existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions

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Abstract
This paper is concerned with the existence of mild solutions for a class of fractional semilinear integro-differential equations having non-instantaneous impulses. The result is obtained by using noncompact semigroup theory and fixed point theorem. The obtained result is illustrated by an example at the end.

Keywords Fractional differential equations, Nonlocal conditions, Fixed point theorem, Noncompact semigroup, Measure of noncompactness
Paper type Original Article

1. Introduction

The objective of this paper is to study the existence of mild solutions to the following abstract integro-differential equations of fractional order with non-instantaneous impulses and nonlocal conditions in a Banach space X:

$$\begin{aligned} {}^cD^q u(t) + Au(t) &= f\left(t, u(t), \int_0^t K(t,s)u(s)ds\right), \quad t \in \cup_{k=0}^{m} (s_k, t_{k+1}], \\ u(t) &= \gamma_k(t, u(t)), \quad t \in \cup_{k=1}^{m} (t_k, s_k], \\ u(0) + g(u) &= u_0, \end{aligned} \tag{1.1}$$

JEL Classification — 34A08, 34A12, 34A37, 34K30, 45J05

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where ${}^cD^q$ is the Caputo fractional derivative of order $q(0 < q < 1)$, $A : D(A) \subset X \to X$ is closed linear operator, $-A$ is the infinitesimal generator of an equicontinuous and uniformly bounded C_0 semigroup $T(t)(t \geq 0)$ on $X, J = [0, a], a > 0$ is a constant, $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} := a$, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \ldots, m$, $f : J \times X \times X \to X$, $g : PC(J, X) \to X$ are given functions satisfying certain assumptions, $\gamma_k : (t_k, s_k] \times X \to X$ are non-instantaneous impulsive functions for all $k = 1, 2, \ldots, m$ and $K \in C(D, \mathbb{R}^+)$ where $D := \{(t, s) : 0 \leq s < t \leq a\}$ and $u_0 \in X$.

In the past decades, many researchers paid attention to study the differential equations with instantaneous impulses, which have been used to describe abrupt changes such as shocks, harvesting and natural disasters. Particularly, the theory of instantaneous impulsive equations have wide applications in control, mechanics, electrical engineering, biological and medical fields. For more details on the differential equations with instantaneous impulses one may see [2,4,7,14,15].

It seems that models with instantaneous impulses could not explain the certain dynamics of evolution process in pharmacotherapy. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in bloodstream and the consequent absorption for the body are gradual and continuous process. Hernández and O'Regan [12] and Pierri et al. [18], initially studied Cauchy problems for first order evolution equations with non-instantaneous impulses. The recent results for evolution equations with non-instantaneous impulses can be found in [1,8,13,19–21] and the references therein.

The nonlocal problem was motivated by physical problems. Indeed it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. For example it is used to represent mathematical models for evolution of various phenomena such as nonlocal neutral networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion (see [16]). The existence results to evolution equations with nonlocal conditions in Banach space were first studied by Byszewski [6]. Deng [9] used the nonlocal condition to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

To the best of our knowledge, there is no work yet reported on fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions (1.1) when the corresponding semigroup $T(t)(t \geq 0)$ is noncompact. Therefore inspired by the previous works, we will study the existence of PC-mild solutions for (1.1) under the assumption that the corresponding C_0 semigroup is noncompact, by using the properties of Kuratowski measure of noncompactness, and ρ-set contraction mapping fixed point theorem (see Lemma 2.10). We conclude this section by summarizing the contents of this paper. In the next section, we will introduce some basic definitions, notations and preliminary lemmas. In Section 3, we will prove existence of mild solutions for the problem (1.1) also we will give an example to illustrate the feasibility of our abstract result.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|$, we use θ to denote the zero function in $PC(J, X)$ and $J = [0, a]$ for any constant $a > 0$. Let $C(J, X)$ be a Banach space of all continuous functions from J into X endowed with supremum norm $\|u\|_C = \sup_{t \in J} \|u(t)\|$. Consider the space $PC(J, X) = \{u : J \to X : u$ is continuous at $t \neq t_k, u(t_{k-}) = u(t_k)$ and $u(t_{k+})$ exists for all $k = 1, 2, \ldots .m\}$, which is a Banach space endowed with supremum norm $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|$. For each finite constant $r > 0$, let $\Omega_r = \{u \in PC(J, X) : \|u(t)\| \leq r, t \in J\}$. Let $L^p(J, X)(1 \leq p < \infty)$ be the Banach space of all X-valued Bochner integrable functions defined on J with norm $\|u\|_{L^p(J,X)} = (\int_0^a \|u(t)\|^p dt)^{\frac{1}{p}}$. Denote $Gu(t) := \int_0^t K(t, s) u(s) ds$, and let $G^* = \sup_{t \in J} \int_0^t K(t, s) ds < \infty$. Let $M = \sup_{t \in J} \|T(t)\|_{\mathcal{L}(X)}$, where $\mathcal{L}(X)$ stands for the Banach space of all linear and bounded operators on X, note that $M \geq 1$. A C_0-semigroup $T(t)(t \geq 0)$ is called equicontinuous if the operator $T(t)$ is continuous by the operator norm for every $t > 0$.

Lemma 2.1 ([10]). *If h satisfies a uniform Hölder continuity with exponent $\beta \in (0, 1]$, then the unique solution of the following linear Cauchy problem:*

$$ {}^cD^q u(t) + Au(t) = h(t), \quad t \in J, \tag{2.1} $$
$$ u(0) = x_0 \in X, $$

is given by

$$ u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1} V(t-s)\, h(s)\, ds, \tag{2.2} $$

where

$$ U(t) = \int_0^\infty \zeta_q(\theta)\, T(t^q\theta)\, d\theta, \quad V(t) = q\int_0^\infty \theta\zeta_q(\theta)\, T(t^q\theta)\, d\theta, \tag{2.3} $$

$$ \zeta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q(\theta^{\frac{-1}{q}}), \rho_q(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!} sin(n\pi q), \theta \in (0,\infty), \tag{2.4} $$

$\zeta_q(\theta)$ is a probability density function defined on $(0, \infty)$.

Remark 2.2. $\zeta_q(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \zeta_q(\theta) d\theta = 1, \quad \int_0^\infty \theta\zeta_q(\theta) d\theta = \frac{1}{\Gamma(1+q)}$.

Lemma 2.3 ([22]). *The operators $U(t)(t \geq 0)$ and $V(t)(t \geq 0)$ have the following properties:*

(i) *For any fixed $t \geq 0$, $U(t)$ and $V(t)$ are strongly continuous.*

(ii) *For any fixed $t \geq 0$, $U(t)$ and $V(t)$ are linear bounded operators, moreover for any $u \in X$,*

$$ \|U(t)u\| \leq M\|u\|, \quad \|V(t)u\| \leq \frac{M}{\Gamma(q)}\|u\|. $$

(iii) *If $T(t)(t \geq 0)$ is an equicontinuous semigroup, then $U(t)$ and $V(t)$ are continuous for $t > 0$ by the operator norm, which means that for $0 < t' < t'' \leq a$, we have*

$$ \|U(t'') - U(t')\| \to 0 \text{ and } \|V(t'') - V(t')\| \to 0 \quad as \quad t'' \to t'. $$

Definition 2.4 ([13]). A function $u \in PC(J, X)$ is said to be a mild solution of the problem (1.1) if $u(0) = u_0 - g(u)$, $u(t) = \gamma_k(t, u(t))$ for all $t \in \cup_{k=1}^m (t_k, s_k]$, and

$$ u(t) = \begin{cases} U(t)(u_0 - g(u)) + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s), Gu(s)) ds, & t \in (0, t_1]; \\ U(t-s_k)\gamma_k(s_k, u(s_k)) + \int_{s_k}^t (t-s)^{q-1} V(t-s) f(s, u(s), Gu(s)) ds, \\ t \in (s_k, t_{k+1}], \ k = 1, 2, \ldots, m. \end{cases} $$

Now, we recall some properties of measure of noncompactness which are useful to prove our main result. For the details about measure of noncompactness, one may see [3,11]. Let $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness of the bounded set.

Lemma 2.5 ([3]). *Let X be a Banach space, and $U\subset C(J,X)$, $U(t)=\{u(t): u\in U\}(t\in J)$. If U is bounded and equicontinuous in $C(J,X)$, then $\alpha(U(t))$ is continuous on J, and $\alpha(U)=\max_{t\in J}\alpha(U(t))$.*

Lemma 2.6 ([11]). *If X be a Banach space and $D=\{u_n\}_{n=1}^{\infty}\subset PC(J,X)$ be a bounded and countable set, then $\alpha(D(t))$ is Lebesgue integrable on J, and*

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right)\le 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 2.7 ([5]). *Let X be a Banach space and U is bounded subset of X, then there exists a countable set $D=\{u_n\}_{n=1}^{\infty}\subset U$ such that $\alpha(U)\le 2\alpha(D)$.*

Lemma 2.8 ([3]). *Let X and E be Banach spaces and $Q: D(Q)\subset E\to X$ is Lipschitz continuous with constant L, then $\alpha(Q(V))\le L\alpha(V)$ for any bounded subset $V\subset D(Q)$.*

Definition 2.9 ([8]). Let X be a Banach space, and S be a nonempty subset of X. A continuous map $Q: S\to X$ is called ρ-set contractive if there exists a constant $\rho\in[0,1)$ such that for every bounded set $\Omega\subset S$,

$$\alpha(Q(\Omega))\le\rho\alpha(\Omega).$$

Lemma 2.10 ([8]). *Let X be a Banach space, $\Omega\subset X$ be a closed bounded and convex subset, and the operator $Q:\Omega\to\Omega$ is ρ-set contractive, then Q has at least one fixed point in Ω.*

3. Main result and example

In this section, we will discuss the existence of mild solutions for the system (1.1), then we will present an example to illustrate our proved result. Let us introduce the required assumptions which are needed to prove our main result:

(H1) For each $t\in J$, the function $f(t,\cdot,\cdot): X\times X\to X$ is continuous and for all $(x,y)\in X\times X$, the function $f(\cdot,x,y): J\to X$ is Lebesgue measurable.

(H2) There exist a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$, a constant $q_1\in(0,q)$, and a function $\phi\in L^{\frac{1}{q_1}}(J,\mathbb{R}^+)$ such that

$$\|f(t,x,y)\|\le\phi(t)\psi(\|x\|),\quad \forall x,y\in X; t\in J.$$

(H3) $g: PC(J,X)\to X$ is continuous and there exists a constant $\alpha^*>0$ such that

$$\|g(x)-g(y)\|\le\alpha^*\|x-y\|,\quad \forall x,y\in PC(J,X).$$

(H4) $\gamma_k:[t_k,s_k]\times X\to X$ are continuous and there exist constants $K_{\gamma_k}>0$, $k=1,2,\ldots,m$ such that

$$\|\gamma_k(t,x)-\gamma_k(t,y)\|\le K_{\gamma_k}\|x-y\|,\quad \forall x,y\in X; t\in[t_k,s_k].$$

(H5) There exist positive constants L_k and $N_k, k=0,1,2,\ldots,m$ such that for any countable sets $D_1,D_2\subset X$,

$$\alpha(f(t,D_1,D_2))\le L_k\alpha(D_1)+N_k\alpha(D_2),\quad \forall t\in(s_k,t_{k+1}], k=0,1,2,\ldots,m.$$

Let us denote:

$$K = \max_{k=1,2,\ldots,m} K_{\gamma_k}, \quad K^* = \max\{K, \alpha^*\},$$
$$L = \max_{k=0,1,2,\ldots,m} (L_k + N_k G^*)(t_{k+1} - s_k)^q. \tag{3.1}$$

Theorem 3.1. *Assume that the semigroup $T(t)(t \geq 0)$ generated by $-A$ is equicontinuous, the functions $g(\theta)$ and $\gamma_k(\cdot, \theta)$ are bounded for $k = 1, 2, \ldots, m$, and the assumptions* (H1)–(H5) *are satisfied, then the system* (1.1) *has at least one PC- mild solution provided that*

$$\max\{\Lambda_1, \Lambda_2\} < 1, \tag{3.2}$$

where $\Lambda_1 = M(\alpha^* + K)$ and $\Lambda_2 = M\,(K^* + \frac{4L}{\Gamma(q+1)})$.

Proof. Define the operator $F : PC(J, X) \to PC(J, X)$ as

$$(Fu)(t) = (F_1 u)(t) + (F_2 u)(t), \tag{3.3}$$

where

$$(F_1 u)(t) = \begin{cases} U(t)(u_0 - g(u)), & t \in [0, t_1]; \\ \gamma_k(t, u(t)), & t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ U(t - s_k)\gamma_k(s_k, u(s_k)), & t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m. \end{cases} \tag{3.4}$$

$$(F_2 u)(t) = \begin{cases} \displaystyle\int_{sk}^{t} t(t-s)^{q-1} V(t-s) f(s, u(s), Gu(s)) ds, \\ \qquad t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, m, \\ 0, \quad \text{otherwise} \end{cases} \tag{3.5}$$

It is easy to see that F is well defined. From Definition 2.4, one can easily see that the *PC*-mild solution of the system (1.1) is equivalent to a fixed point of the operator F defined by (3.3). Now, we will prove that the operator F has a fixed point.

Let $u \in \Omega_R$ for some $R > 0$, $q_2 = \frac{q-1}{1-q_1} \in (-1, 0)$ and $M_1 = \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J, \mathbb{R}^+)}$, by using Hölder inequality and (H2), we obtain

$$\int_0^t \|(t-s)^{q-1} f(s, u(s), Gu(s))\| ds \leqslant \left(\int_0^t (t-s)^{q_2} ds \right)^{1-q_1} \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J, \mathbb{R}^+)}$$
$$\leqslant \frac{M_1}{(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}. \tag{3.6}$$

Now, we divide the proof into the following steps:

Step I: We prove that there exists a constant $R > 0$ such that $F(\Omega_R) \subset \Omega_R$.

If this is not true, then for each $r > 0$, there will exist $u_r \in \Omega_r$ and $t_r \in J$ such that $\|(Fu_r(t_r))\| > r$. If $t_r \in [0, t_1]$, then by (3.3), (3.6), and (H3) we have

$$\begin{aligned}\|(Fu_r)(t_r)\| &\leq M(\|u_0\| + \alpha^*\|u_r - \theta\| + \|g(\theta)\| + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}}a^{(1+q_2)(1-q_1)} \\ &\leq M(\alpha^* r + \|u_0\| + \|g(\theta)\| + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}}a^{(1+q_2)(1-q_1)}.\end{aligned} \tag{3.7}$$

If $t_r \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, then by (3.4) and (H4), we obtain

$$\begin{aligned}\|(Fu_r)(t_r)\| &= (\gamma_k(t_r, u_r(t_r))) \\ &\leq K_{\gamma_k}\|u_r(t_r)\| + \|\gamma_k(t_r, \theta)\| \\ &\leq K_{\gamma_k} r + \beta,\end{aligned} \tag{3.8}$$

where $\beta = \max_{k=1,2,\ldots,m}\{\sup_{t\in J}\|\gamma_k(t,\theta)\|\}$. If $t_r \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, then by (3.3), (3.6), and (H4) we have

$$\begin{aligned}\|(Fu_r)(t_r)\| &\leq M(K_{\gamma_k} r + \beta) + M\int_{s_k}^{t_r}(t_r - s)^{q-1}\|f(s, u_r(s), Gu_r(s))\|ds \\ &\leq M(K_{\gamma_k} r + \beta) + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}}a^{(1+q_2)(1-q_1)}\end{aligned} \tag{3.9}$$

Combining (3.7)–(3.9) with the fact $r < \|(Fu_r)(t_r)\|$, we obtain

$$r < \|(Fu_r)(t_r)\| \leq M(\alpha^* r + \|u_0\| + \|g(\theta)\| + M(Kr + \beta) + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}}a^{(1+q_2)(1-q_1)}. \tag{3.10}$$

Dividing both sides of (3.10) by r and taking limit as $r \to \infty$, we have

$$1 \leq M(\alpha^* + K), \tag{3.11}$$

which contradicts (3.2).

Step II: We prove that the operator $F_1 : \Omega_R \to \Omega_R$ is Lipschitz continuous.

For $t \in [0, t_1]$ and $u, v \in \Omega_R$, using (3.4) and (H3) we have

$$\|(F_1u)(t) - (F_1v)(t)\| \leq M\|g(u) - g(v)\| \leq M\alpha^*\|u - v\|. \tag{3.12}$$

For $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$ and $u, v \in \Omega_R$, by (3.4) and the assumption (H4), we obtain

$$\|(F_1u)(t) - (F_1v)(t)\| \leq K_{\gamma_k}\|u(t) - v(t)\| \leq MK\|u - v\|. \tag{3.13}$$

For $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ and $u, v \in \Omega_R$, using (H4), we have

$$\|(F_1u)(t) - (F_1v)(t)\| \leq M\|\gamma_k(s_k, u(s_k)) - \gamma_k(s_k, v(s_k))\| \leq MK\|u - v\|. \tag{3.14}$$

From (3.12)–(3.14), we obtain

$$\|F_1u - F_1v\| \leq MK^*\|u - v\|, \tag{3.15}$$

where $K^* := m\{K, \alpha^*\}$.

Step III: In this step, we prove that F_2 is continuous on Ω_R.

Let $\{u_n\}$ be a sequence in Ω_R such that $\lim_{n\to\infty} u_n = u$ in Ω_R. By the continuity of nonlinear term f with respect to second and third variables, for each $s \in J$, we have

$$\lim_{n\to\infty} f(s, u_n(s), Gu_n(s)) = f(s, u(s), Gu(s)). \tag{3.16}$$

So, we can conclude that

$$\sup_{t\in J} \| f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s)) \| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.17}$$

For $s \in [s_k, t]$ and $t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, m$, by (3.16) and (3.17), we obtain

$$\begin{aligned}
&\|(F_2u_n)(t) - (F_2u)(t)\| \\
&\leq \frac{M}{\Gamma(q)} \int_{s_k}^{t} (t-s)^{q-1} \| f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s)) \| ds \qquad (3.18) \\
&\leq \frac{Ma^q}{\Gamma(q+1)} \sup_{t\in J} \| f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s)) \| \\
&\to 0 \quad \text{as} \quad n \to \infty. \qquad (3.19)
\end{aligned}$$

Hence,

$$\|F_2u_n - F_2u\| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.20}$$

which means that F_2 is continuous on Ω_R.

Step IV: Now, we show $F_2 : \Omega_R \to \Omega_R$ is equicontinuous.

For any $u \in \Omega_R$ and $s_k \leq t' < t'' \leq t_{k+1}$ for $k = 0, 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\|(F_2u)(t'') - (F_2u)(t')\| = \| \int_{s_k}^{t''} (t''-s)^{q-1} V(t''-s) f(s, u(s), Gu(s)) ds \\
&- \int_{s_k}^{t'} (t'-s)^{q-1} V(t''-s) f(s, u(s), Gu(s)) ds \| \\
&\leq \| \int_{t'}^{t''} (t''-s)^{q-1} V(t''-s) f(s, u(s), Gu(s)) ds \| \\
&\quad + \| \int_{s_k}^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}] V(t''-s) f(s, u(s), Gu(s)) ds \| \\
&\quad + \| \int_{s_k}^{t'} (t'-s)^{q-1} [V(t''-s) - V(t'-s)] f(s, u(s), Gu(s)) ds \| \\
&\quad = I_1 + I_2 + I_3,
\end{aligned}$$

where,

$$I_1 = \| \int_{t'}^{t''} (t''-s)^{q-1} V(t''-s) f(s, u(s), Gu(s)) ds \|,$$

$$I_2 = \| \int_{s_k}^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}] V(t''-s) f(s, u(s), Gu(s)) ds \|,$$

$$I_3 = \| \int_{s_k}^{t'} (t'-s)^{q-1}[V(t''-s) - V(t'-s)]f(s,u(s),Gu(s))ds\|.$$

Now, we only need to check that I_1, I_2 and I_3 tend to 0 independently of $u \in \Omega_R$ when $t'' \to t'$. By (3.6), we have

$$I_1 \leq \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}(t''-t')^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad t'' \to t'.$$

For I_2, by (H2), Lemma 2.3, Hölder inequality, and [22], we get that

$$I_2 \leq \frac{M}{\Gamma(q)}\left(\int_{s_k}^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}]^{\frac{1}{1-q_1}} ds\right)^{1-q_1} \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R})}$$

$$\leq \frac{M_1 M}{\Gamma(q)}\left(\int_{s_k}^{t'} [(t'-s)^{q_2} - (t''-s)^{q_2}]ds\right)^{1-q_1}$$

$$\leq \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}[(t')^{1+q_2} - (t'')^{1+q_2} + (t''-t')^{1+q_2}]^{1-q_1}$$

$$\leq \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}(t''-t')^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad t'' \to t'.$$

For $t' = s_k$, it is easy to see that $I_3 = 0$. For $t' > s_k$ and $\epsilon > 0$ small enough, by (H2), Lemma 2.3, and the equicontinuity of $T(t)$, we estimate

$$I_3 \leq \| \int_{s_k}^{t'-\epsilon} (t'-s)^{q-1}[V(t''-s) - V(t'-s)]f(s,u(s),Gu(s))ds\|$$

$$+\| \int_{t'-\epsilon}^{t'} (t'-s)^{q-1}[V(t''-s) - V(t'-s)]f(s,u(s),Gu(s))ds\|$$

$$\leq \int_{s_k}^{t'-\epsilon} \|(t'-s)^{q-1}f(s,u(s),Gu(s))\|ds \sup_{s\in[s_k,t'-\epsilon]} \|V(t''-s) - V(t'-s)\|$$

$$+\frac{2M}{\Gamma(q)}\int_{t'-\epsilon}^{t'} \|(t'-s)^{q-1}f(s,u(s),Gu(s))\|ds$$

$$\leq \frac{M_1}{(1+q_2)^{1-q_1}}((t')^{1+q_2} - \epsilon^{1+q_2})^{1-q_1} \sup_{s\in[s_k,t'-\epsilon]} \|V(t''-s) - V(t'-s)\|$$

$$+\frac{2M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad t'' \to t'.$$

As a result, $\|(F_2u)(t'') - (F_2u)(t')\| \to 0$ independently of $u \in \Omega_R$ as $t'' \to t'$, which means that $F_2 : \Omega_R \to \Omega_R$ is equicontinuous.

Step V: We show that $F : \Omega_R \to \Omega_R$ is a ρ-set contractive map.

For any bounded set $D \subset \Omega_R$, by Lemma 2.7, we know that there exists a countable set $D_0 = \{u_n\} \subset D$ such that

$$\alpha(F_2(D)) \leq 2\alpha(F_2(D_0)). \tag{3.21}$$

Since $F_2(D_0) \subset F_2(\Omega_R)$ is bounded and equicontinuous, by Lemma 2.5, we get

$$\alpha(F_2(D_0)) = \max_{t \in [s_k, t_{k+1}], k=0,1,2,\ldots,m} \alpha(F_2(D_0)(t)). \tag{3.22}$$

For every $t \in [s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, by Lemma 2.6, the assumption (H5) and (3.1), we have

$$\begin{aligned} \alpha(F_2(D_0)(t)) &= \alpha\left(\left\{ \int_{s_k}^{t} (t-s)^{q-1} V(t-s) f(s, u_n(s), Gu_n(s)) ds \right\}\right) \\ &\leq \frac{2M}{\Gamma(q)} \int_{s_k}^{t} (t-s)^{q-1} \alpha(\{f(s, u_n(s), Gu_n(s))\}) ds \\ &\leq \frac{2M}{\Gamma(q)} \int_{s_k}^{t} (t-s)^{q-1} [L_k \alpha(D_0(s)) + N_k \alpha(GD_0(s))] ds. \end{aligned} \tag{3.23}$$

Meanwhile, we have

$$\alpha(GD_0(s)) \leq \alpha(GD_0) \leq \|G\| \alpha(D_0) \leq G^* \alpha(D_0) \leq G^* \alpha(D). \tag{3.24}$$

Therefore,

$$\alpha(F_2(D_0)(t)) \leq \frac{2M}{\Gamma(q+1)} (L_k + N_k G^*)(t_{k+1} - s_k)^q \alpha(D) \leq \frac{2ML}{\Gamma(q+1)} \alpha(D). \tag{3.25}$$

From (3.21) and (3.25), we obtain

$$\alpha(F_2(D)) \leq \frac{4ML}{\Gamma(q+1)} \alpha(D). \tag{3.26}$$

From (3.15) and Lemma 2.8, we know that for any bounded set $D \subset \Omega_R$,

$$\alpha(F_1(D)) \leq MK^* \alpha(D). \tag{3.27}$$

Therefore, by (3.26) and (3.27), we obtain

$$\alpha(F(D)) \leq \alpha(F_1(D)) + \alpha(F_2(D)) \leq M\left(K^* + \frac{4L}{\Gamma(q+1)}\right)\alpha(D) = \Lambda_2 \alpha(D). \tag{3.28}$$

Now combining (3.28) with (3.2) and Definition 2.9, we get that $F : \Omega_R \to \Omega_R$ is a ρ-set-contractive map with $\rho = \Lambda_2$. Hence Lemma 2.10 implies that F has at least one fixed point $u \in \Omega_R$, which is a PC-mild solution of (1.1). □

Next, we present an example to illustrate our main result.

Example. Consider the following fractional partial differential system with non-instantaneous impulses and nonlocal conditions:

$$
\begin{cases}
{}^{c}D^{\frac{1}{2}}u(t,x) + \dfrac{\partial^2}{\partial x^2}u(t,x) & \\
= \dfrac{1}{25}\dfrac{e^{-t}}{1+e^{t}}u(t,x) + \displaystyle\int_0^t \dfrac{1}{50}e^{-s}u(s,x)ds, & x\in(0,1), t\in(0,\frac{1}{3}]\cup(\frac{2}{3},1]; \\
u(t,0) = u(t,1) = 0, & t\in[0,1]; \\
u(t,x) = \dfrac{e^{-(t-\frac{1}{3})}}{4}\dfrac{|u(t,x)|}{1+|u(t,x)|}, & x\in(0,1), t\in(\frac{1}{3},\frac{2}{3}]; \\
u(0,x) + \displaystyle\sum_{i=1}^{2}\dfrac{1}{3^i}u(\frac{1}{i},x) = u_0(x), & x\in[0,1].
\end{cases}
\tag{3.29}
$$

Let $X = L^2[0,1]$ and $Au = u''$ with $D(A) = \{u\in X : u, u'$ are absolutely continuous and $u''\in X, u(0) = u(1) = 0\}$. It is well known by [17], that $-A$ generates an equicontinuous C_0-semigroup $T(t)(t\geq 0)$ on X, and $\|T(t)\|\leq 1$, for any $t\geq 0$. Let $a = t_2 = 1, t_0 = s_0 = 0$, $t_1 = \frac{1}{3}, s_1 = \frac{2}{3}$. By putting

$$
u(t) = u(t,\cdot),
$$

$$
f(t,u(t),Gu(t)) = \frac{1}{25}\frac{e^{-t}}{1+e^{t}}u(t,\cdot) + \int_0^t \frac{1}{50}e^{-s}u(s,\cdot)ds,
$$

$$
Gu(t) = \int_0^t \frac{1}{50}e^{-s}u(s,\cdot)ds,
$$

$$
\gamma_1(t,u(t)) = \frac{e^{-(t-\frac{1}{3})}}{4}\frac{|u(t,\cdot)|}{1+|u(t,\cdot)|},
$$

$$
g(u) = \sum_{i=1}^{2}\frac{1}{3^i}u\left(\frac{1}{i},\cdot\right),
$$

the parabolic partial differential equation (3.29) can be rewritten into the abstract form of (1.1) for $m = 1$. It is easy to verify that the assumptions (H1)–(H5) and condition (3.2) hold with

$$
q = \frac{1}{2}, M = 1, \ \ \phi(t) = \frac{1}{25}\frac{e^{-t}}{1+e^{t}} + \frac{1}{50}, \ \ \psi;(r) = r,
$$

$$
\alpha^* = \frac{4}{9}, \ \ K = K_{\gamma_1} = \frac{1}{4}, \ \ L = 0.02, \ \ \Lambda_1 = 0.69 < 1, \ \ \Lambda_2 = 0.53 < 1.
$$

Therefore, Theorem 3.1 is applicable, so the system (3.29) has at least one PC-mild solution.

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